ECEG335 Detection & Estimation
Solution Set 4

4.9 Observe $\{\underline{X}_i\}_{i=0}^{N-1}$ i.i.d, with $\underline{X}_i \sim f_\theta(\underline{x}_i)$

$H_0: \theta \le \theta_0$

$H_1: \theta > \theta_0$

here

$$f_\theta(\underline{x}_i) = c(\theta)\, h(\underline{x}_i) \exp\left[T(\theta)\, t(\underline{x}_i)\right]$$

let $T(\theta)$ be nondecreasing in θ. Then for any $\theta_1 > \theta_0 \ge \theta$

$$l(\underline{x}_0, \dots, \underline{x}_{N-1}) = \frac{\prod_{i=0}^{N-1} f_{\theta_1}(\underline{x}_i)}{\prod_{i=0}^{N-1} f_\theta(\underline{x}_i)} = \frac{c^N(\theta_1)}{c^N(\theta)} \exp\left[\{T(\theta_1) - T(\theta)\} \sum_{i=0}^{N-1} t(\underline{x}_i)\right]$$

- θ is a scalar.
- $\sum_{i=0}^{N-1} t(\underline{x}_i)$ is a scalar
- $l(\underline{x}_0, \dots, \underline{x}_{N-1}) = \lambda(\sum t(\underline{x}_i))$ is nondecreasing in $\sum t(\underline{x}_i)$ if $T(\theta_1) - T(\theta) > 0$.

by the Karlin Rubin theorem, a threshold test on $\sum t(\underline{x}_i)$ is UMP for H_0 vs. H_1.

4.10 normal distn. $\underline{x}_i \sim N(\theta \underline{m}, R)$, $\underline{m}$, R known.

H_0: $\theta \leq \theta_0$

H_1: $\theta > \theta_0$

$$f_\theta(\underline{x}_i) = \frac{1}{\sqrt{2\pi}^N} \frac{1}{\sqrt{\det R}} \exp\left[-\frac{1}{2}(\underline{x}_i - \theta\underline{m})^T R^{-1}(\underline{x}_i - \theta\underline{m})\right]$$

use 4.9, with

$$c(\theta) = \frac{1}{\sqrt{2\pi}^N} \frac{1}{\sqrt{\det R}} \exp\left[-\frac{\theta^2}{2} \underline{m}^T R^{-1} \underline{m}\right]$$

$$h(\underline{x}_i) = \exp\left[-\frac{1}{2} \underline{x}_i^T R^{-1} \underline{x}_i\right]$$

$$t(\underline{x}_i) = \underline{m}^T R^{-1} \underline{x}_i$$

$$\pi(\theta) = \theta$$

gamma distn.

scalar x: $$f(x) = \left(\frac{x-\mu}{\beta}\right)^{\gamma-1} \exp\left[-\frac{x-\mu}{\beta}\right] \frac{1}{\beta\Gamma(\gamma)}$$

$$x > \mu, \; \gamma > 0, \; \beta > 0$$

γ is the shape parameter.

μ is the location parameter.

β is the scale parameter.

rewrite in exponential form

$$f(x) = \exp\left[-\log\beta - \log\Gamma(\gamma) + (\gamma-1)\log(x-\mu) - (\gamma-1)\log(\beta) - \frac{1}{\beta}x + \frac{\mu}{\beta}\right]$$

For μ known gamma distn is a member of the exponential family, with $\pi_1(\theta)t_1(x) = (\gamma-1)\log(x-\mu)$

$$\pi_2(\theta)t_2(x) = -\frac{1}{\beta}x$$

(a) Now, since both μ and γ are known, and we are testing for scale:

$$H_0: \beta \le \beta_0$$
$$H_1: \beta > \beta_0$$

From 4.9, with $\theta = \beta$, and

$$c(\theta)\, h(x_i) = \exp\left[-\log\beta - \log\Gamma(\gamma) + (\gamma-1)\log(x_i-\mu) - (\gamma-1)\log\beta + \frac{\mu}{\beta}\right]$$

$$\pi(\theta)\, t(x_i) = -\frac{1}{\beta}x_i$$

Since $-\frac{1}{\beta}$ increases with β, the conditions in 4.9 are met.

