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Averaging Theory for Functional Differential Equations

Brad Lehman[1] Steven P. Weibel [2]

Abstract

Recent work by the authors has established new averaged models for time-dependent functional differential equations (FDE's). In this paper, we present new theorems extending the existing theory to infinite time intervals and a more general class of FDE's. Improvements over the classical theory are illustrated in simulations.

1 Introduction

While averaging theory for ordinary differential equations (ODE's) has established itself as a highly-developed tool in the stability analysis of a wide variety of engineering systems[1, 2, 3, 4, 5, 12, 17, 22, 21], there is at present a much less extensive literature on the averaging of *functional differential equations* (FDE's). FDE's include differential equations with delays, or *delay differential equations.* The existing theory has its roots in the 1960's, when authors such as Fodcuk[7], Halanay[8, 9], Hale[10], Medvedev[18], and Volosov[20] developed a method of averaging for time-varying functional differential equations that admit a small parameter. The most general of these results is given in Hale[10], where the FDE

$$\dot{x}(t) = \epsilon f(t, x_t), \; x_{t_0} = \phi \tag{1}$$

is considered, where $\epsilon \geq 0$. Suppose that, for any constant vector c, we define $\tilde{c}(\theta) = c$, $\theta \in [-r, 0]$. Then the work of [10] (as well as [7, 8, 9, 18, 20]) gives conditions under which solutions of (1) can be approximated by the averaged autonomous ODE

$$\dot{\xi}(t) = \epsilon F_{av}(\tilde{\xi}^t); \; \tilde{\xi}^t(\theta) = \xi(t), \; \theta \in [-r, 0] \tag{2}$$

where $\xi(t_0) = \phi(t_0)$ and

$$F_{av}(\psi) \equiv \lim_{T\to\infty} \frac{1}{T} \int_t^{t+T} f(s, \psi) ds. \tag{3}$$

Recent work[14] has shown that (1) can be approximated on finite time intervals by the averaged system

$$\dot{z}(t) = \epsilon F_{av}(z_t); \; z_{t_0} = \phi, \tag{4}$$

where F_{av} is given in (3). Notice that (4) is an FDE and not an ODE. Numerical simulations indicate that (4) is usually a more accurate approximation of (1) than the classical averaged model given by (2). In this paper, we expand upon the results of [14] by proving the closeness of solutions of (1) and (4) on infinite time intervals when the initial functions of both systems lie in a basin of exponential stability of an equilibria, and prove a finite time interval averaging theorem analogous to the main result of [14] for a class of FDE's which possess additional functional dependence on the small parameter ϵ. In addition, we show how using the method of *moving averages* it is possible to recover the classical averaging result (2) from (4).

There has been a recent interest from the controls community (see [11, 12, 13, 17]) in averaging FDE's of the form of $\dot{x}(t) = f(t/\epsilon, x_t)$. FDE's have found applications in vibrational control [12, 17] and periodic control design [19]. Where improvements in averaging theory for FDE's may have the greatest impact, however, is in the synthesis of adaptive identification algorithms for systems with delays. While this topic has been briefly discussed in [15, 14], a complete theory has yet to emerge. Also, note that analogous theorems for discrete-time systems follow almost directly from the analysis presented here.

