ECEG335 Detection and Estimation Theory

Solution Set 1 **Fall, 2004**

Problem 2.2.a

$$\begin{aligned}
\mathbf{X} &= [\mathbf{x}_1 \ \mathbf{x}_2 \ \ \mathbf{x}_3 \ ], \quad \mathbf{Q} = [\mathbf{q}_1 \ \mathbf{q}_2 \ \ \mathbf{q}_3 \ ], \quad \mathbf{R} = [\mathbf{r}_1 \ \mathbf{r}_2 \ \ \mathbf{r}_3 \ ] \\
\mathbf{x}_1 &= [3 \ 1 \ 2]^T, \quad \|\mathbf{x}_1\|^2 = 14, \quad \mathbf{q}_1 = \frac{1}{\sqrt{14}}[3 \ 1 \ 2]^T, \quad \mathbf{r}_1 = [\sqrt{14} \ 0 \ 0]^T, \\
\mathbf{x}_2 &= [2 \ 3 \ 1]^T, \quad \mathbf{y}_2 = \frac{1}{14}[-5 \ 31 \ \ -8]^T, \quad \|\mathbf{y}_2\| = \frac{1}{14}\sqrt{1050}, \\
&\mathbf{q}_2 = \frac{1}{\sqrt{1050}}[-5 \ 31 \ \ -8]^T, \quad \mathbf{r}_2 = \left[11/\sqrt{14} \ \ \sqrt{1050}/14 \ \ 0\right]^T, \\
\mathbf{x}_3 &= [1 \ 2 \ 3]^T, \quad \mathbf{y}_3 = [-1.20 \ 0.24 \ 1.68]^T, \\
\|\mathbf{y}\| &= 2.08, \quad \mathbf{q}_3 = [-0.58 \ 0.12 \ 0.81]^T, \quad \mathbf{r}_3 = [2.9399 \ 1.0184 \ 2.0785]^T
\end{aligned}$$

Problem 2.2.b

$$
\begin{aligned}
\mathbf{X} &= \begin{bmatrix} 3 & 2 & 1 \\ 1 & 3 & 2 \\ 2 & 1 & 3 \end{bmatrix}, \\
\hat{\mathbf{x}}_1 &= \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}, \quad \mathbf{v}_1 = \begin{bmatrix} -0.7414 \\ 1 \\ 2 \end{bmatrix} \\
\mathbf{Q}_1^T &= \hat{\mathbf{Q}}_1^T = \begin{bmatrix} 0.8018 & 0.2673 & 0.5345 \\ 0.2673 & 0.6396 & -0.7207 \\ 0.5345 & -0.7207 & -0.4414 \end{bmatrix}, \\
\mathbf{Q}_1^T\mathbf{X} &= \begin{bmatrix} 3.7417 & 2.9399 & 2.9399 \\ 0 & 1.7327 & -0.6156 \\ 0 & -1.5345 & -2.2312 \end{bmatrix}, \\
\mathbf{v}_2 &= \begin{bmatrix} -0.5818 \\ -1.5345 \end{bmatrix}, \\
\hat{\mathbf{Q}}_2^T &= \begin{bmatrix} 0.7486 & -0.6630 \\ -0.6630 & -0.7486 \end{bmatrix}, \\
\mathbf{Q}_2^T &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.7486 & -0.6630 \\ 0 & -0.6630 & -0.7486 \end{bmatrix}, \\
\mathbf{Q}_2^T\mathbf{Q}_1^T\mathbf{X} &= \begin{bmatrix} 3.7417 & 2.9399 & 2.9399 \\ 0 & 2.3146 & 1.0184 \\ 0 & 0 & -2.0785 \end{bmatrix} = \mathbf{R}, \\
\mathbf{Q} &= \mathbf{Q}_1\mathbf{Q}_2.
\end{aligned}
$$

Problem 2.6

Denote $\mathbf{X}_n = [\mathbf{x}_1 \ \ldots \mathbf{x}_n]$. Note that $\mathbf{X}_n$ is n by n, has rank n, and is invertible. Consider $\mathbf{G} = \mathbf{X}_n^H \mathbf{X}_n$, and note that $\mathbf{G}$ is invertible since $\{\mathbf{x}_i\}$ are linearly independent. Define $\hat{\mathbf{x}} = \mathbf{X}_n(\mathbf{X}_n^H \mathbf{X}_n)^{-1}\mathbf{X}_n^H \mathbf{X}$, which is a linear combination of the $\{\mathbf{x}_i\}$. The proof is complete if $\mathbf{x} = \hat{\mathbf{x}}$. The result follows from the fact that $(\mathbf{X}_n^H \mathbf{X}_n)^{-1} = \mathbf{X}_n^{-1}\mathbf{X}_n^{-H}$.

Problem 2.8

First we find $\mathbf{P}_{k-1}$. Define $\mathbf{Q}_{k-1} = [\mathbf{q}_1 \ \mathbf{q}_2 \ \ldots \ \mathbf{q}_{k-1}]$. We propose that $\mathbf{P}_{k-1} = \mathbf{Q}_{k-1}(\mathbf{Q}_{k-1}^H \mathbf{Q}_{k-1})^{-1}\mathbf{Q}_{k-1}^H$. First note that $\mathbf{P}_{k-1}^2 = \mathbf{P}_{k-1}$, so that it is a projector. Next note that $\mathbf{Q}_{k-1}^H \mathbf{Q}_{k-1} = \mathbf{I}_{k-1}$, and so

$$\begin{aligned}\mathbf{y}_k &= \mathbf{x}_k - \textstyle\sum_{n=1}^{k-1} \mathbf{q}_n \mathbf{q}_n^H \mathbf{x}_k, \\ &= \mathbf{x}_k - \mathbf{Q}_{k-1}\mathbf{Q}_{k-1}^H \mathbf{x}_k, \\ &= (\mathbf{I} - \mathbf{P}_{k-1})\mathbf{x}_k.\end{aligned}$$

