EXTENSIONS OF AVERAGING THEORY FOR POWER ELECTRONIC SYSTEMS

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***Abstract**—This paper extends averaging theory for power electronic systems to include feedback controlled converters. New averaging techniques based on the integral equation description provide theoretical justification for commonly used averaging methods. The new theory provides a basis for answering fundamental questions about the averaging approximation. A ripple estimate expression is presented, along with the simulation results for a feedback controlled boost converter.*

I. INTRODUCTION

State space averaging techniques are commonly used in the analysis and control of pulse width modulated (PWM) power electronic systems [1,2,3]. However, it was not until recently that rigorous mathematical justification [3,4] was given which theoretically explained the applications of these averaging techniques. As [3,5] have pointed out, the theoretical development of PWM systems lags far behind the many practical control applications.

In [3], classical Russian averaging techniques [6,7] are shown to be applicable to several types of PWM power electronic systems, such as open loop dc–dc converters. Besides using these classical averaging techniques to prove stability, [3] also gives a ripple estimate for improving the accuracy of the averaging technique, even for systems with large ripple. However, the application of the results of [3] are limited to systems with time discontinuities[1].

In fact, the classical averaging theory used in [3] is not applicable when there are state discontinuities. This is significant because all feedback controlled converters are state discontinuous. In [3], the argument is made that smooth commutation models can be used in place of the discontinuous Heaviside unit step function to avoid any state discontinuity in the mathematical system model. In essence, this idea was introduced by Filoppov [8] to justify what is meant by solutions to state discontinuous differential equations. However, it has never been rigorously shown that these techniques can be used to justify averaging approximations.

It is the purpose of this paper to introduce new averaging techniques which are general enough to encompass both time discontinuity and large classes of state discontinuity. This paper is, therefore, an extension of [3] and, in particular, this paper will address the following questions raised in [3], only now for feedback controlled converters with state discontinuity:

1. *Under what conditions do averaged models give useful system approximations?*
2. *Are the stability properties of the original and approximate system the same?*
3. *How is the error of the approximation estimated, and how is it improved?*
4. *Is the approximation valid for large signals?*
5. *What is the lower limit on the switching speed for averaging to be valid?*

[1] In this paper, a system with "time discontinuity" is described by a differential equation whose right–hand side is discontinuous with respect to time. A system with "state discontinuity" is described by a differential equation whose right–hand side is discontinuous with respect to a state variable.

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Section II reviews some of the mathematical issues associated with state discontinuous systems. The primary theoretical contribution of this paper is contained in two theorems presented in Section III. Section IV discusses the practical implications of the results of Section II, and gives numerical examples and computer simulations.

II. THEORETICAL PRELIMINARIES

The difficulty in mathematically justifying averaging approximation techniques of state discontinuous differential equations can be best explained through an example. Consider the state discontinuous differential equation

$$\dot{x}(t) = f(x) + bu(d(x) - tri\ (t,T)) \tag{2.1}$$

where $x \in \Re^n$, $b \in \Re^n$, $f: \Re^n \rightarrow \Re^n$ and $d: \Re^n \rightarrow \Re$ are both continuous functions with $0 \leq d(x) \leq 1$, and $u(\cdot)$ is the Heaviside step function, i.e., $u(s) = 1$ for $s \geq 0$ and $u(s) = 0$ for $s < 0$. The function $tri(t,T) = \frac{t}{T} - floor\ (\frac{t}{T}) = \frac{t \bmod T}{T}$ is shown in Figure 1. Equation (2.1) is a typical representation of a feedback controlled PWM Buck converter [3].

The usual condition for a unique solution of (2.1) to exist is that the right hand side satisfy a Lipschitz condition. (A function, f(x) is said to be Lipschitz with constant k>0 if $\| f(x) - f(y) \| \leq k \| x - y \|$ for any $x \in \Re^n$, $y \in \Re^n$.) However (2.1) is not Lipschitz since it is discontinuous with respect to x. Hence, standard approaches fail when trying to prove the existence of a unique solution — which implies that formal averaging approximations of (2.1) cannot, in general, be directly derived. There is an extensive amount of literature on differential inclusions which shows how one can redefine what is meant by a unique solution to (2.1) (see Filippov [8]).