2 Preliminaries

Let $\mathbb{R}^n$ be $n-$dimensional Euclidean space. Let $\mathcal{C} = \mathcal{C}([-r, 0], \mathbb{R}^n)$, $r \geq 0$, denote the space of continuous functions that map $[-r, 0]$ into $\mathbb{R}^n$. If $x(t)$ is a continuous function defined on $[t_0 - r, L]$, then we define $x_t \in \mathcal{C}$ by setting $x_t(\theta) = x(t + \theta)$, $\theta \in [-r, 0]$ for each $t_0 \leq t \leq L$, where $L > t_0$. For each $\psi \in \mathcal{C}$, let $||\psi||$ denote $sup\{|\psi(\theta)| : \theta \in [-r, 0]\}$, where $|\cdot|$ is a norm of $\mathbb{R}^n$. For any $D \subset \mathbb{R}^n$, let $\mathcal{C}(D) = \mathcal{C}([-r, 0], D)$. The functional $f : \mathbb{R} \times \mathcal{C} \to \mathbb{R}^n$ is always assumed to be continuous. Let $\phi(t)$ be a continuous function on $t \in [t_0-r, t_0]$, and assume in (1) that $x(t) = \phi(t)$ on this interval. Then (1) has a solution which is denoted as $x(t) = x(t; t_0, \phi)$. The parameter ϵ is always assumed to be nonnegative. (We also sometimes write $x_{t_0} = \phi_{t_0} = \phi \in \mathcal{C}$, in a standard mild abuse of notation.) Likewise, the solution of (4) is denoted as $z(t) = z(t; t_0, \phi)$ for $z_{t_0} = \phi$. All derivatives are assumed to be right-hand derivatives. As already introduced in (2), let $\tilde{\xi}^s(\theta) = \xi(s)$. Finally, let $\mathbb{Z}^+$ denote the set of nonnegative real integers.

Definition 1 Suppose that $f : \mathbb{R} \times \mathcal{C} \to \mathbb{R}^n$ is continuous and is uniformly bounded such that $|f(t, \psi)| \leq M$ for all (t, ψ) on $\mathbb{R} \times \mathcal{C}(D)$. Assume further that f is locally Lipschitz, i.e., for any (t, ψ^1, ψ^2) in $\mathbb{R} \times \mathcal{C}(D) \times \mathcal{C}(D)$ there exists a $K > 0$ such that $|f(t, \psi^1) - f(t, \psi^2)| \leq K||\psi^1 - \psi^2||$. Furthermore, suppose that the average in (3) exists uniformly for all $(t, \psi) \in \mathbb{R} \times \mathcal{C}(D)$. Then f is said to be a **KBM-functional**.

[1]Department of Electrical and Computer Engineering, Northeastern University, Boston MA 02115. The first author gratefully acknowledges the support of an NSF Presidential Faculty Fellowship through grant CMS 9596268.

[2]Department of Electrical and Computer Engineering, Northeastern University, Boston, MA 02115.

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Definition 2 Suppose that $x(t) = x(t; t_0, \phi)$ is the solution to (1) with initial function $\phi \in \mathcal{C}$. The **moving average** of $x(t)$ is denoted by $\bar{x}(t)$ and is defined as

$$\overline{x}(t) \equiv \begin{cases} \phi(t), & \text{for } t \in [t_0 - r, t_0) \\ \frac{1}{T}\int_t^{t+T} x(s)ds, & \text{for } t \geq t_0, \end{cases}$$

where $T > 0$.

Definition 3 Consider a functional

$$f : \mathbb{R} \times \mathcal{C}([-r, 0], \mathbb{R}^p) \to \mathbb{R}^n.$$

The **local average** of f, denoted by f_T is defined by

$$f_T(t, \psi) \equiv \frac{1}{T}\int_0^T f(t+s, \psi)ds$$

where $T > 0$ and p is a non-negative integer. We define the *locally averaged FDE*

$$\dot{y}(t) = \epsilon f_T(t, y_t). \qquad (5)$$

As usual, denote the solution of (5) with the initial function $y_{t_0} = \phi \in \mathcal{C}$ as $y(t) = y(t; t_0, \phi)$.

Using the moving averages and local averages to bound the difference between solutions of (1) and (4), it is possible to prove the following theorem, which is proved in its entirety in [14].