Now we solve the second part of the problem. Since

$$\mathbf{y}_k = \mathbf{x}_k - \mathbf{Q}_{k-1}\mathbf{Q}_{k-1}^H \mathbf{x}_k,$$

by definition, all we must do is express $\mathbf{Q}_{k-1}$ in terms of $\mathbf{X}_{k-1} = [\mathbf{x}_1 \ \mathbf{x}_2 \ \ldots \ \mathbf{x}_{k-1}]$. Since $\mathbf{Q}_{k-1} = \mathbf{X}_{k-1}(\mathbf{X}_{k-1}^H \mathbf{X}_{k-1})^{-1}\mathbf{X}_{k-1}^H \mathbf{Q}_{k-1}$, it follows that $\mathbf{Q}_{k-1}\mathbf{Q}_{k-1}^H = \mathbf{X}_{k-1}(\mathbf{X}_{k-1}^H \mathbf{X}_{k-1})^{-1}\mathbf{X}_{k-1}^H$, and $\mathbf{y}_k$ may be expressed as

$$\mathbf{y}_k = \mathbf{x}_k - \mathbf{X}_{k-1}(\mathbf{X}_{k-1}^H \mathbf{X}_{k-1})^{-1}\mathbf{X}_{k-1}^H \mathbf{x}_k$$

This equation clearly shows that if $\mathbf{y}_k = \mathbf{0}$, then $\mathbf{x}_k$ is a linear combination of $\{\mathbf{x}_i\}$. Further, if $\mathbf{x}_k$ is a linear combination of the $\{\mathbf{x}_i\}$, then the right-hand side must be zero by necessity.

Problem 2.10.a

$$\begin{aligned}\mathbf{x}_1 &= \mathbf{y}_1 = [1 \ 2 \ 3]^T, \quad \|\mathbf{y}_1\| = \sqrt{14}, \quad \mathbf{q}_1 = \tfrac{1}{\sqrt{14}}[1 \ 2 \ 3]^T, \quad \mathbf{r}_1 = [\sqrt{14} \ 0 \ 0]^T, \\ \mathbf{x}_2 &= [4 \ 5 \ 6]^T, \quad \mathbf{y}_2 = \tfrac{1}{7}[12 \ 3 \ -6]^T, \quad \|\mathbf{y}_2\| = \sqrt{189}/7, \\ &\mathbf{q}_2 = \tfrac{1}{\sqrt{189}}[12 \ 3 \ -6]^T, \quad \mathbf{r}_2 = [32/\sqrt{14} \ \sqrt{189}/7 \ 0]^T, \\ \mathbf{x}_3 &= [7 \ 8 \ 0]^T, \quad \mathbf{y}_3 = [-1.5 \ 3 \ -1.5]^T, \quad \|\mathbf{y}_3\| = \sqrt{13.5}, \\ &\mathbf{q}_3 = \sqrt{\tfrac{2}{27}}[-1.5 \ 3 \ -1.5]^T, \quad \mathbf{r}_3 = [23/\sqrt{14} \ 108/\sqrt{189} \ \|\mathbf{y}_3\|]^T.\end{aligned}$$

Problem 2.10.b

$$
\begin{aligned}
\mathbf{v}_1 &= \begin{bmatrix} -2.7417 \\ 2 \\ 3 \end{bmatrix} \\
\hat{\mathbf{Q}}_1 &= \mathbf{Q}_1 = \begin{bmatrix} 0.2673 & 0.5345 & 0.8018 \\ 0.5345 & 0.6101 & -0.5849 \\ 0.8018 & -0.584 & 0.1227 \end{bmatrix}, \\
\mathbf{Q}_1^T\mathbf{X} &= \begin{bmatrix} 3.7417 & 8.5524 & 6.1470 \\ 0 & 1.6791 & 8.6222 \\ 0 & 1.0187 & 0.9334 \end{bmatrix}, \\
\mathbf{v}_2 &= \begin{bmatrix} -0.2849 \\ 1.0187 \end{bmatrix}, \\
\hat{\mathbf{Q}}_2^T &= \begin{bmatrix} 0.8550 & 0.5187 \\ 0.5187 & -0.8550 \end{bmatrix}, \\
\mathbf{Q}_2^T &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.8550 & 0.5187 \\ 0 & 0.5187 & -0.8550 \end{bmatrix}, \\
\mathbf{Q}_2^T\mathbf{Q}_1^T &= \mathbf{R} = \begin{bmatrix} 3.7417 & 8.5524 & 6.1470 \\ 0 & 1.9640 & 7.8558 \\ 0 & 0 & 3.6743 \end{bmatrix}, \\
\mathbf{Q} &= \mathbf{Q}_1\mathbf{Q}_2 = \begin{bmatrix} 0.2673 & 0.8729 & -0.4082 \\ 0.5345 & 0.2182 & 0.8165 \\ 0.8018 & -0.4364 & -0.4083 \end{bmatrix}.
\end{aligned}
$$

Problem 2.17

To show that $\mathbf{U}$ is orthogonal, we expand $\mathbf{U}^H\mathbf{U}$ as

$$
\mathbf{U}^H\mathbf{U} = \underbrace{\mathbf{D}^{-H}\,\underbrace{\mathbf{A}^H\,\underbrace{\mathbf{X}^H\mathbf{X}}_{\mathbf{G}}\,\mathbf{A}}_{\mathbf{D}^2}\,\mathbf{D}^{-1}}_{I\ if\ \mathbf{D}\ is\ real}.
$$