While, in general, "standard" solutions to (2.1) are not known to exist, under the proper conditions (see Section II–A), there are a finite number of jumps in the right–hand side of (2.1) on any finite time interval and each jump (switch) is norm bounded, due to the fact that $0 \leq u(\cdot) \leq 1$. This implies that (under these conditions) the right–hand side of (2.1) is Lebesgue integrable for all $t \geq t_0$ and that the solution of the integral equation

$$x(t; t_0, x(t_0)) = x(t) \equiv x(t_0) + \int_{t_0}^{t} [f(x(s)) + bu(d(x(s)) - tri\ (s,T))]ds\ , \tag{2.2}$$

is unique and satisfies state differential equation (2.1) almost everywhere. Hence, when no chattering occurs in the system, the "standard" solution to (2.1) can be derived and will be equal to the solution of integral equation (2.2) almost everywhere.

Furthermore, when there is no chattering, $x(t; t_0, x(t_0)) = x(t)$, as given by (2.2), is a continuous function, which depends continuously on its switching period, T. Using this fact, [4] develops approximation techniques by examining (2.2) instead of (2.1). This work by H. Sira–Ramirez shows that the solution of (2.2) can be accurately approximated by an autonomous averaged system by letting $T \rightarrow 0$. In [4], it is shown that there always exists a sufficiently small sampling period T, for which the deviations between the actual

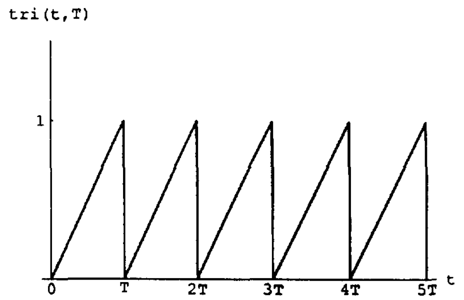

Figure 1. *Tri* (t, T) with $t_0 = 0$.

PWM controlled responses (of an integral equation) and those of an averaged model, under identical initial conditions, remain arbitrarily close to each other.

Therefore, it seems reasonable to approach the problem of approximating the dynamics of (2.1) by using classical averaging techniques on integral equation (2.2), since averaging provides answers to *questions 1 – 5* in the Introduction. By performing averaging on an integral equation instead of a differential equation, this paper will show that the difficulties due to many types of state discontinuities are eliminated. The techniques used rely on the recently developed averaging theory presented in [9,10].

III. AVERAGING OF STATE DISCONTINUOUS POWER ELECTRONIC SYSTEMS

In general form, feedback pulse width modulated systems considered in this paper will be modelled by the integral equation

$$x(t; t_0, x(t_0)) = x(t) = x(t_0) + \int_{t_0}^{t} \left[f_0(x(s))ds + \sum_{i=1}^{N} f_i(x(s))u(d_i(x(s)) - tri(s, T)) \right] ds, \quad (3.1)$$

where it will always be assumed that $x \in \Re^n$, t_0 denotes initial time and, $f_i : \Re^n \to \Re^n$ are locally Lipschitz functions, i.e. there exists an open neighborhood $\Omega \subset \Re^n$ such that for every $x_1 \in \Omega$, $x_2 \in \Omega$, there are constant positive k_i satisfying $\| f_i(x_1) - f_i(x_2) \| \le k_i \| x_1 - x_2 \|$. The functions $d_i : \Re^n \to \Re$ are the duty ratios and will also be assumed locally Lipschitz in Ω with Lipschitz constant m_i. Furthermore, they will always satisfy $0 \le d_i(x) \le 1$.