Theorem 1 (Averaging on Finite Time Intervals) *Suppose the f is a KBM-functional and that (1), (4) and (5) have the same continuous initial function, $\phi \in \mathcal{C}(D)$, on $t \in [t_0 - r, t_0]$. Let $L > 0$ be a constant that is independent of ϵ. Assume that $x(t)$, the solution to (1), lies in D for $t \in \left[t_0 - r, t_0 + \frac{L}{\epsilon} + \frac{1}{\sqrt{\epsilon}}\right]$, and that $z(t)$ and $y(t)$, solutions to (4) and (5) respectively, lie in D for $t \in \left[t_0 - r, t_0 + \frac{L}{\epsilon}\right]$. Then $|x(t) - z(t)| = \mathcal{O}(Q(\epsilon))$ for all $t \in \left[t_0 - r, t_0 + \frac{L}{\epsilon}\right]$, where $Q(\epsilon) \to 0$ as $\epsilon \to 0$ and*

$$Q(\epsilon) \equiv \frac{M\sqrt{\epsilon}}{2} + L\gamma\left(\frac{1}{\sqrt{\epsilon}}\right)e^{KL} + [2\epsilon Mr(1+\epsilon) + M\sqrt{\epsilon}(1/2 + KL)]e^{KL},$$

and where $\gamma(t) \equiv \sup_{\psi \in \mathcal{C}(D)} \sup_{t \geq t_0} |f_T(t, \psi) - F_{av}(\psi)|$.

3 Averaging Theory on Infinite Time Intervals

Theorem 2 (Hovering Theorem) *Assume the hypotheses of Theorem 1 are true for all $t \geq t_0 - r$. Let z_e be an exponentially stable equilibrium of (4), and let the initial function $\phi(t)$ lie in the domain of exponential stability of z_e, where $x(t) = z(t) = \phi(t)$ for $t \in [t_0 - r, t_0]$. Then*

$$\sup_{t \geq t_0} |x(t) - z(t)| = \mathcal{O}(Q(\epsilon)).$$

where $Q(\epsilon) \to 0$ as $\epsilon \to 0$, and is given in Theorem 1.

Proof: Without loss of generality, let $t_0 = 0$. For $t \in [-r, 0]$, $x(t) = z(t) = \phi(t)$, therefore $|x(t) - z(t)| = 0$ on this interval. Consider a partitioning of the time axis into the union of intervals $\bigcup_{n=0}^{\infty} I_{n,n+1}$, where $n \in \mathbb{Z}^+$ and

$$I_{n,n+1} \equiv \left\{t : t \in \mathbb{R}, \frac{nL}{\epsilon} \leq t \leq \frac{(n+1)L}{\epsilon}\right\},$$

and where $L > 0$ is a constant whose value is to be determined. On each interval $I_{n,n+1}$, define $z(n,t)$ as the solution of (4) with initial function $z(n,t) = x(t)$ for $t \in \left[\frac{nL}{\epsilon} - r, \frac{nL}{\epsilon}\right]$. On each interval $I_{n,n+1}$, we can then write

$$\begin{aligned} |x(t) - z(t)| &= |x(t) - z(n,t) + z(n,t) - z(t)| \\ &\leq |x(t) - z(n,t)| + |z(n,t) - z(t)|. \end{aligned}$$

The strategy of this proof is to use $z(n,t)$ as an intermediate solution with which we can bound $x(t)$ and $z(t)$ on the time interval $t \in I_{n,n+1}$. Since the choice of n is arbitrary, we can allow $n \to \infty$ and hence we have shown that $x(t) - z(t)$ is bounded on every time interval. By Theorem 1, we have for any fixed L that $|x(t) - z(n,t)| \leq Q(\epsilon)$ on $I_{n,n+1}$, where $Q(\epsilon) \to 0$ as $\epsilon \to 0$. Since $z(0,t) = z(t)$, this gives closeness on $I_{0,1}$. For $n \geq 1$, we make use of the definition of exponential stability and its continuity properties (see [6]) to write for $t \geq \frac{nL}{\epsilon}$

$$\begin{aligned} &|z(n,t) - z(t)| \\ &\quad\leq me^{-\epsilon\alpha(t - nL/\epsilon)} \sup_{s \in \left[\frac{nL}{\epsilon} - r, \frac{nL}{\epsilon}\right]} |z(n,s) - z(s)| \\ &\quad\leq me^{-\epsilon\alpha(t - nL/\epsilon)} \sup_{s \in \left[\frac{nL}{\epsilon} - r, \frac{nL}{\epsilon}\right]} [|z(n,s) - z(n-1,s)| \\ &\qquad + |z(s) - z(n-1,s)|], \end{aligned} \qquad (6)$$

