6.8 $X_t = S_t + n_t \qquad \{n_t\}_{t=0}^{N-1} \text{ iid} \quad n_0 \sim \mathcal{N}(0, \sigma^2)$

$S_t = A\cos(\omega t - \phi)$.

let $\underline{X} = [X_0 \dots X_{N-1}]^T$

$\underline{S} = [S_0 \dots S_{N-1}]^T$

$\underline{\theta} = [A, \phi, \omega, \sigma^2]$

$$\log f_{\underline{X}|\underline{\theta}}(\underline{X}|\underline{\theta}) = \log \frac{1}{\sqrt{2\pi}^N} \frac{1}{\sqrt{\sigma^{2N}}} \exp\left[-\frac{1}{2}\frac{1}{\sigma^2}\|\underline{X}-\underline{S}\|^2\right]$$

$$\therefore \log f_{\underline{X}|\underline{\theta}} = K - \frac{N}{2}\log(\sigma^2) - \frac{1}{2}\frac{1}{\sigma^2}\sum_{t=0}^{N-1}(X_t - S_t)^2$$

$$\frac{\partial}{\partial A}\log f_{\underline{X}|\underline{\theta}} = \frac{1}{\sigma^2}\sum_{t=0}^{N-1}(X_t - S_t)\cdot\frac{\partial S_t}{\partial A}$$

$$\frac{\partial^2}{\partial A^2}\log f_{\underline{X}|\underline{\theta}} = \frac{1}{\sigma^2}\sum -\left(\frac{\partial S_t}{\partial A}\right)^2 + (X_t - S_t)\cancel{\frac{\partial^2}{\partial A^2}S_t}^{\,0}$$

$$= -\frac{1}{\sigma^2}\sum_{t=0}^{N-1}\left(\frac{\partial S_t}{\partial A}\right)^2$$

$$-E\frac{\partial^2}{\partial A^2}\log f_{\underline{X}|\underline{\theta}} = \frac{1}{\sigma^2}\sum(\partial S_t/\partial A)^2$$

$$\frac{\partial^2}{\partial\omega\partial A}\log f_{\underline{X}|\underline{\theta}} = \frac{1}{\sigma^2}\sum -\left(\frac{\partial S_t}{\partial\omega}\right)\left(\frac{\partial S_t}{\partial A}\right) + (X_t - S_t)\frac{\partial^2}{\partial\omega\partial A}S_t$$

$$-E\frac{\partial^2}{\partial\omega\partial A}\log f_{\underline{X}|\underline{\theta}} = \frac{1}{\sigma^2}\sum(\partial S_t/\partial\omega)(\partial S_t/\partial\omega)$$

$$\partial^2/\partial\phi\partial A \log f_{\underline{X}|\underline{\theta}} = \frac{1}{\sigma^2}\sum -\left(\frac{\partial S_t}{\partial\phi}\right)(\partial S_t/\partial A) + (X_t - S_t)\frac{\partial^2 S_t}{\partial\phi\partial A}$$

$$-E(\;''\;) = \frac{1}{\sigma^2}\sum(\partial S_t/\partial\phi)(\partial S_t/\partial A)$$

similarly

$$-E\left[\partial^2/\partial\sigma^2\partial A \log f_{\underline{X}|\underline{\theta}}\right] = 0$$

$$-E\left[\frac{\partial^2}{\partial\phi^2}\log f_{\underline{X}|\theta}\right] = \frac{1}{\sigma^2}\sum(\partial S_t/\partial\phi)^2$$

$$-E\left[\frac{\partial^2}{\partial\phi\partial\omega}\log f_{\underline{X}|\theta}\right] = \frac{1}{\sigma^2}\sum\left(\frac{\partial S_t}{\partial\phi}\right)\left(\frac{\partial S_t}{\partial\omega}\right)$$

$$-E\left[\frac{\partial^2}{\partial\sigma^2\partial\phi}\log f_{\underline{X}|\underline{\theta}}\right] = 0$$

$$-E\left[\frac{\partial^2}{\partial\omega^2}\log f_{\underline{X}|\underline{\theta}}\right] = \frac{1}{\sigma^2}\sum\left(\frac{\partial S_t}{\partial\omega}\right)^2$$

$$-E\left[\frac{\partial^2}{\partial\sigma^2\partial\omega}\log f_{\underline{X}|\underline{\theta}}\right] = 0$$

and

$$\frac{\partial}{\partial\sigma^2}\log f_{\underline{X}|\underline{\theta}} = -\frac{N}{2}\frac{1}{\sigma^2} + \frac{1}{2}\frac{1}{(\sigma^2)^2}\sum(X_t - S_t)^2$$

$$\frac{\partial^2}{\partial\sigma^2\partial\sigma^2}\log f_{\underline{X}|\underline{\theta}} = \frac{N}{2}\frac{1}{\sigma^4} - \frac{1}{(\sigma^2)^3}\sum(X_t - S_t)^2$$

so

$$-E\left[\frac{\partial^2}{\partial\sigma^2\partial\sigma^2}\log f_{\underline{x}|\underline{\theta}}\right] = \frac{N}{(\sigma^2)^2} - \frac{N}{2}\frac{1}{(\sigma^2)^2} = \frac{N}{2\sigma^4}$$