Along with (3.1) consider the corresponding "averaged" integral equation

$$y(t; t_0, y(t_0)) = y(t) = y(t_0) + \int_{t_0}^{t} f_0(y(s))ds + \sum_{i=1}^{N} \int_{t_0}^{t} f_i(y(s))d_i(y(s))ds, \quad (3.2)$$

where f_i, and d_i are as previously defined and $y \in \Re^n$. This section will discuss the conditions under which solutions to (3 . 2) can approximate solutions to (3 . 1). Since (3 . 2) is both continuous and autonomous, its analysis is much simpler than that of discontinuous and time–varying (3 . 1) .

A. Chattering

By representing state discontinuous differential equations by a corresponding integral equation, it is possible to rigorously explain averaging approximations in power electronic systems. However, it will always be necessary to assume that the models under consideration have a finite number of right–hand side state discontinuities on any bounded time interval and that each discontinuity is Lebesgue integrable. This, is not always true for power electronic systems. For example, when systems are switching infinitely often (chattering), there exists no compact time interval in which the right–hand side of the differential equation is continuous. Hence, a unique solution to a corresponding integral equation will not exist in the usual sense unless the theory of differential inclusions [8] is used.

In this paper, we will always assume that the system is not chattering. The physical implication of this assumption is that power electronic switches turn on and off only once each PWM switching period. Conditions for guaranteeing this are presented in [11] and will not be discussed here.

B. Theoretical Results

We begin this section by outlining the procedure to be taken in order to justify the approximation of (3.1) by (3.2):

Given a non–autonomous, integral equation (such as (2.1) or (3.1)) $x(t) = x(t_0) + \int_{t_0}^{t} g(s, x(s), T)ds$, consider the corresponding autonomous "averaged" integral equation $y(t) = y(t_0) + \int_{t_0}^{t} \bar{g}(y(s))ds$, where $\bar{g}(\cdot)$ is an "average value" of $g(t, \cdot, \cdot)$, and $\bar{g}(\cdot)$ does not depend on time, t, or on the switching period, T.

<u>*Step 1:*</u> Take the difference between the two integral equations to obtain

$$\| x(t) - y(t) \| \le \| x(t_0) - y(t_0) \| + \left\| \int_{t_0}^{t} [g(s, x(s), T) - \bar{g}(y(s))]ds \right\| .$$

<u>*Step 2:*</u> Show that for any $\delta > 0$, however small, and any $L > t_0$, however large, there will always exist a $T_0 = T_0(\delta, L)$ and a constant $K > 0$ such that, for $0 < T \le T_0$,

$$\left\| \int_{t_0}^{t} g(s, x(s), T) - \bar{g}(y(s))ds \right\| \le \delta + K \int_{t_0}^{t} \| x(s) - y(s) \| ds ,$$

for any $t \in [t_0, L]$.

<u>*Step 3:*</u> Immediately from *step 1*, *step 2* and Gronwall's inequality, this implies that for $t \in [t_0, L]$, $L > t_0$, and $0 < T \le T_0$

$$\| x(t) - y(t) \| \le \left(\| x(t_0) - y(t_0) \| + \delta \right) e^{K(t - t_0)}$$

where $\delta \to 0$ as $T \to 0$. This implies that on any arbitrarily large but bounded time interval, if $x(t_0) = y(t_0)$, then $x(t)$ and $y(t)$ can remain arbitrarily close to each other for sufficiently small switching period.

<u>*Step 4:*</u> Assume that $x(t_0) = y(t_0)$ and that $y(t)$ approaches a uniformly asymptotically stable equilibrium point, y_e. Then there will always exist a sufficiently small $T_0 = T_0(\delta)$ such that, for $0 < T \le T_0$,

$$\| x(t) - y(t) \| < \delta , \quad t \ge t_0 .$$

Furthermore, this result will remain valid for initial conditions which satisfy $\| x(t_0) - y(t_0) \| \le \beta$, where $\beta > 0$ is sufficiently small.

