negative binomial distn

$$P(x) = \binom{x+r-1}{r-1} \pi^r (1-\pi)^x \qquad 0 \le \pi \le 1$$

x = # failed in indep trials $\qquad x = 0, 1, 2, \dots$

r = # successes

π = success prob. $\qquad r = 1, \dots$

Consider testing $\theta_0 = r$ versus $\theta_1 = r+n$ for π known. The likelihood ratio is

$$\frac{(x+r+n-1)!}{(r+n-1)!\,x!} \cdot \frac{(r-1)!\,x!}{(x+r-1)!} \pi^n = \underbrace{K \cdot (x+r+n-1)\cdots(x+r)}_{\text{increases in } x}$$

Consider testing $\theta_0 = \pi$ versus $\theta_1 = \pi - \Delta$, for r known. The likelihood ratio is

$$\left(\frac{\pi-\Delta}{\pi}\right)^r \left(\frac{1-\pi+\Delta}{1-\pi}\right)^x = \left(1 + \frac{\Delta}{1-\pi}\right)^x \cdot K$$

since $\Delta > 0$, $1 + \frac{\Delta}{1-\pi} > 1$, and the likelihood ratio increases in x.

Poisson distn

$$p(x) = \frac{\lambda^x e^{-\lambda}}{x!}, \quad x = 0, 1, 2 \quad \lambda > 0$$

Use problem 4.9, with

$$c(\theta) = e^{-\lambda}$$

$$\pi(\theta) = \log \lambda$$

$$t(x) = x$$

$$h(x) = \frac{1}{x!}$$

4.20 By the result in 4.19, the likelihood ratio is monotone ↑ in $\sum_{i=0}^{N-1} t(\underline{x}_i)$, since $Q(\theta)$ is non-decreasing. Since θ and $\sum_{i=0}^{N-1} t(\underline{x}_i)$ are both scalars, the Karlin-Rubin Thm shows that a threshold test on $\sum_{i=0}^{N-1} t(\underline{x}_i)$ is UMP for:

$$H_0: \theta \le \theta_0$$
$$H_1: \theta > \theta_0 .$$

namely:

$$\varphi\left(\sum_{i=0}^{N-1} t(\underline{x}_i)\right) = \begin{cases} 0, & \Sigma t < \ell \\ \gamma, & \Sigma t = \ell \\ 1, & \Sigma t > \ell \end{cases}$$

$$\alpha = \gamma P_{\theta_0}(\Sigma t = \ell) + P_{\theta_0}[\Sigma t > \ell]$$

4.23 $\underline{x} = \mu H \underline{\theta} + \underline{w}\sigma$, $\underline{w} \sim \mathcal{N}(\underline{0}, I)$, $\langle A \rangle = \langle H \rangle^{\perp}$

a) $$QQ^T = \begin{bmatrix} (H^TH)^{-\frac{1}{2}} & 0 \\ 0 & (A^TA)^{-1/2} \end{bmatrix} \underbrace{\begin{bmatrix} H^T \\ A^T \end{bmatrix} \cdot \begin{bmatrix} H & A \end{bmatrix}}_{\begin{bmatrix} (H^TH) & \\ & (A^TA) \end{bmatrix}} \begin{bmatrix} (H^TH)^{-\frac{1}{2}} & 0 \\ 0 & (A^TA)^{-\frac{1}{2}} \end{bmatrix}$$

$$= I$$

(b) $$Q\underline{x} = \begin{bmatrix} \underline{x}_1 \\ \underline{x}_2 \end{bmatrix} \sim \mathcal{N}\left(\mu \begin{bmatrix} (H^TH)^{\frac{1}{2}}\theta \\ 0 \end{bmatrix}, \sigma^2 I\right)$$

(c) $\Leftarrow$ let $\underline{y} = Q\underline{x}$, $\underline{y} \sim$ "

Consider $\underline{z} = H(H^TH)^{-1/2}\underline{x}_1 + A(A^TA)^{-1/2}\underline{x}_2$

Show that $\underline{z} = \underline{x}$.

$$\underline{z} = HH^{\#}\underline{x} + AA^{\#}\underline{x} = \underline{x}\text{, since } \langle H \rangle^{\perp} = \langle A \rangle.$$

4.24 $\dfrac{\underline{y}^T P_H \underline{y}}{\underline{y}^T\underline{y}} = \dfrac{M(\underline{y})}{M(\underline{y})+1}$, where $M(\underline{y}) = \dfrac{\underline{y}^T P_H \underline{y}}{\underline{y}^T (I-P_H)\underline{y}}$

Show that $M(\underline{y})$ is invariant to gains.

$$M(\sigma\underline{y}) = \frac{\sigma\underline{y}^T P_H \sigma\underline{y}}{\sigma\underline{y}^T (I-P_H)\sigma\underline{y}} = M(\underline{y})$$

Show that $M(\underline{y})$ is invariant to rotations in $\langle H\rangle$.

$Q_H = U_H Q U_H^T + (I-P_H)$, $Q^T Q = I$

$$P_H Q_H \underline{y} = U_H Q U_H^T \underline{y} \Rightarrow (Q_H\underline{y})^T P_H (Q_H\underline{y}) = \underline{y} P_H \underline{y}$$

$$(I-P_H)Q_H\underline{y} = (I-P_H)\underline{y} \Rightarrow (Q_H\underline{y})^T (I-P_H)(Q_H\underline{y}) = \underline{y}^T(I-P_H)\underline{y}$$

so $M(Q_H\underline{y}) = M(\underline{y})$.