Available online at www.sciencedirect.com
SCIENCE DIRECT®
ACADEMIC PRESS
Journal of Magnetic Resonance 163 (2003) 121–123
JMR Journal of Magnetic Resonance
www.elsevier.com/locate/jmr

SNR-optimality of sum-of-squares reconstruction for phased-array magnetic resonance imaging[☆]

Erik G. Larsson,[a,*] Deniz Erdogmus,[a] Rui Yan,[a] Jose C. Principe,[a] and Jeffrey R. Fitzsimmons[b]

[a] *Department of Electrical and Computer Engineering, University of Florida, Gainesville, FL 32611, USA*
[b] *Department of Radiology, University of Florida, Gainesville, FL 32610, USA*

Received 6 December 2002; revised 16 April 2003

Abstract

We consider the commonly used "Sum-of-Squares" (SoS) reconstruction method for phased-array magnetic resonance imaging with unknown coil sensitivities. We show that the signal-to-noise ratio (SNR) in the image produced by SoS is asymptotically (as the input SNR $\to \infty$) equal to that of maximum-ratio combining, which is the best unbiased reconstruction method when the coil sensitivities are known. Finally, we discuss the implications of this result.

Keywords: Magnetic resonance imaging; Sum-of-squares; Signal-to-noise ratio; Statistical properties

1. Introduction

Magnetic resonance imaging (MRI) has become an extremely important instrument for the diagnosis of a number of important and serious diseases. The development of MRI was sparked in the early 1950s when Bloch and Purcell were awarded the Nobel Prize for their discovery of nuclear magnetic resonance. Since then, the MRI technology has undergone significant development, and several researchers have suggested to use systems with multiple receiver coils to improve the imaging speed and quality. The first implementation of such a so-called *phased-array* system is probably due to Roemer et al. [1], but the ideas of using multiple detectors for MRI can be traced back to the late 1980s. A good summary of the early phased-array MRI technology is provided in the review paper [2]. More recently, a substantial body of research has focussed on sophisticated techniques for phase encoding together with the use of gradient coils with the primary aim of increasing the imaging speed. This work includes the *sensitivity encoding for fast MRI* (SENSE) technique [3] and *simultaneous acquisition of spatial harmonics* (SMASH) imaging [4]. Owing to the last decade's intensive research on the topic, phased-array MR imaging is now becoming a mature field and arrays with up to 16 elements have been designed and used for imaging experiments [5].

In principle, with phased-array technology, an increase in imaging speed equal to the number of parallel coils can be achieved. However, the use of large coil arrays imposes a number of difficulties, in particular for high field strengths. Most importantly, since the coil sensitivities are typically unknown variables (which are very difficult to model for high magnetic fields), optimal and artifact-free image reconstruction is a challenge. The most commonly used method for image reconstruction is the so-called "sum-of-squares" (SoS) method, which effectively computes the root-mean-square average of the images associated with the different coils.

The SoS method does not utilize any knowledge of the coil sensitivities, or otherwise use any a priori information about the system. In addition to SoS,

[☆] This work was partially supported by the NSF Grant ECS-9900394.
[*] Corresponding author. Fax: 1-352-392-0044.
E-mail address: larsson@dsp.ufl.edu (E.G. Larsson).

1090-7807/03/$ - see front matter
doi:10.1016/S1090-7807(03)00132-0

a number of alternative techniques for image reconstruction with phased-array coils have appeared during the last decade. For example, a clever method by Debbins et al. [6] adds the images *coherently*, after their relative phase was properly adjusted (for instance, by using a hardware phase shifter—in which case a standard single-channel receiver can be used). Bydder et al. [7] proposed a method that attempts to estimate the coil sensitivities from the image, and report that the resulting images have somewhat less variance than the SoS reconstruction. Walsh et al. [8] used adaptive filters to improve the SNR in the reconstructed image. Finally, we should note Kellman and McVeigh [9] proposed a method to use the phased array for ghost artifact cancellation, which is an important related problem.

In this short paper, we study the SoS method and show that it gives asymptotically the same SNR[1] as maximum ratio combining (which is also referred to as optimal linear combining), which can only be used when the coil sensitivities are perfectly known. Therefore, from a pure SNR point of view, SoS is optimal in the asymptotic regime and knowledge of the coil sensitivities cannot improve the SNR in the reconstructed image. We therefore believe that work on signal processing methods for phased-array image reconstruction should to a larger extent focus on bias[2] reduction, rather than SNR improvement.