<u>*Theorem 3.1:*</u> *Let $x(t)$ and $y(t)$ denote the solutions to* (3 . 1) *and* (3 . 2) *respectively. Then for any constants $L > t_0$ and $\eta > 0$, there exists a $T_0 = T_0(\eta, L) > 0$ and a constant $K > 0$ such that, for $0 < T \le T_0$,*

$$\| x(t) - y(t) \| \le \left(\| x(t_0) - y(t_0) \| + \eta \right) \exp\{ K(t - t_0) \}$$

for all $t \in [t_0, L]$.

<u>Remark 3.1:</u> The main trick of the proof of Theorem 3.1 is to construct $N + 1$ piecewise constant functions $\bar{x}_i(t)$, $i = 0, 1, ..., N$,

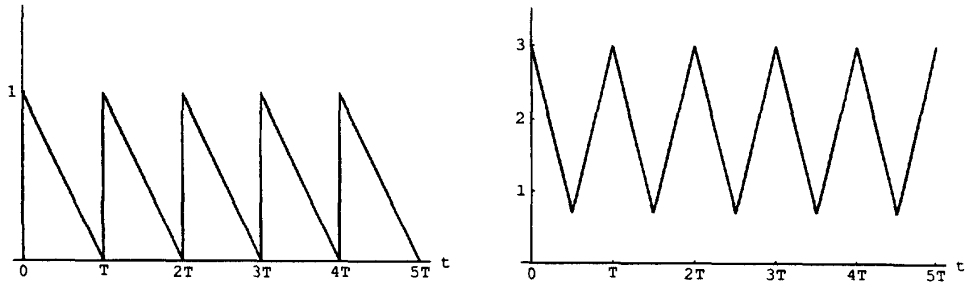

Figure 2. Other possible triangle waves

which accurately approximate $x(t)$ on $t \in [t_0, L]$. Such functions can always be constructed since $x(t)$ is continuous [12].

In order to simplify notation, define the following operators

$$(\mathcal{J}x)(t) \equiv x(t_0) + \int_{t_0}^{t} f_0(x(s))ds + \sum_{i=1}^{N} \int_{t_0}^{t} f_i(x(s))u(d_i(x(s)) - tri(s,T))ds;$$

$$(\mathcal{W}y)(t) = y(t_0) + \int_{t_0}^{t} f_0(y(s))ds + \sum_{i=1}^{N} \int_{t_0}^{t} f_i(y(s))d_i(y(s))ds .$$

Then, we have,

$$\begin{aligned} \| x(t) - y(t) \| &= \| (\mathcal{J}x)(t) - (\mathcal{W}y)(t) \| \\ &\leq \| (\mathcal{J}x)(t) - (\mathcal{J}\bar{x})(t) \| + \| (\mathcal{J}\bar{x})(t) - (\mathcal{W}\bar{x})(t) \| \\ &\quad + \| (\mathcal{W}\bar{x})(t) - (\mathcal{W}x)(t) \| + \| (\mathcal{W}x)(t) - (\mathcal{W}y)(t) \| . \end{aligned} \tag{3.3}$$

Now, *step 2* of the averaging algorithm must be performed. Each term on the right-hand side of (3 . 3) is considered separately. By constructing $\bar{x}_i(t)$ to approximate $x(t)$ with arbitrary accuracy, the quantities $\| (\mathcal{J}x)(t) - (\mathcal{J}\bar{x})(t) \|$ and $\| (\mathcal{W}\bar{x})(t) - (\mathcal{W}x)(t) \|$ can be made arbitrarily small. In essence, this is due to the Fundamental Theorem of Calculus which states that any integral can be estimated by the sums of the areas of rectangles. Since $\bar{x}_i(t)$ is piecewise constant, $(\mathcal{J}\bar{x})(t)$ and $(\mathcal{W}\bar{x})(t)$ represent nothing more than areas under the curve of a piecewise constant function which is equivalent to summing the areas of rectangles. Of course, due to the discontinuities that appear in $\mathcal{J}(\cdot)$, more advanced theoretically arguments must be made in order to justify these approximations.