where $m > 1$ and $\alpha > 0$. Without loss of generality, assume $L/\epsilon \geq r$. By the definition of $z(n,t)$ and by Theorem 1, we have for $n \geq 1$

$$\begin{aligned} &\sup_{s \in \left[\frac{nL}{\epsilon} - r, \frac{nL}{\epsilon}\right]} |z(n,s) - z(n-1,s)| \\ &\qquad = \sup_{s \in \left[\frac{nL}{\epsilon} - r, \frac{nL}{\epsilon}\right]} |x(s) - z(n-1,s)| \leq Q(\epsilon). \end{aligned}$$

Let $\Delta_{n-1} \equiv \sup_{s \in \left[\frac{nL}{\epsilon} - r, \frac{nL}{\epsilon}\right]} |z(s) - z(n-1,s)|$, then by (6)

$$|z(t) - z(n,t)| \leq me^{-\epsilon\alpha\left(t - \frac{nL}{\epsilon}\right)}[\Delta_{n-1} + Q(\epsilon)], \qquad (7)$$

for all $t \geq \frac{nL}{\epsilon}$, which implies that

$$\begin{aligned} &\sup_{s \in \left[\frac{(n+1)L}{\epsilon} - r, \frac{(n+1)L}{\epsilon}\right]} |z(s) - z(n,s)| \\ &\qquad \leq m \sup e^{-\epsilon\alpha\left(s - \frac{nL}{\epsilon}\right)}[\Delta_{n-1} + Q(\epsilon)], \end{aligned}$$

which further implies

$$\begin{aligned} \Delta_n &\leq me^{-\epsilon\alpha\left(\frac{L}{\epsilon} - r\right)}[\Delta_{n-1} + Q(\epsilon)] \\ &= me^{-\alpha L}e^{\epsilon\alpha r}[\Delta_{n-1} + Q(\epsilon)]. \end{aligned}$$

Now, select L sufficiently large that $me^{\epsilon\alpha r}e^{-\alpha L} = K_L < 1$. Noting that $\Delta_0 = 0$, we have

$$\Delta_n \leq K_L[\Delta_{n-1} + Q(\epsilon)] \leq \frac{K_L Q(\epsilon)}{1 - K_L},$$

Therefore, in (7), for $t \in I_{n,n+1}$ and $n \geq 0$,

$$\begin{aligned} |z(t) - z(n,t)| &\leq me^{-\epsilon\alpha(t-\frac{nL}{\epsilon})}\left[\frac{K_L Q(\epsilon)}{1-K_L} + Q(\epsilon)\right] \\ &\leq \frac{mQ(\epsilon)}{1-K_L}e^{-\epsilon\alpha(t-\frac{nL}{\epsilon})} \leq \frac{mQ(\epsilon)}{1-K_L}. \end{aligned}$$

Then on $I_{n,n+1}$,

$$\begin{aligned} |x(t) - z(t)| &\leq |x(t) - z(n,t)| + |z(n,t) - z(t)| \\ &\leq Q(\epsilon) + \frac{mQ(\epsilon)}{1-K_L} \\ &\leq Q(\epsilon)\left[1 + \frac{m}{1-K_L}\right]. \end{aligned} \tag{8}$$

This guarantees that $|x(t) - z(t)| = \mathcal{O}(Q(\epsilon))$ on any interval $I_{n,n+1}$, where $n \geq 0$ is arbitrary. □

4 Uniformly Continuous Functional Dependence on ϵ

Consider the FDE

$$\dot{u}(t) = \epsilon g(t, u_t, \epsilon); \quad u_{t_0} = h. \tag{9}$$

where $u_t \in \mathcal{C}$ and $g : \mathbb{R} \times \mathcal{C} \times \mathbb{R} \to \mathbb{R}^n$. Furthermore assume that

$$g(t, u_t, 0) = f(t, u_t) \tag{10}$$

where f is as defined in (1). In the next theorem, we relate the solutions of (9) to the solutions of (4). Notice that it is no longer required that (9) and (4) have the same initial function and that g can explicitly depend on ϵ. As usual, let $u(t) = u(t; t_0, h)$ denote the solution to (9).