(b) Next suppose that μ and β are known, and we are testing for shape:

$$H_0: \gamma \le \gamma_0$$
$$H_1: \gamma > \gamma_0$$

Use 4.9, with $\theta = \gamma$, and

$$c(\theta)\, t(x_i) = (\gamma-1)\log(x_i-\mu)$$

Since $(\gamma-1)$ is increasing in γ and $t(x_i) = \log(x_i-\mu)$ is a scalar, the conditions of 4.9 are met.

(c) Finally, if we are testing for location, with everything else known, we have

$$\theta = -\mu$$

$$l(x) = \exp\left[\frac{1}{\beta}(\theta_0-\theta_1) + (\gamma-1)\log\left[\frac{x+\theta_1}{x+\theta_0}\right]\right]$$

since $\frac{x+\theta_1}{x+\theta_0}$ increases in x for $\theta_1 > \theta_0$, the K-R theorem applies.

β - dbn.

$$f(x) = \frac{(x-a)^{p-1}(b-x)^{q-1}}{B(p,q)\,(b-a)^{p+q-1}} \qquad a \le x \le b,\ p>0,\ q>0$$

p & q are shape parameters

rewrite $f(x)$ in exponential form

$$f(x) = \exp\left[(p-1)\log(x-a) + (q-1)\log(b-x) - \log(B(p,q)) - (p+q-1)\log(b-a)\right] \qquad a \le x \le b,\ p>0,\ q>0.$$

(a) When testing for $\theta = p$, with all else known, the UMP test applies, since

$$\pi(\theta) = \theta - 1$$

$$t(x_i) = \log(x_i - a).$$

(b) similar result when $\theta = q$

(c) suppose we are testing $\theta = a$, with all else known. Then

$$\ell(x) = \frac{f_{\theta_1}(x)}{f_{\theta_0}(x)} = \left(\frac{x-a_1}{x-a_0}\right)^{p-1}\left(\frac{b-a_0}{b-a_1}\right)^{p+q-1}$$

$$= \left(\frac{x+\theta_1}{x+\theta_0}\right)^{p-1}\left(\frac{b+\theta_0}{b+\theta_1}\right)^{p+q-1}$$

Since $\left(\frac{x+\theta_1}{x+\theta_0}\right)^{p-1}$ increases in x when $\theta_1 > \theta_0$ the K-R Thm applies.

(d) When $\theta = b$, w.r. all else known, the likelihood ratio is

$$\underbrace{\left(\frac{\theta_1 - x}{\theta_0 - x}\right)^{g-1}}_{\text{increase in } x \text{ when } \theta_1 > \theta_0} \left(\frac{\theta_0 - a}{\theta_1 - a}\right)^{p-g-1}$$

The K-R Thm applies when $\theta = b$.

binomial distr

$$p_s(x) = \binom{N}{x} s^x (1-s)^{N-x}, \quad x = 0, 1, \ldots$$

$$= \exp\left[\log\binom{N}{x} + x \log\frac{s}{1-s} + N\log(1-s)\right]$$

When $\theta = s$, $\pi(\theta) = \log\frac{\theta}{1-\theta}$, $t(x_i) = x_i$.
The conditions of problem 4.9 are met.

Consider testing $\theta = N$ versus $\theta = N+n$, with s known.
The likelihood ratio is

$$\frac{\binom{N+n}{x}}{\binom{N}{x}} (1-s)^n = \frac{(N+n)!}{(N+n-x)!}\frac{(N-x)!}{N!} (1-s)^n$$

$$= K(N,n,s) \frac{1}{(N+n-x)(N+n-x-1)\cdots(N-x+1)}$$

The log likelihood ratio is

$$\underbrace{\ell(x)}_{\text{increases in } x.} = \log K - \sum_{j=0}^{n-1} \underbrace{\log(N+n-j-x)}_{\text{decreases in } x}$$