also

$$\frac{\partial S_t}{\partial A} = \cos(\omega t - \phi)$$

$$\frac{\partial S_t}{\partial \phi} = A\sin(\omega t - \phi)$$

$$\frac{\partial S_t}{\partial \omega} = -tA\sin(\omega t - \phi)$$

so

$$J_{\underline{x}}(\underline{\theta}) = \frac{1}{\sigma^2}\begin{bmatrix} \sum\left(\frac{\partial S_t}{\partial A}\right)^2 & \sum\left(\frac{\partial S_t}{\partial \phi}\right)\left(\frac{\partial S_t}{\partial A}\right) & \sum\left(\frac{\partial S_t}{\partial \omega}\right)\left(\frac{\partial S_t}{\partial A}\right) & 0 \\ \sum\left(\frac{\partial S_t}{\partial \phi}\right)\left(\frac{\partial S_t}{\partial A}\right) & \sum\left(\frac{\partial S_t}{\partial \phi}\right)^2 & \sum\left(\frac{\partial S_t}{\partial \phi}\right)\left(\frac{\partial S_t}{\partial \omega}\right) & 0 \\ \sum\left(\frac{\partial S_t}{\partial \omega}\right)\left(\frac{\partial S_t}{\partial A}\right) & \sum\left(\frac{\partial S_t}{\partial \phi}\right)\left(\frac{\partial S_t}{\partial \omega}\right) & \sum\left(\frac{\partial S_t}{\partial \omega}\right)^2 & 0 \\ 0 & 0 & 0 & \frac{N}{2\sigma^2} \end{bmatrix}$$

from section 6.12

$$SNR = NA^2/2\sigma^2$$

$\sin(x-y) = \sin(x)\cos(y) - \cos(x)\sin(y)$

$\sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y)$

$\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y)$

$\cos(x-y) = \cos(x)\cos(y) + \sin(x)\sin(y)$

$$\sum\left(\frac{\partial S_t}{\partial A}\right)^2 = \frac{1}{2}\sum\left(1+\cos(2\omega t - 2\phi)\right)$$

$$= \frac{N}{2} + \frac{1}{2}\underbrace{\sum_{t=0}^{N-1}\cos(2\omega t - 2\phi)}_{\sim \Delta\int_0^{2\pi}\cos(x)\,dx \;=\; 0}$$

Further

$$\sum\left(\frac{\partial S_t}{\partial \phi}\right)\left(\frac{\partial S_t}{\partial A}\right) = \sum A\cos(\omega t - \phi)\sin(\omega t - \phi)$$

$$= \frac{A}{2}\underbrace{\sum\sin(2\omega t - 2\phi)}_{\sim\Delta\int_0^{2\pi}\sin(x)\,dx \;=\; 0} + 0$$

$$\sum \left(\frac{\partial s_t}{\partial \phi}\right)^2 = \sum A^2 \sin^2(\omega t - \phi) = A^2\Big(N - \underbrace{\sum \cos^2(\omega t - \phi)}_{\sim N/2}\Big) \sim \frac{N}{2}A^2$$

$$\begin{aligned}
\sum t \sin^2(\omega t - \phi) &= \sum t - \sum t \cos^2(\omega t - \phi) \\
&= \sum \frac{t}{2} + \sum \frac{t}{2} \cos(2\omega t - 2\phi) \\
&= \sum \frac{t}{2} + \frac{1}{4} \frac{\partial}{\partial \omega} \underbrace{\sum \sin(2\omega t - 2\phi)}_{\sim \Delta \int_0^{2\pi} \sin(x)\,dx = 0} \\
&\sim \frac{1}{2} \sum t
\end{aligned}$$

$$\begin{aligned}
\sum t \sin(\omega t - \phi) \cos(\omega t - \phi) &= \frac{1}{2} \sum t \sin(2\omega t - 2\phi) \\
&= -\frac{1}{4} \frac{\partial}{\partial \omega} \underbrace{\sum \cos(2\omega t - 2\phi)}_{\approx \Delta \int_0^{2\pi} \cos(x)\,dx = 0} \\
&\approx 0.
\end{aligned}$$

large N approximation to $J_{\underline{x}}(\underline{\theta})$.

$$J_{\underline{x}}(\underline{\theta}) = \frac{1}{\sigma^2} \begin{bmatrix} N/2 & 0 & 0 & 0 \\ 0 & A^2 N/2 & -\frac{A^2}{2}\sum t & 0 \\ 0 & -\frac{A^2}{2}\sum t & \frac{A^2}{2}\sum t^2 & 0 \\ 0 & 0 & 0 & \frac{N}{2\sigma^2} \end{bmatrix}$$

