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Nonlinear Control Designs for Systems with Bifurcations with Applications to Stabilization and Control of Compressors

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Abstract: This paper studies the effects of applying both nonlinear feedback and time-varying control algorithms in the form of inlet flow disturbances by introducing variations in the shut-off head of a single stage axial compressor. The closed loop controller utilizes nonlinear feedback in order to stabilize rotating stall. Using bifurcation analysis, it is possible to analytically guarantee stability of local bifurcated solutions near the stall point. Physically, this means that the system will no longer "jump" to a large amplitude oscillatory mode when bifurcating at the stall point. Thus the possible operating region of the axial compressor is enlarged. The open loop control strategy provides oscillatory input to the shut-off head coefficient. For some model systems, this type of strategy is shown to be effective in enhancing stability margins and improving efficiency. Methods of implementing both types of control laws using variations in inlet guide vanes and bleed valves and by blowing air into the compressor are under investigation.

1. INTRODUCTION

The purpose of this paper is to explore control designs aimed at improving the efficiency and stability of axial flow compressors. While a detailed understanding of such designs would require extensive experimental studies, it has been shown that calculations using a simple three dimensional model often provide a fairly good representation of the actual dynamics of a single stage axial compressor, [9]. Both open loop and closed loop control designs will be presented. Calculations for both types of control provide evidence that a well known problem of compressor stall is partially improved by the control laws.

The approach taken in the design of closed loop controllers takes on the philosophy introduced by [10]. Nonlinear feedback control is proposed in the shut-off head coefficient in order to stabilize rotating stall. Using analysis similar to [10], it is shown that these control laws can guarantee the stability of local bifurcated solutions near the stall point. Physically, this means that the system will no longer "jump" to a steady state with non-zero first harmonic when near the stall point. Thus the possible operating region of the axial compressor is enlarged due to the increase in the size of the domain of attraction.

The open loop control approach provides an oscillatory input to the compressor shut-off valve coefficient with the hope that such open-loop forcing can be tuned to enhance stability margins and improve efficiency. There is growing literature on the uses of such control strategies. It has been observed by many authors, for instance, that the inverted pendulum can be stabilized by a high-frequency, small amplitude vertical oscillation of the point of suspension. Arnold attributes this discovery to Bogolyubov and Kapitsa. (See [1], p. 153.) Experimental work (e.g. with the inverted pendulum) shows that the stabilization effects of oscillatory forcing can be quite robust, even though the mathematical machinery needed to understand the phenomenon is somewhat intricate. Very recent work ([3]) has shown that such oscillatory open-loop control can be used to stabilize super-critical rotations of the hanging chains studied in [2]. Because these effects do not depend on a precise knowledge of the physical parameters of the system, the possibility of using such open-loop designs is attractive in settings where it is difficult to obtain accurate measurements of the variables needed to close a feedback loop. Such considerations have been a significant part of our motivation for studying these designs for controlling axial compressors.

Implementation of control laws can be carried out in a variety of ways. Bleed valves or actuated, variable pitch inlet guide vanes as studied in the thesis by Haynes ([11]) may be used. Alternatively, air can be injected into the compressor along the lines suggested in the work of D'Andrea *et al.*, ([4]). In the work reported below, it is shown that oscillatory variation of the inlet pressure (shut-off head) will lead to improved stability margins in a neighborhood of the compressor stall point. Simulations indicate that such oscillatory variation of the inlet

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pressure may also be used as a stall-recovery strategy.

2. MODELS

Because of the difficulty encountered in analyzing the qualitative features in high-fidelity models of the fluid dynamics of compressors, a number of simplified models have been proposed in recent years. Reference [8], for instance, discusses a number of models, the most complex of which consists of three simultaneous nonlinear partial differential equations for pressure rise, and average and disturbed values of the flow coefficient where the independent variables are time and angular position around the annulus of the compressor. This is then simplified to a single mode Galerkin approximation, leading to a three dimensional system of ordinary differential equations having essentially the same form as the following third-order set of ODEs

$$\frac{m\alpha+1}{\alpha}\frac{dA_1}{d\tau} = \frac{3H\Phi}{2W^3}(2W-\Phi)A_1 - \frac{3H}{8W^3}A_1^3 , \qquad (1)$$

$$l_c\frac{d\Phi}{d\tau} = -\Psi+\psi_{c_0}+\frac{H\Phi^2}{2W^3}(3W-\Phi) + \frac{3H}{4W^3}(W-\Phi)A_1^2 + u(t), \qquad (2)$$

$$l_c\frac{d\Psi}{d\tau} = \frac{1}{4B^2}\left[\Phi-\gamma\sqrt{\Psi}\right] . \qquad (3)$$

