ECEG335 Detection and Estimation Theory

Solution Set 2 **Fall, 2004**

Problem 3.11

By setting $\mathbf{x}_t = [x_0, \dots x_t]^T$, then $\mathbf{x}_t$ may be constructed from

$$\begin{aligned}
\mathbf{x}_t &= \mathbf{H}_t\theta + \mathbf{n}_t, \\
\mathbf{H}_t &= \begin{bmatrix} \mathbf{H}_{t-1} \\ \mathbf{c}_t^T \end{bmatrix}, \\
\mathbf{c}_t &= \begin{bmatrix} 0 & \dots & \dots & 0 \\ 1 & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \mathbf{c}_{t-1} + \begin{bmatrix} h_t \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \\
\mathbf{H}_{-1} &= [\,], \\
\mathbf{c}_{-1} &= \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}.
\end{aligned}$$

Let $\mathbf{\Lambda}_t^2 = \sigma^2 \mathbf{I}_{t+1}$. From class notes, $\mathbf{H}_t^T \mathbf{\Lambda}_t^2 \mathbf{x}_t$ is minimal and sufficient. From (3.64) in the text,

$$\mathbf{H}_t^T \mathbf{\Lambda}_t^2 \mathbf{x}_t = \mathbf{H}_{t-1}^T \mathbf{\Lambda}_{t-1}^2 \mathbf{x}_{t-1} + \frac{1}{\sigma^2}\mathbf{c}_t^T \mathbf{x}_t.$$

Problem 3.12

Since $\{\mathbf{x}_i\}$ is an i.i.d. set of vectors,

$$f_\theta(\mathbf{x}_0, \dots \mathbf{x}_{M-1}) = \left(\prod_{i=0}^{M-1} c(\theta)\right)\left(\prod_{i=0}^{M-1} a(\mathbf{x}_i)\right) exp\left[\sum_{k=1}^{k} \Pi_i(\theta)\left(\sum_{j=0}^{M-1} t_i(\mathbf{x}_j)\right)\right].$$

From this it is clear that the joint distribution of the M vectors belongs to the $k-$parameter exponential family. Intuitively, the sample mean of each sufficient statistic for $\mathbf{x}_0$ is sufficient for the random sample.

Problem 3.13d

$$p_\theta(\mathbf{x}_0) = \underbrace{(1-\theta)^N}_{c(\theta)} \underbrace{\binom{N+x_0-1}{x_0} 1_{\{0,1,\ldots N\}}(x_0)}_{a(x_0)} \quad \underbrace{\mathrm{e}^{x_0 \log\theta}}$$

$$\begin{aligned} t_1(x_0) &= x_0 \\ \Pi_1(\theta) &= \log\theta \end{aligned}$$

This is a $1-$parameter exponential family. From problem 3.12, $\sum_{i=0}^{M-1} x_i$ is sufficient for θ, and it is also complete. The characteristic function of this statistic is

$$E\left[e^{jv}\sum_{i=0}^{M-1} x_i\right] = \frac{(1-\theta)^{MN}}{(1-\theta e^{jv})^{MN}}.$$

Note that the sufficient statistic has the same distribution as that for x_0, by replacing N with NM.

Problem 3.13e

$$p_\theta(\mathbf{x}_0) = e^{-\theta}\frac{1}{x_0!}1_{\{0,1,2,\ldots\}}(x_0)e^{x_0\log\theta}$$

This is a 1-parameter exponential family. The statistic $\sum_{i=0}^{M-1}$ is sufficient for θ, and is Poission distributed with parameter $M\theta$.