Likewise, because $f_i(\cdot)$ and $d_i(\cdot)$ have been assumed Lipschitz, it is not too difficult to show that for any $t \in [t_0, L]$

$$\| (\mathcal{W}x)(t) - (\mathcal{W}y)(t) \| \leq \| x(t_0) - y(t_0) \| + K \int_{t_0}^{t} \| x(s) - y(s) \| d\cdot$$

Now, the only term left to consider in (3.3) is $\| (\mathcal{J}\bar{x})(t) - (\mathcal{W}\bar{x})(t) \|$. However, this term only considers the difference between the integrals of piecewise constant functions, which as [9,10] show, is very small under the conditions of Theorem 1.

Remark 3.2: When $x(t_0) = y(t_0)$, Theorem 3.1 guarantees that there will always exist a sufficiently small switching period such that for any $\eta > 0$, however small, $\| x(t) - y(t) \| < \eta$ on any finite time interval. This bound is true, even when (3.1) or (3.2) are unstable. For the case when solutions are bounded, however more powerful theorems can be stated.

Remark 3.3: The choice of T_0 is best found through numerical simulation, since theoretical estimates are often extremely conservative. One reason for poor theoretical estimates of T_0 is that Theorem 3.1 does not distinguish between stable and unstable systems. For unstable systems, it is possible that solutions to (3.1) and (3.2) grow exponentially, making it difficult to estimate the difference, $\| x(t) - y(t) \|$.

For general systems, from the proof of Theorem 3.1 and from basic averaging theory, it can be derived that T_0 is sufficiently small if all three of the following conditions are satisfied:

(i) there exists no chattering in the system;
(ii) $T_0 << k_i$, where k_i are the Lipschitz constants for $f_i(\cdot)$;
(iii) $T_0 << m_i$, where m_i are the Lipschitz constants for $d_i(\cdot)$.

This is not to say that for every system in question, the switching period must be chosen so that (i) – (iii) are satisfied. For example, if solutions to (3.2) decay exponentially to an equilibrium point, then condition (ii) can be substantially relaxed. It is important to remark that condition (i) must always be fulfilled or else the solutions of (3.1) will not be defined in the usual sense. This answers *question 5* posed in the Introduction.

Remark 3.4: Based on the Theorem 3.1 and the above discussion, it is possible to determine general conditions which suggest the improvement of the accuracy of approximation between the original (3.1) and the approximate (3.2) system (answering *question 1* in the Introduction). Clearly, the approximation becomes better as the switching period becomes smaller, but also, as Remark 3.3 notes, the approximations will tend to improve for systems with smaller Lipschitz constants, i.e. the smaller k_i and m_i. Additionally, if, as Theorem 3.2 suggests below, the averaged system is stable, then the averaging approximations will also improve. In particular, the closer the solution of (3.2) is to its equilibrium point, the better the approximation will become. Conversely, if the averaged system is unstable, the averaging approximation tends to worsen. Finally, a necessary condition for the solutions of (3.2) to approximate the solutions of (3.1) is that the initial conditions of the two systems must be chosen in appropriate neighborhoods.

Remark 3.5: One of the main advantages of the averaging technique is that nonlinearities are maintained in the averaged system. Hence, the approximation of (3.1) by (3.2) is valid even when the states, x, become large, which would not be true if a linearization technique were to be used. This answers *question 4* in the Introduction.

The proof of the following theorem is almost identical to Proposition 4 of [3] and Theorem 2.3 of [10].

Theorem 3.2: Let $x(t)$ and $y(t)$ denote the solutions to (3.1) and (3.2) respectively, and let $y_s \in \Omega$ ($y_s \neq y(t_0)$) denote a uniformly asymptotically stable equilibrium point. Suppose that

$y(t) \to y_s$ *as* $t \to \infty$.