Here A_1 is the amplitude of the first harmonic of angular disturbance of axial-flow coefficient; Φ denotes the annulus-averaged axial flow coefficient; Ψ is the plenum-to-atmosphere pressure rise coefficient. The quantity γ is proportional to the throttle opening. The variable $u(t)$ is an input, which may be thought of as a controlled inlet flow disturbance, the design of which we shall discuss below. The parameter values given in the following table have been provided by C.N. Nett of United Technologies Research Center [9].

Parameter	*Value*	*Description*
α	1/3	internal compressor lag
l_c	6.0	overall compressor length
m	2.0	exit duct length factor
H	0.32	compressor characteristic height factor
W	0.18	compressor characteristic width factor
ψ_{c_0}	0.23	shut-off head
B	0.1	throttle characteristic

Table 1: Values of Compressor Parameters

This model retains many of the important qualitative features of the full p.d.e. models and has been studied by a number of researchers. (See, e.g. [6] and [10].) A more detailed model of rotating stall can be developed without passing to the full p.d.e. description using a spatial discretization such as proposed in [7]. On the other hand, an even simpler system which seems to exhibit many of the essential qualitative features is described by the (2-d) system

$$\dot{x} = x[\lambda a(y) - x^2] \qquad (4)$$

$$\dot{y} = -\frac{y^2}{\gamma^2} + 1 - yx^2 + u(t). \qquad (5)$$

Here γ, λ are parameters, and $u(\cdot)$ is a again a control input. The zeros of the scalar function $a(\cdot)$ determine the bifurcation set of the open loop system, and we consider one interesting choice in Section 4.

3. BIFURCATION OF STEADY STATE SOLUTIONS

Setting the left hand sides of the system of differential equations (1)-(3) equal to zero, we obtain steady-state equations, and using (3) to eliminate Ψ from (2), we obtain a pair of cubic polynomial equations in Φ and A_1.

$$A_1\left(-\Phi+\frac{A_1^2}{8\,W}+\frac{\Phi^2}{2\,W}\right)=0, \qquad (6)$$

$$-\psi_{co}+\Phi^2\left(\gamma^{-2}-\frac{3\,H}{2\,W^2}\right)+\frac{3\,A_1^2\,H\,\Phi}{4\,W^3} + \frac{H\,\Phi^3}{2\,W^3}-\frac{3\,A_1^2\,H}{4\,W^2} = 0. \qquad (7)$$

Bezout's theorem suggests that the number of solutions should be nine, and this number may be confirmed for typical values of γ using standard computer algebra software. Only those solutions in which both A_1 and Φ are real and Φ is nonnegative are physically meaningful, however. Figure 1 plots the locus of the steady state operating points as a function of γ, the throttle setting for the parameters given in Table 1. The dashed portions of the curve indicate unstable equilibria, while the solid portion represent stable steady state operating points. As γ decreases through the value 0.3856..., the axisymmetric equilibrium ($A_1 = 0$) loses stability through a pitchfork bifurcation. This is the so-called *stall inception point*. For $\gamma < 0.3856...$, the compressor will enter fully developed stall as it tends to a stable steady state in which $A_1 \neq 0$. Note that for γ slightly larger than 0.3856..., there are simultaneously two stable equilibrium operating points, the axisymmetric equilibrium with $A_1 = 0$ and the non-axisymmetric solution, $A_1 \neq 0$, corresponding to fully developed stall. In this parameter regime, the axisymmetric equilibrium is not stable with respect to all possible perturbations, and indeed as the throttle setting γ_0 is reduced beyond 0.38596..., (the stall inception point), the axisymmetric equilibrium loses stability. In order to return the system to stable axisymmetric flow, it is now necessary to substantially increase γ_0 so that the axisymmetric equilibrium is either globally stable or else has a sufficiently large domain of attraction that perturbations of