Theorem 3 *Let L be as in Theorem 1 and let the assumptions of Theorem 1 be true. Assume that for $(t, \psi, \epsilon) \in \mathbb{R} \times \mathcal{C}(D) \times [0, \epsilon_1]$, $\epsilon_1 > 0$, the function g is continuous with respect to all its arguments and uniformly continuous with respect to ϵ. Assume, further, that $g(t, \psi, 0) = f(t, \psi)$, where f is a KBM-functional. Finally, assume that both $u(t; t_0, h)$ and $u(t; t_0, \phi)$, the solutions of (9) with initial functions h and ϕ respectively, remain in D for $\left[t_0 - r, t_0 + \frac{L}{\epsilon}\right]$. Then for any $\eta > 0$ there exists a $\beta_0 = \beta_0(\eta, \sigma, L)$ and an $\epsilon_0 = \epsilon_0(\eta, \sigma, \beta, L, \epsilon_1)$ such that, for $0 \leq \beta \leq \beta_0$ and $0 \leq \epsilon \leq \epsilon_0$,*

$$|u(t; t_0, h) - z(t; t_0, \phi)| \leq \eta$$

for all $t \in \left[t_0 - r, t_0 + \frac{L}{\epsilon}\right]$, where $\beta = \sup_{s \in [t_0 - r, t_0]} |\phi(s) - h(s)|$ and $z(t; t_0, \phi)$ denotes the solution to (4).

Proof: The theorem may be proved by bounding $|u(t; t_0, h) - z(t; t_0, \phi)|$ as follows:

$$\begin{aligned} &|u(t; t_0, h) - z(t; t_0, \phi)| \\ &\quad \leq |u(t; t_0, \phi) - u(t; t_0; h)| + |x(t; t_0, \phi) - u(t; t_0, \phi)| \\ &\quad\quad + |x(t; t_0, \phi) - z(t; t_0, \phi)|, \end{aligned}$$

where $x(t; t_0, \phi)$ denotes the solution to (1), and making use of the continuity of $u(\cdot)$, $x(\cdot)$, and $z(\cdot)$. Details of the proof of this theorem may be found in [16]. □

In some problems, it may be useful to retain higher order terms in ϵ in the hope that the additional ϵ dependence will improve the accuracy of the averaged model.

Theorem 4 *Let L, K, M, ϵ, and ϵ_0 be positive numbers, and let $D \subset \mathbb{R}^m$. Let the average $w(t)$ be defined as the solution to*

$$\dot{w}(t) = \epsilon G_{av}(w_t, \epsilon); \; w_{t_0} = \phi(t), \tag{11}$$

where

$$G_{av}(\psi, \epsilon) = \lim_{T \to \infty} \frac{1}{T} \int_t^{t+T} g(s, \psi, \epsilon) ds, \tag{12}$$

and let the local average $v(t)$ be defined as the solution to

$$\dot{v}(t) = \epsilon g_T(t, v_t, \epsilon); v_{t_0}(t) = \phi(t), \tag{13}$$

where $g_T(t, \psi, \epsilon) = \frac{1}{T}\int_0^T g(t + s, \psi, \epsilon) ds$ and $T \in \mathbb{R}$ with $T > 0$. Assume that for all $(t, \psi^i, \epsilon) \in [t_0 - r, t_0 + L/\epsilon] \times \mathcal{C}(D) \times [0, \epsilon_0]$ the following hold:

1. *g is continuous with respect to all its arguments;*
2. *$\left|g(t, \psi^1, \epsilon)\right| \leq M$;*
3. *$\left|g(t, \psi^1, \epsilon) - g(t, \psi^2, \epsilon)\right| \leq K\|\psi^1 - \psi^2\|$; and*
4. *limit (12) exists uniformly, i.e. $|g_T(t, \psi, \epsilon) - G_{av}(\psi, \epsilon)| \leq \gamma(T)$, where $\gamma(T) > 0$ is independent of ϵ with $\lim_{T \to \infty} \gamma(T) \to 0$.*

Assume that (9), (11), and (13) have the same continuous initial function and let the solutions of these systems be denoted by $u(t)$, $w(t)$, and $v(t)$ respectively. Assume further that $u(t)$ remains in D for all $t \in \left[t_0 - r, t_0 + \frac{L}{\epsilon} + \frac{1}{\sqrt{\epsilon}}\right]$ and that $v(t)$ and $w(t)$ remain in D for all $t \in [t_0 - r, t_0 + L/\epsilon]$. Then for all $(t, \epsilon) \in [t_0 - r, t_0 + L/\epsilon] \times [0, \epsilon_0]$,

$$|u(t; t_0, \phi) - w(t; t_0, \phi)| = \mathcal{O}(Q(\epsilon)),$$

where $Q(\epsilon) \to 0$ as $\epsilon \to 0$ and $Q(\epsilon)$ is defined in Theorem 1.

Proof: The proof is identical to the proof of Theorem 1.

Remark 1 By permitting G_{av} to depend nonlinearly on ϵ, it is hoped that $w(t)$ better approximates $u(t)$ than $z(t)$ as shown in Theorem 3. However, there are no guarantees since all bounds derived tend to be conservative. Additionally, Assumption 4 in Theorem 4 may be restrictive and difficult to verify.

5 Eliminating the Delay by Averaging

Theorem 5 *Let the assumptions of Theorem 1 be true. Suppose that $\xi(t)$ is defined as the solution to ODE (2) with initial condition $\xi(t_0) = \phi(t_0)$, where ϕ is the initial function for (1). Further assume that $\xi(t) \in D$ for $t \in \left[t_0 - r, t_0 + \frac{L}{\epsilon}\right]$. Then*

$$|x(t) - \xi(t)| \leq Q(\epsilon) + \epsilon Mr(2 + KL)e^{KL},$$

where $Q(\epsilon)$ is as defined in Theorem 1, and $x(t)$ is the solution to (1).

Proof: To prove the theorem, we bound $|x(t)-\xi(t)|$ as follows:

$$|x(t)-\xi(t)| \leq |x(t)-z(t)|+|z(t)-\xi(t)|.$$

By Theorem 1, $|x(t)-z(t)| \leq Q(\epsilon)$. Therefore, all that remains is to compute a bound for $|z(t)-\xi(t)|$.

Let $\tilde{\xi}^s(\theta)=\xi(s)$. Then, recalling (4) and (2), we can write for $t \geq t_0$

$$|z(t)-\xi(t)| = \epsilon \int_{t_0}^{t} [F_{av}(z_s)-F_{av}(\tilde{\xi}^s)]ds.$$

Noting that $z(t), \xi(t) \in D$ for $t \in [t_0, t_0+L/\epsilon]$ and recalling that for $\psi \in \mathcal{C}(D)$, $|F_{av}(\psi)| \leq M$, on the interval $t \in [t_0, t_0+r]$ we have

$$|z(t)-\xi(t)| \leq \epsilon \int_{t_0}^{t_0+r} 2M ds \leq 2\epsilon Mr.$$

Now consider the interval $t \in [t_0+r, t_0+L/\epsilon]$. Let $t_1 = t_0+r$. Then

$$|z(t)-\xi(t)| \leq 2\epsilon Mr + \epsilon \int_{t_1}^{t} |F_{av}(z_s)-F_{av}(\xi_s)|\, ds + \epsilon \int_{t_1}^{t} \left|F_{av}(\xi_s)-F_{av}(\tilde{\xi}^s)\right| ds. \tag{14}$$