Then there are constants $\beta_0(\eta)$ and $T_0(\eta)$ such that, for any $\eta > 0$, any $\| x(t_0) - y(t_0) \| < \beta$, $0 \leq \beta < \beta_0 < \eta$, and any $0 < T \leq T_0$,

$$\| x(t) - y(t) \| < \eta, \quad \textit{for all } t \geq t_0 .$$

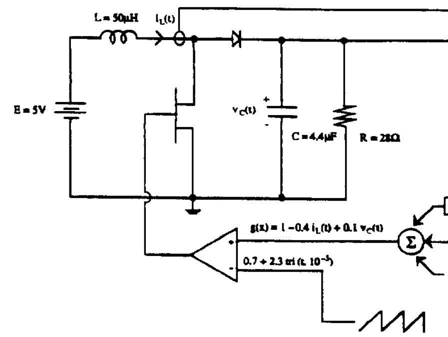

Figure 3. Feedback control boost converter.

Remark 3.6: The above theorem gives conditions in which the interval in Theorem 3.1 can be made infinite. For the case when $y(t)$ approaches a uniformly asymptotically stable equilibrium point, y_s the difference, $\| x(t) - y(t) \|$, can be made arbitrarily small, for all $t \geq t_0$, assuming $\| x(t_0) - y(t_0) \|$ is sufficiently small.

Remark 3.7: Suppose $f_i(\cdot)$ and $d_i(\cdot)$ have continuous partial derivatives. Then, for an equilibrium point, y_s, of (3.2) to be uniformly asymptotically stable, it is possible to check that

$$Det\left\{ sI - \frac{\partial f_0(y_s)}{\partial y} - \sum_{i=1}^{N}\left[\frac{\partial f_i(y_s)}{\partial y}\, d_i(y_s) - f_i(y_s)\,\frac{\partial d_i(y_s)}{\partial y}\right]\right\} = 0$$

have all solutions with $Re(s) < 0$.

Remark 3.8: Theorem 3.2 guarantees that, under the proper conditions, when (3.2) is stable, then so is (3.1). Unlike (3.2), however, the solution to (3.1) will not in general approach an equilibrium point as $t \to \infty$, since (3.1) is a time varying integral equation. In general, the solution to (3.1) will (assuming it is stable) approach a periodic orbit which oscillates in the vicinity of the equilibrium point of (3.2). This partially answers *question 2* in the Introduction.

Remark 3.9: In Theorems 3.1 and 3.2, the feedback signals are compared with $tri(t,T)$, as shown in Figure 1. However, all the above theorems remain valid for triangle waves as shown in Figure 2 also, provided that they are rescaled to vary between zero and one (see Section IV). Furthermore, it is not necessary to compare each $d_i(\cdot)$ with the same function with the same period. For instance, in (3.1), we might have $u(d_i(\cdot) - tri(\cdot, T_i))$ instead of $u(d_i(\cdot) - tri(\cdot, T))$, where T_i might not equal T_j, for $i \neq j$. As long as each T_i is sufficiently small, all previous results remain valid.

C. Ripple Estimate

It is often desirable to obtain an estimate on the ripple of the system, which will be denoted in this paper as $\Psi(t, T, \cdot)$. Then, practical applications of averaging tell us that a better approximation of the solution to (3.1) will be given by

$$x(t) \approx y(t) + \Psi(t, T, y(t)),$$

where $x(t)$ and $y(t)$ are the solutions of (3.1) and (3.2) respectively, T is the switching period, and $\Psi(t, T, \cdot)$ is the ripple estimate which is obtained by the following algorithm:

Consider only the right hand sides of (3.1) and (3.2). Let $x(t_0) = y(t_0)$, and replace every $x(s)$ and $y(s)$ in (3.1) and (3.2) by the constant $c \in \Re^n$. Now take the difference between (3.1) and (3.2) to obtain

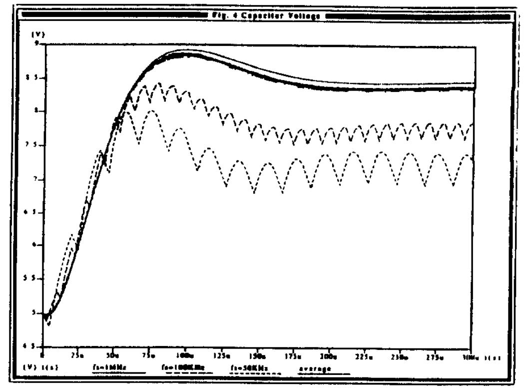

Figure 4. Simulated start up transient response of capacitor voltage for (4.1) and (4.2) for different values of switching frequency.