Bifurcations of Compressor Equilibria

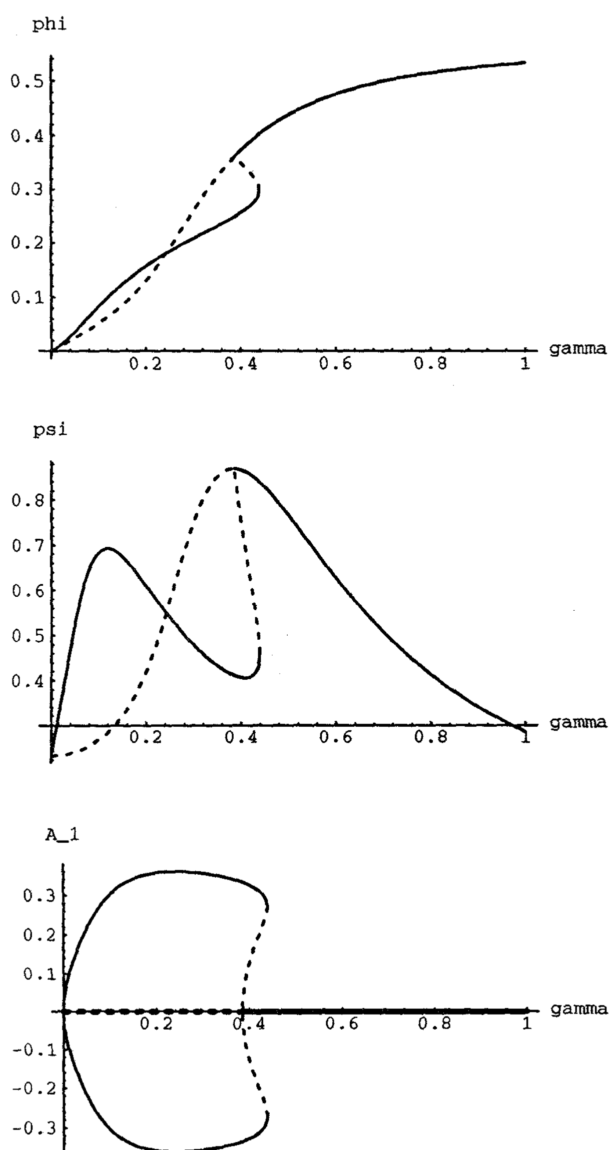

Figure 1: Bifurcation diagram depicting steady-state operating points (Φ, Ψ, A_1) as a function of γ.

the state will tend to settle toward the axisymmetric solution. This asymmetry in the loss and recapture of stability for the axisymmetric equilibrium as γ_0 is varied has been referred to in the literature as a "hysteresis loop".

The analysis of bifurcations of equilibria in parameterized systems of differential equations may be carried out using the Lyapunov-Schmidt technique, as outlined for the compressor model in [10]. This technique provides a coordinate system, $(x_1, \ldots, x_n)$, in terms of which the effects of the bifurcation are confined to the x_1-coordinate. The goal of the analysis is to isolate, in addition to the bifurcation parameter γ, a second parameter β, whose value determines whether the bifurcation is subcritical or supercritical. At the pitchfork bifurcation of (1)-(3) (or (4)-(5)), for instance, with bifurcation parameter γ set equal to its critical value γ_0, we obtain a normal form equation for the x_1-coordinate:

$$\dot{x}_1 = \beta x_1^3 + Higher\ Order\ Terms. \qquad (8)$$

When $\beta < 0$, $x_1 = 0$ of (8) is stable, and hence, this will imply that the local bifurcated solutions near the rotating stall inception point are stable. In this case, the bifurcation is said to be ***supercritical.*** When $\beta > 0$, the local bifurcated solutions are unstable, and the bifurcation is said to be ***subcritical.*** Details of the way in which this circle of ideas applies to various models of compressor dynamics are briefly discussed in the next section.

Remark 1 The subcritical bifurcation is viewed as more problematical in the application under study, because it indicates the possibility of multiple (locally stable) modes of operation for values of the parameter γ just above the bifurcation (stall) point. As we have noted above, locally stable solutions to (6)-(7) in which $A_1 \neq 0$ correspond to the compressor operating in rotating stall. The existence of such locally stable solutions coexisting with locally stable "axisymmetric" solutions ($A_1 = 0$) indicates the possibility of perturbations in the operating state leading to system trajectories which tend toward the rotating stall equilibrium. This motivates our search for control laws which eliminate the occurrence of multiple locally stable equilibria without affecting the location of the stall inception point in the throttle parameter setting.