2. Data model

Consider a phased-array MRI system with N coils and let s_k be the observed pixel value from coil k

$$s_k = \rho c_k + e_k, \quad k = 1, 2, \ldots, N, \tag{1}$$

where ρ is the (real-valued) object density (viz. the MR contrast), c_k is the (in general complex-valued) sensitivity associated with coil k for the image voxel under consideration, and e_k is zero-mean noise with variance σ^2. We assume in this paper that all signals have been prewhitened to account for the noise correlation; as is well known, such prewhitening is easily accomplished by premultiplying the received data with the inverse of the Hermitian square root of the noise covariance matrix.

[1] SNR refers to the ratio between the signal power and the noise power; if a reconstructed pixel is given by $\hat{s} = \alpha s + e$, where α is a scaling factor, s is the true pixel value, and e is noise with zero mean and variance σ^2, then the SNR is equal to $|\alpha s|^2/\sigma^2$.

[2] Bias refers to the difference between the true image, and the average (over the noise) of the reconstructed image, i.e., using the notation of footnote 1, the bias in the reconstructed pixel $\hat{s}$ is given by $E[\hat{s} - s] = (\alpha - 1)s$ where $E[\cdot]$ stands for statistical expectation.

3. Analysis of reconstruction methods

3.1. Maximum-ratio combining

If the coil sensitivities c_k are known, the optimal estimate of ρ can be shown to be

$$\tilde{\rho} = \Re\left(\frac{\sum_{k=1}^{N} c_k^* s_k}{\sum_{k=1}^{N} |c_k|^2}\right) = \rho + \Re\left(\frac{\sum_{k=1}^{N} c_k^* e_k}{\sum_{k=1}^{N} |c_k|^2}\right), \tag{2}$$

where $(\cdot)^*$ stands for the complex conjugate and $\Re(\cdot)$ denotes the real part. A neat and self-contained derivation of this result can be found in, for example, [1,2], although it also follows directly by using some standard results on minimum variance estimation theory [10]. We can easily establish that $\tilde{\rho}$ is unbiased, i.e., $E[\tilde{\rho}] = \rho$, where $E[\cdot]$ stands for statistical expectation. Then the SNR in $\tilde{\rho}$ is equal to [1,2,10]

$$\begin{aligned} SNR_{\text{opt}} &= \frac{\rho^2}{E[|\tilde{\rho} - \rho|^2]} = \frac{\rho^2}{E\left[\left(\Re\left(\frac{\sum_{k=1}^{N} c_k^* e_k}{\sum_{k=1}^{N} |c_k|^2}\right)\right)^2\right]} \\ &= 2 \cdot \frac{\rho^2}{\sigma^2} \cdot \sum_{k=1}^{N} |c_k|^2 \end{aligned} \tag{3}$$

assuming the real and imaginary parts of the noise are uncorrelated and have the same variance.

3.2. Sum-of-squares reconstruction

The SoS method is applicable when $\{c_k\}$ are unknown. The reconstructed pixel is obtained via

$$\hat{\rho} = \sqrt{\sum_{k=1}^{N} |s_k|^2}. \tag{4}$$

(This SoS estimate can be interpreted as an optimal linear combination according to Eq. (2) but with c_k replaced by $s_k/\sqrt{\sum_{k=1}^{N} |s_k|^2}$ [7].) Clearly, if the noise level goes to zero the SoS estimate converges to $\hat{\rho} \to \rho\sqrt{\sum_{k=1}^{N} |c_k|^2}$ which is in general *not* equal to ρ. Therefore, SoS reconstruction typically yields severely biased images, *even in the noise-free case*. Unless c_k is constant for all coils (which is certainly *not* the case in practice), this bias depends on the coil number k and hence it cannot be corrected for if c_k is unknown. Also, $\{c_k\}$ are typically not constant over an entire image, and therefore the bias will be location-dependent, which may imply serious artifacts in the image.

We next analyze the statistical properties of the SoS method. For a high input SNR, the expression for $\hat{\rho}$ in Eq. (4) can be written as

$$\begin{aligned}\hat{\rho} &= \sqrt{\sum_{k=1}^{N} |\rho c_k + e_k|^2} \\ &= \sqrt{\sum_{k=1}^{N} [\rho^2 |c_k|^2 + 2\rho \Re(c_k^* e_k) + |e_k|^2]} \\ &\approx \rho \sqrt{\sum_{k=1}^{N} |c_k|^2} \sqrt{1 + \frac{2\sum_{k=1}^{N} \Re(c_k^* e_k)}{\rho \sum_{k=1}^{N} |c_k|^2}} \\ &\approx \rho \sqrt{\sum_{k=1}^{N} |c_k|^2} \left[1 + \frac{\sum_{k=1}^{N} \Re(c_k^* e_k)}{\rho \sum_{k=1}^{N} |c_k|^2}\right] \\ &= \rho \sqrt{\sum_{k=1}^{N} |c_k|^2} + \frac{\sum_{k=1}^{N} \Re(c_k^* e_k)}{\sqrt{\sum_{k=1}^{N} |c_k|^2}}. \end{aligned} \tag{5}$$