$$\Gamma(t, T, c) = \sum_{i=1}^{N} f_i(c) \int_t [u(d_i(c) - tri(t, T)) - d_i(c)]dt,$$

where $\int_t h(t)dt$ denotes the indefinite integral of $h(t)$ (mathematically referred to as the primitive). The ripple estimate is given as

$$\Psi(t, T, c) = \Gamma(t, T, c) - \frac{1}{T}\int_0^T \Gamma(t, T, c)dt.$$

Replacing c by y(t) yields $\Psi(t, T, y(t))$. Performing integrations, using (3.1) and (3.2), an estimate on the ripple is computed to be

$$\begin{aligned}\Psi(t, T, y(t)) &= T\sum_{i=1}^{N} f_i(y(t))\{ [u(d_i(y(t)) - tri(t,T)) - d_i(y(t))]\, tri(t, T) \\ &\quad + [1 - u(d_i(y(t)) - tri(t, T))]\, d_i(y(t)) \\ &\quad + \frac{1}{2}\, d_i(y(t))\, [d_i(y(t)) - 1] \}. \qquad (3.3)\end{aligned}$$

As the switching period becomes smaller, the amplitude of $\Psi(t, T, \cdot)$ will also become smaller, and ripple of the system will almost become negligible. Additionally, an adjustment on the initial condition can be made by solving solving the equation $x(t_0) = y(t_0) + \Psi(t, T, y(t_0))$, for $y(t_0)$ in terms of $x(t_0)$.

IV. APPLICATION EXAMPLE

Consider the PWM boost converter with feedback control structure as shown in Figure 3. Open loop operation of this device was considered in [3]. Assuming the converter is operating in the continuous conduction mode, the closed loop (rescaled) system description is given by

$$\begin{aligned} x(t) = x(t_0) &+ \int_{t_0}^{t} [A_0 x(s) + b]ds \\ &+ \int_{t_0}^{t} A_1 x(s) u(d(x(s)) - tri(s, T))ds \end{aligned}$$

$$d(x) = V_{REF} - k_1 x_1(t) - k_2 x_2(t), \qquad (4.1)$$

$$A_0 = \begin{bmatrix} 0 & -\frac{1}{L} \\ \frac{1}{C} & -\frac{1}{RC} \end{bmatrix},\ A_1 = \begin{bmatrix} 0 & \frac{1}{L} \\ -\frac{1}{C} & 0 \end{bmatrix},\ b = \begin{bmatrix} \frac{E}{L} \\ 0 \end{bmatrix},$$

where the components of $x(t) = [i_L(t), v_C(t)]^T$ are the inductor current and capacitor voltage. Note, that since the triangle wave in Figure

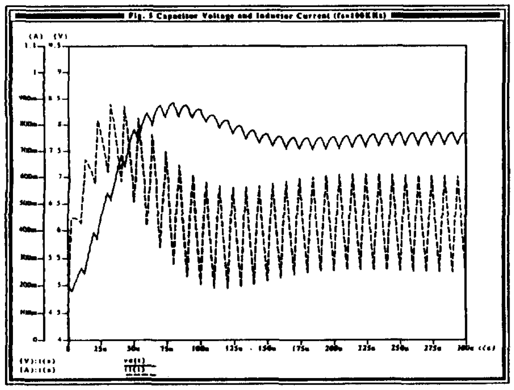

Figure 5. Simulated start up transient response of both capacitor voltage and inductor current for (4.1) when switching frequency equals 100 KHz.