4. Closed Loop Control

In this section, we consider the design of feedback control laws for systems with pitchfork bifurcations, with the aim of finding designs to guarantee that the bifurcation is supercritical, thereby eliminating the possibility of multiple locally stable equilibria near the bifurcation point. The theory of this type of control is still under development, but several preliminary results may be noted. The analysis is fairly explicit in the case of the system (4)-(5), for which we have the following:

Proposition 1 *In the system (4)-(5), let a(y)=y(1-y). Let λ be given and consider the parameterized family of feedback control laws $u = (1 - \mu/\lambda)x^2 y$. Substituting this into (5), the equilibrium analysis of the resulting closed-loop system shows there remains a pitchfork bifurcation at $\gamma = 1$ for all values of μ. For all μ in the range $0 < \mu < 2$, this bifurcation is supercritical.*

We also describe a quadratic control law which guarantees stability of the local bifurcated solutions of the equilibrium equations (6)-(7) near the stall point. Consider the quadratic control law of the form

$$u = KA_1^2 \tag{9}$$

where the feedback gain K is a real parameter.

Applying the Lyapunov-Schmidt reduction to (1)-(3), we obtain the explicit value of the parameter β appearing in (8) as:

$$\beta = \left(\frac{3H}{W^3}\right)\left(\frac{\alpha}{m\alpha+1}\right)\left[(W - \Phi_s)a_1 - \frac{1}{8}\right] \tag{10}$$

where

$$a_1 = \frac{-\left(\frac{1}{2}\right)\left(-3HW + 3H\Phi_s - 4KW^3\right)\gamma_o}{3H\Phi_s^2\gamma_o - 6H\Phi_s W\gamma_o - 4W^3\sqrt{\Psi_s}}.$$

Note that if $\gamma = \gamma_o$ then $\Phi_s = 2W$ and $\Psi_s = \left(\frac{2w}{\gamma_o}\right)^2$. Note also that $\beta = 4.878 > 0$, when $K = 0$ for our parameters, which indicates that the local bifurcated solutions are unstable.

Proposition 2 *The bifurcation at (x_s, γ_o) can be guaranteed to be a supercritical pitchfork bifurcation by a purely quadratic feedback control in the form of (9) by choosing K sufficiently large.*

Numerical calculations were carried out to verify the predicted change of bifurcation behavior. Figure 2 plots the steady state locus with feedback control (9) and $K = 5$. As is clearly shown, the local bifurcations are now stable. Additionally, we have eliminated the hysteresis loop of the stable system equilibria. The unusually large pressure rise for small values of the throttle setting (γ) reflects a high level of effectiveness of the control law applied to (1)-(3). This may be physically unrealistic. Irrespective of the control law which is applied, it is doubtful that our three-state model remains valid for small values of γ. A cubic feedback law has also been shown to be effective in rendering the pitchfork bifurcations of the system (1)-(3) supercritical. Letting $u = K\Phi A^2$, we find a similar range of gain values K for which the bifurcation is supercritical. For instance, when $K = 11$, the bifurcation is supercritical, and there is a smaller relative pressure rise, Ψ, as the throttle setting γ is reduced than was noticed for the quadratic feedback.