Since

$$\begin{aligned}
\sum t^2 \sin^2(\omega t - \phi) &= \sum t^2 - \sum t^2 \cos^2(\omega t - \phi) \\
&= \frac{1}{2}\sum t^2 + \frac{1}{2}\sum t^2 \cos(2\omega t - 2\phi) \\
&= \frac{1}{2}\sum t^2 - \frac{1}{2} \frac{\partial^2}{\partial \omega^2} \frac{1}{4} \underbrace{\sum \cos(2\omega t - 2\phi)}_{\approx 0} \\
&\approx \frac{1}{2} \sum t^2
\end{aligned}$$

$$\det J_{\underline{x}} = \frac{N}{2\sigma^2} \det \frac{1}{\sigma^2}\begin{bmatrix} \frac{NA^2}{2} & -\frac{A^2}{2}\Sigma t & 0 \\ -\frac{A^2}{2}\Sigma t & \frac{A^2}{2}\Sigma t^2 & 0 \\ 0 & 0 & \frac{N}{2\sigma^2} \end{bmatrix}$$

$$= \frac{N}{2\sigma^2} \cdot \frac{N}{2\sigma^4} \quad \frac{A^4}{\sigma^4}\left[\frac{N}{4}\Sigma t^2 - \left(\frac{1}{2}\Sigma t\right)^2\right]$$

$$\det J_x = \frac{A^4 N^2}{16(\sigma^2)^5}\left[N\Sigma t^2 - (\Sigma t)^2\right]$$

$$[J_x^{-1}]_{1,1} = \frac{\det \frac{1}{\sigma^2}\begin{bmatrix} A^2N/2 & -\frac{A^2}{2}\Sigma t & 0 \\ -\frac{A^2}{2}\Sigma t & \frac{A^2}{2}\Sigma t^2 & 0 \\ 0 & 0 & \frac{N}{2\sigma^2} \end{bmatrix}}{\det J_x} = \frac{2\sigma^2}{N}$$

$$[J_x^{-1}]_{2,2} = \frac{\det \frac{1}{\sigma^2}\begin{bmatrix} N/2 & 0 & 0 \\ 0 & \frac{A^2}{2}\Sigma t^2 & 0 \\ 0 & 0 & \frac{N}{2\sigma^2} \end{bmatrix}}{\det J_x} = \frac{\frac{N^2 A^2}{8(\sigma^2)^4}\Sigma t^2}{\frac{N^2 A^4}{16(\sigma^2)^5}\left[N\Sigma t^2 - (\Sigma t)^2\right]}$$

$$[J_x^{-1}]_{2,2} = \frac{2\sigma^2}{A^2}\frac{1}{\left[N - \frac{(\Sigma t)^2}{\Sigma t^2}\right]}$$

$$[J_x^{-1}]_{3,3} = \frac{\det \frac{1}{\sigma^2}\begin{bmatrix} \frac{N}{2} & 0 & 0 \\ 0 & A^2N/2 & 0 \\ 0 & 0 & \frac{N}{2\sigma^2} \end{bmatrix}}{\det J_x} = \frac{\frac{A^2N^3}{(\sigma^2)^4}\frac{1}{8}}{\frac{N^2 A^4}{16(\sigma^2)^5}\left[N\Sigma t^2 - (\Sigma t)^2\right]}$$

$$[J_x^{-1}]_{3,3} = 2\sigma^2 \Big/ A^2\left[\Sigma t^2 - \frac{1}{N}(\Sigma t)^2\right]$$

$$[J_x^{-1}]_{4,4} = \frac{\det \frac{1}{\sigma^2}\begin{bmatrix} N/2 & 0 & 0 \\ 0 & A^2 N/2 & -\frac{A^2}{2}\Sigma t \\ 0 & -\frac{A^2}{2}\Sigma t & \frac{A^2}{2}\Sigma t^2 \end{bmatrix}}{\det J_x}$$

$$= \frac{\frac{N}{2\sigma^2}\left[\frac{N}{4}\Sigma t^2 - \left(\frac{1}{2}\Sigma t\right)^2\right]\frac{A^4}{\sigma^4}}{\det J_x}$$

$$= \frac{\frac{A^4 N^2}{8(\sigma^2)^3}\left[\Sigma t^2 - \frac{1}{N}(\Sigma t)^2\right]}{\frac{N^2 A^4}{16(\sigma^2)^5}\left[N\Sigma t^2 - (\Sigma t)^2\right]}$$

$$= \frac{2}{N}\sigma^4$$

Large-N approximations to the variance bounds

A: $[J_x^{-1}]_{1,1} = \frac{2\sigma^2}{N} = A^2\,\frac{2\sigma^2}{NA^2} = \frac{1}{SNR}A^2$

ϕ: $[J_x^{-1}]_{22} = \frac{1}{SNR}\,\frac{1}{\left[1 - \frac{1}{N}\frac{(\Sigma t)^2}{\Sigma t^2}\right]}$

ω: $[J_x^{-1}]_{3,3} = \frac{1}{SNR}\,\frac{N}{\left[\Sigma t^2 - \frac{1}{N}(\Sigma t)^2\right]}$

σ^2: $[J_x^{-1}]_{4,4} = \frac{1}{SNR}\,\sigma^2 A^2$

$A=1$, $\omega=1/20$, $\phi=0$, $\sigma^2 = NA^2/2\,SNR$