3 varies from 0.7 V to 3 V, it is necessary to rescale the system into (4.1) so that Theorems 1 and 2 can be applied. This is easily done scaling the duty ration function using the minimum ($trimin = 0.7$ V) and maximum ($trimax = 3.0$ V) values of the triangle wave:

$$d(x) = \frac{g(x) - trimin}{trimax - trimin},$$

where $g(x)$ is defined in Figure3. For this specific $g(x)$, we have $V_{REF} = 0.3/2.3$, $k_1 = 0.4/2.3$ and $k_2 = -0.1/2.3$.

The corresponding average of (4.1) is then given by

$$y(t) = y(t_0) + \int_{t_0}^{t} [A_0 y(s) + b] ds + \int_{t_0}^{t} A_1 y(s) d(y(s)) ds . \quad (4.2)$$

Application of Theorems 3.1 and 3.2 is now immediate upon noting that, using the previous notation, $f_0(x) = A_0 x + b$, $f_1(x) = A_1 x$, and $N = 1$. Figure 4 illustrates the switching and averaged trajectories of the capacitor voltage for different switching periods. As the frequency of the system, f_s, increases, or equivalently as the switching period decreases (since $f_s = T^{-1}$), the approximation of $x(t)$ by $y(t)$ improves. For example, when $f_s = 50$ KHz, system (4.1) has a capacitor voltage that, in steady state, oscillates about (approximately) 7.3 V.

The averaged system, on the other hand, approaches (approximately) 8.5 V. As the frequency of the system increases (the switching period decreases) the capacitor voltage for (4.2) more closely approximates the capacitor voltage of (4.1). For $f_s = 1$MHz, system (4.1) has steady state capacitor voltage which oscillates about (approximately) 8.4 V, representing a significant improvement. Additionally, for larger frequency, the amplitude of the ripple decreases. This further verifies Theorems 1 and 2 which state that the approximation between the averaged system and the original system improves as the switching period decreases. Similar results can be obtained for the inductor current.

Using (3.3), it is possible to directly compute an estimate on the ripple of the system as

$$\begin{aligned} &\Psi(t, T, y(t)) \\ &\quad = T A_1 y(t) \{ [u(d(y(t)) - tri(t, T)) - d(y(t))] \, tri(t, T) \\ &\qquad + [1 - u(d(y(t)) - tri(t, T))] \, d(y(t)) \\ &\qquad + \frac{1}{2} d(y(t)) [d(y(t)) - 1] \} . \end{aligned}$$

Figure 5 plots the capacitor voltage and inductor current of the original system (4.1) when $f_s = 100$ KHz. A comparison of these plots can be made with Figure 6, which shows the improvement of the

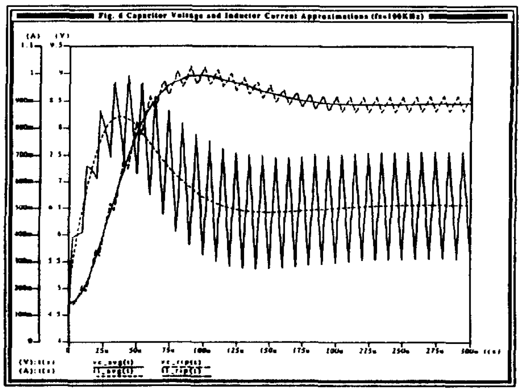

Figure 6. Simulated start up transient response of both capacitor voltage and inductor current for (4.2) and with ripple correction when switching frequency equals 100 KHz.

averaging technique by approximating $x(t)$ by $x(t) = y(t) + \Psi(t, T, y(t))$ and updating the initial condition, $y(t_0)$, by solving (given $x(t_0)$) the nonlinear equation

$x(t) =$

$$\begin{bmatrix} 1 & \frac{T}{2L}[d^2(y(t_0)) - d(y(t_0))] \\ -\frac{T}{2C}(d^2(y(t_0)) - d(y(t_0))] & 1 \end{bmatrix} y(t) .$$

Figure 6 indicates that the "shape" of solutions to averaged system (4.2) added to the ripple estimate closely resembles the "shape" of solutions to the original system (plus, perhaps, a dc–offset). Therefore, the ripple estimate may provide important system information, even at a low frequency (large switching period).

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