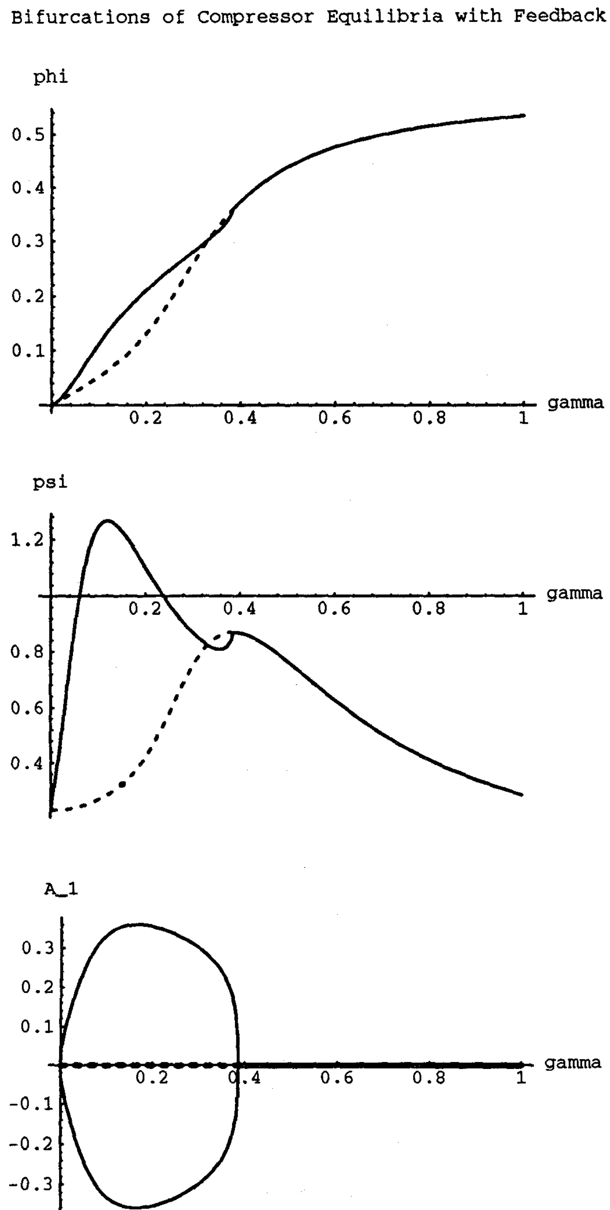

Figure 2: Bifurcation diagram depicting steady-state operating points (Φ, Ψ, A_1) as a function of γ for the closed-loop system with quadratic feedback law (9) and gain $K = 5$. As noted in the text, the equilibrium versions of the three-state models may not retain physical accuracy for small values of γ.

5. Open Loop Controllers

In this section, we study open-loop, zero mean oscillatory control laws $u(\cdot)$ applied to a general class of parameterized nonlinear systems of the form

$$\dot{x}(t) = F_\gamma(x(t)) + b\,u(t), \tag{11}$$

where $b \in \mathbf{R}^n$, and for each value of the real parameter γ, $F_\gamma(\cdot)$ is an analytic vector on $\mathbf{R}^n$. This class includes both the systems (1)-(3) and (4)-(5) as special cases. Suppose that in (11) $u(t) = \frac{1}{\epsilon} f\left(\frac{t}{\epsilon}\right)$, where $\epsilon > 0$, $f(t+T) = f(t)$ and $\frac{1}{T}\int_0^T f(t)dt = 0$. The following proposition describes the averaged dynamics under certain conditions of practical interest.

Proposition 3 *Consider the system (11) with $u(t) = \frac{1}{\epsilon} f\left(\frac{t}{\epsilon}\right)$, where $f(\cdot)$ is a p.w.-continuous, T-periodic function as described above. Let $h(\tau) = \frac{1}{l_c}\int^\tau f(s)\,ds$ and suppose that that $\frac{1}{T}\int_0^T h^i(t)dt = 0$. Let*

$$h_k = \frac{1}{T}\int_0^T h^k(\tau)\,d\tau.$$

Then under the influence of the input $u(\cdot)$, the system (11) will execute stable motions in a neighborhood of any asymptotically stable equilibrium of

$$\begin{aligned} \dot{z} &= F_\gamma(z) + \frac{1}{2}[[F_\gamma, b], b]\,h_2 \\ &+ \frac{1}{4!}[[[[F_\gamma, b], b], b], b]\,h_4 + \ldots, \end{aligned} \tag{12}$$

where $[\cdot,\cdot]$ denotes the usual Lie bracket of vector fields: $[f,g] = \frac{\partial f}{\partial x} g - \frac{\partial g}{\partial x} f$.

A brief sketch of the proof goes as follows. Introduce a new variable y by writing $x(t) = y(t) + b\,h(t/\epsilon)$. Rewriting the equation (11) in terms of the variable y, and averaging all coefficients over the period T, the result then follows from averaging theory. We omit the details.

Suppose, for the sake of concreteness, $h(\tau) = \eta\cos\tau$. Then

$$h_j = \begin{cases} 0 & \text{if } j\text{=odd} \\ \eta^2 \frac{j-1}{j} h_{j-2} & \text{if } j \text{ =even.} \end{cases}$$

This means that

$$h_{2k} = 2\binom{2k-1}{k}\left(\frac{\eta}{2}\right)^{2k}$$

for positive integers k. With this choice of h, the right-hand side of (12) can be abbreviated to

$$\begin{aligned} F_\gamma(z,\eta) &= F_\gamma(z) + \frac{1}{4}[[F_\gamma, b], b]\,\eta^2 \\ &+ \frac{3}{8\cdot 4!}[[[[F_\gamma, b], b], b], b]\,\eta^4 + \ldots. \end{aligned}$$