In the first approximation, the higher-order term is discarded, while a first-order Taylor series expansion is used in the second approximation. Clearly, $E[\hat{\rho}] \neq \rho$ in general and thus we see again that SoS gives biased images. The SNR in $\hat{\rho}$ is obtained as

$$SNR_{\text{SoS}} = \frac{\left(\rho\sqrt{\sum_{k=1}^{N} |c_k|^2}\right)^2}{E\left[\left|\frac{\sum_{k=1}^{N} \Re(c_k^* e_k)}{\sqrt{\sum_{k=1}^{N} |c_k|^2}}\right|^2\right]} = 2 \cdot \frac{\rho^2}{\sigma^2} \cdot \sum_{k=1}^{N} |c_k|^2, \tag{6}$$

which is equal to the SNR for optimal combining with known coil sensitivities (see Eq. (3)). Therefore, from a pure SNR point of view, SoS is optimal for high input SNR.

We stress that our comparison contrasts SoS with maximum-ratio-combining, assuming in both cases that the measured data have been prewhitened to account for noise correlation (cf. the remark at the end of Section 2). Our result should not be confused by comparisons between prewhitened and non-prewhitened image combination (such as the calculation by Roemer et al. [1, pp. 209–210], which showed that combination without taking the noise correlation into account achieves 90% of the SNR corresponding to prewhitened combination; assuming a 41% noise correlation).

4. Concluding remarks

The analysis in this paper shows that for high input SNR, the simple and commonly used SoS method closely approximates the theoretical upper bound (i.e., the SNR given by maximum-ratio-combining using perfect knowledge of the coil sensitivities). This result explains why it is often difficult to enhance the SNR in reconstructed phased-array images by using signal processing methods, and it raises the question as to whether SoS is the "best possible" reconstruction method. As SoS gives biased results in general—and as the SNR generally correlates only weakly with image quality as perceived by the human eye—we would argue that it may *not* be the best possible method; yet the results in this paper suggest that the difference, at least in terms of SNR, between an "optimally" reconstructed image and the SoS image may be small in many cases. In summary, we believe that image improvement via signal processing may also focus on quantities other than the SNR—such as bias or information theoretic measures, and that work on optimal image reconstruction for phased-array MRI is a ripe ground for research. Our results in this area will be presented elsewhere.

References

[1] P.B. Roemer, W.A. Edelstein, C.E. Hayes, S.P. Souza, O.M. Mueller, The NMR phased array, Magnetic Resonance in Medicine 16 (1990) 192–225.
[2] S.M. Wright, L.L. Wald, Theory and application of array coils in MR spectroscopy, NMR in Biomedicine 10 (1997) 394–410.
[3] K.P. Pruessmann, M. Weiger, M.B. Scheidegger, P. Boesiger, SENSE: sensitivity encoding for fast MRI, Magnetic Resonance in Medicine 42 (1999) 952–962.
[4] J.A. Bankson, M.A. Griswold, S.M. Wright, D.K. Sodickson, SMASH imaging with an eight element multiplexed RF coil array, Magnetic Resonance Materials in Physics, Biology and Medicine (MAGMA) 10 (2000) 93–104.
[5] J.R. Porter, S.M. Wright, A. Reykowski, A 16-element phased-array head coil, Magnetic Resonance in Medicine 40 (1998) 272–279.
[6] J.P. Debbins, J.P. Felmlee, S.J. Riederer, Phase alignment of multiple surface coil data for reduced bandwidth and reconstruction requirements, Magnetic Resonance in Medicine 38 (1997) 1003–1011.
[7] M. Bydder, D.J. Larkman, J.V. Hajnal, Combination of signals from array coils using image-based estimation of coil sensitivity profiles, Magnetic Resonance in Medicine 47 (2002) 539–548.
[8] D.O. Walsh, A.F. Gmitro, M.W. Marcellin, Adaptive reconstruction of phased array MR imagery, Magnetic Resonance in Medicine 43 (2000) 682–690.
[9] P. Kellman, E.R. McVeigh, Ghost artifact cancellation using phased array processing, Magnetic Resonance in Medicine 46 (2001) 335–343.
[10] S.M. Kay, Fundamentals of Statistical Signal Processing: Estimation Theory, Prentice-Hall, Englewood Cliffs, NJ, 1993.