Problem 3.13f

$$F_\theta(x_0) = 1_{[0,\infty]}(x_0)\int_0^{x_0} e^{-y/\theta_1}y^{\theta_2-1}dy\frac{1}{\Gamma(\theta_2)\theta_1^{\theta_2}}$$

The pdf is found to be

$$f_\theta(x_0) = \frac{1}{\Gamma(\theta_2)\theta_1^{\theta_2}}\frac{1}{x_0}1_{[0,\infty]}(x_0)e^{-\frac{1}{\theta_1}x_0+\theta_2\log x_0}.$$

This is a 2-parameter exponential family, and $\left(\sum_{i=0}^{M-1} x_i \ , \ \sum_{i=0}^{M-1} \log x_i\right)$ **is sufficient for** (θ_1, θ_2). **Equivalently,** $\left(\sum_{i=0}^{M-1} x_i \ , \ \prod_{i=0}^{M-1} x_i\right)$ **is sufficient.**

Problem 3.13g

$$f_\theta(x_0) = 1_{[0,1]}(x_0)\frac{\Gamma(\theta_1+\theta_2)}{\Gamma(\theta_1)\Gamma(\theta_2)}x_0^{\theta_1}(1-x_0)^{\theta_2}\frac{1}{1-x_0}\frac{1}{x_0}.$$

This is a 2-parameter exponential family, and $\left(\sum_{i=0}^{M-1} \log(x_i) \ , \ \sum_{i=0}^{M-1} \log(1-x_i)\right)$ **is sufficient for** θ. **Equivalently,** $\left(\prod_{i=0}^{M-1}(x_i) \ , \ \prod_{i=0}^{M-1}(1-x_i)\right)$ **is sufficient.**

Problem 3.14d

It is not obvious how to proceed, so, let's begin by considering the simplest case, $M = 1$. **Here we are looking for a mapping** $W_N(x_0)$ **such that its average is** θ. **In other words,**

$$\begin{aligned} \textstyle\sum_{k=0}^{\infty} W_N(k) \binom{N+k-1}{k} (1-\theta)^N \theta^k &= \theta \\ \textstyle\sum_{k=0}^{\infty} W_N(k) \underbrace{\binom{N+k-1}{k}}_{\frac{N+k-1}{k}\binom{N+(k-1)-1}{k-1}} (1-\theta)^N \theta^{k-1} &= 1. \end{aligned}$$

This last equality is true for $W_N(k) = k/(N+k-1)$, **and** $W_N(x_0)$ **is unbiased for** θ.

For M **samples we use the fact that the sufficient statistic has the same distribution as** x_0, **by replacing** N **with** NM. **So, an unbiased estimator for** θ **is** $W_{NM}(\sum_{i=0}^{M-1} x_i)$. **From the answer to 3.13d, the characteristic function may be differentiated to find the mean for** x_0, $N\theta/(1-\theta)$, **and the variance for** x_0, $N\theta/(1-\theta)^2$. **The variance of our estimator may be found by computing the second moment for** $W_{MN}(\sum_{i=0}^{M-1} x_i)$, **and the variance is** $\theta/(1-\theta)^2/(MN)$

Problem 3.14e

$Ex_0 = Var(x_0) = \theta$. **As a result,** $\frac{1}{M}\sum_{i=0}^{M-1} x_i$ **is MVUB, and the minimumm variance is** $\frac{\theta}{M}$**.**

Problem 3.15

(a.)

$$C = {\theta_1}^{-\theta_0}(1-\theta_1)$$

(b.)

$$p_\theta(\mathbf{x}) = (1-\theta_1)^N \theta_1^{\sum_{i=0}^{N-1} x_i} \underbrace{\prod_{i=0}^{N-1} 1_{\{0,1,\ldots\}}(x_i - \theta_0)}_{1_{\{0,1,\ldots\}}(x_0-\theta_0)}$$

(c.)

If θ_0 **is known, then** $\sum_{i=0}^{N-1} x_i$ **is sufficient for** θ_1 **by the factorization theorem.**

(d.)

If θ_1 **is known, then** x_0 **is sufficient for** θ_0**. Equivalently, any one of the** x_i **is sufficient.**

(e.)

The characteristic function of $S_N = \sum_{i=0}^{N-1} x_i$ **is**

$$ES_N = \frac{e^{jvN\theta_0} C^N}{(1-\theta_1 e^{jv})^N}.$$