An important question regarding parameterized control systems of the form (11) is how stability and bifurcation characteristics of equilibrium solutions of the free motions (unforced dynamics) change under the influence of control. In particular, we may ask the question of how the bifurcations of $F_\gamma(z,\eta) = 0$ depend on η. We know that the essential qualitative features of a pitchfork bifurcation do not generally persist for a one-parameter perturbation. (See [5] for details.) Hence it is noteworthy that under this type of oscillatory forcing, the pitchfork bifurcation is preserved in the averaged versions of both (1)-(3) and (4)-(5).

Proposition 4 *Let $f(\cdot)$ be a p.w.-continuous, T-periodic function, and assume that $h(t)$ is defined as above, such that $\frac{1}{T}\int_0^T h^i(t)dt = 0$ when i = odd. Then if (11) is specialized to (1)-(3) with $u(t) = \frac{1}{\epsilon} f\left(\frac{t}{\epsilon}\right)$, the corresponding averaged system (12) specializes to*

$$\begin{aligned} \frac{dz_1}{d\tau} &= \left(\frac{\alpha}{m\alpha+1}\right)\left[\frac{3H}{W^2}z_2 - \frac{3H}{2W^3}\left(z_2^2+\sigma^2\right)\right]z_1 \\ &-\frac{3H}{8W^3}z_1^3 \end{aligned} \tag{13}$$

$$\begin{aligned} \frac{dz_2}{d\tau} &= \frac{1}{l_c}\Big[-z_3 + \psi_{co} + \frac{3H}{2W^2}\left(z_2^2+\sigma^2\right) \\ &- \frac{Hz_2}{2W^3}\left(z_2^2+3\sigma^2\right) + \frac{3H}{4W^3}(W-z_2)\,z_1^2\Big] \end{aligned} \tag{14}$$

$$\frac{dz_3}{d\tau} = \frac{1}{4B^2 l_c}\left[z_2 - \gamma\sqrt{z_3}\right] \tag{15}$$

where $\sigma^2 \equiv \frac{1}{T}\int_0^T h^2(t)dt$. Under this oscillatory control law, there exists an $\epsilon_o > 0$ such that for all $\epsilon > \epsilon_0$ the asymptotic stability characteristics of (1)-(3) are the same as those of (13)-(15) in the sense that to each equilibrium solution of (13)-(15) there corresponds a periodic solution of the nonautonomous system (1)-(3) having the same asymptotic stability properties.

The proof of the above proposition follows again from the method of averaging, and we omit the details. It is important to note that for small values of σ^2, this averaged system (13)-(15) also undergoes a pitchfork bifurcation as γ varies. As σ^2 increases, the pitchfork bifurcation becomes supercritical, indicating that these zero mean vibrations are having favorable effects. In fact, for sufficiently large σ^2, the bifurcation disappears.

We conclude with some remarks regarding the conclusions one can draw from the averaged dynamics (13)-(15) together with a more detailed analysis of the periodically force system (1)-(3).

Remark 2 Suppose $h(s) = \eta\sin s$. As η is increased from 0, the interval of values of γ for which there are multiple stable equilibria of the averaged system decreases to zero, at which point the pitchfork bifurcation of the averaged system becomes supercritical. With the given parameter values, for instance, one can show that the bifurcation is supercritical when $\eta = 0.2$. To observe the stabilizing effects, however, we must take the frequency of oscillation sufficiently high. This means that the amplitude of the actual forcing, $\frac{1}{\epsilon}f(\frac{t}{\epsilon})$, must also be fairly high. The physical effect of this is that the shut-off head must oscillate between positive and negative values. Implementing such a control law would be more complex than, say, simply varying the pitch of inlet guide vanes. Nevertheless, the theory seems sufficiently attractive that we have begun studying closely related open loop strategies. Specifically, our recent work has been aimed at the use of rapidly decaying oscillations as open loop controls. Simulations indicate that it is possible to use decaying, always positive variations in the shut-off head to steer stalled states back to axisymmetric flow. The attractive feature of this type of control is that its implementation requires that a relatively large volume of air be injected only over a short time interval. Such methods could be used in conjunction with recently proposed methods for stall detection to provide simple and robust means for preventing or recovering from compressor stall.

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