For the first integral, we have for $t \geq t_1$

$$\int_{t_1}^{t} |F_{av}(z_s)-F_{av}(\xi_s)|\, ds \leq K \int_{t_1}^{t} \|z_s-\xi_s\| ds.$$

For the second integral, we have for $t \geq t_1$

$$\int_{t_1}^{t} \left|F_{av}(\xi_s)-F_{av}(\tilde{\xi}^s)\right| ds \leq K \int_{t_1}^{t} \sup_{\sigma \in [-r,0]} |\xi(s+\sigma)-\xi(s)|\, ds.$$

However, for $s \geq t_0+r$,

$$\sup_{\sigma \in [-r,0]} |\xi(s+\sigma)-\xi(s)| = \sup_{\sigma \in [-r,0]} \left| \epsilon \int_{s}^{s+\sigma} F_{av}(\tilde{\xi}^\tau) d\tau \right| \leq \epsilon Mr.$$

Substituting this into (14) yields for $t \in [t_1, t_0+L/\epsilon]$

$$|z(t)-\xi(t)| \leq \epsilon[2Mr+KLMr] + \epsilon K \int_{t_1}^{t} \|z_s-\xi_s\| ds.$$

This implies (see [16]) that for $t \in [t_1, t_0+L/\epsilon]$

$$\sup_{s \in [t_1,t]} |z(s)-\xi(s)| \leq \epsilon Mr(2+KL)e^{KL}.$$

This inequality is true on $t \in [t_0, t_1]$ also. Hence, for $t \in \left[t_0, t_0+\frac{L}{\epsilon}\right]$,

$$\sup_{s \in \left[t_0, t_0+\frac{L}{\epsilon}\right]} |z(s)-\xi(s)| \leq \epsilon Mr(2+KL)e^{KL}. \qquad \square$$

In proving Theorem 5, the approach taken is that the "averaging out" of the delay might introduce additional error. That is, this paper suggests the possibility of two different upper bounds in ϵ. First, there is an ϵ_0 such that for $0 \leq \epsilon \leq \epsilon_0$, the new averaged model (4) becomes a good approximation of (1). Next, there is a value ϵ_1 such that for $0 \leq \epsilon \leq \epsilon_1$, the delay can be ignored so (2) becomes an accurate approximation of (1). Since all theorems and lemmas in this paper yield only sufficient conditions, the approaches taken fail in determining accurate approximations of ϵ_0 and ϵ_1. Numerical simulations, however, have indicated that (4) gives a better approximation to (1).

6 Examples and Applications

6.1 Example - Improvements over the "Classical" Method of Averaging for Delay Systems

Consider the nonautonomous scalar differential delay equation given by

$$\dot{x}(t) = \epsilon\left(-4\cos^2(t)x(t-r)+x(t)\right), \tag{15}$$

where $x(t)=1$ on the interval $t \in [-r,0]$. By the classical averaging methods of [7, 8, 9, 10, 18, 20], the average (15) can be written

$$\dot{y}(t) = \epsilon\left(-2y(t)+y(t)\right) = -\epsilon y(t),\ y(0)=1, \tag{16}$$

which is a simple linear ODE in which the origin is uniformly asymptotically stable. Therefore, for sufficiently small ϵ, the classical approach to averaging described in Section 1 predicts that

$$|x(t)-y(t)| < \eta(\epsilon)$$

for $t \geq 0$, where $\eta(\epsilon) \to 0$ as $\epsilon \to 0$. That is, for sufficiently small ϵ, $x(t) \approx y(t) = e^{-\epsilon t}$.

The averaged model (4) retains the delay term and can be written

$$\dot{z}(t) = \epsilon\left(-2z(t-r)+z(t)\right), \tag{17}$$

where $z(t)=1$ on the interval $t \in [-r,0]$. By Theorem 1, $|x(t)-z(t)| \leq Q(\epsilon)$ for $t \in \left[0, \frac{L}{\epsilon}\right]$ where $Q(\epsilon) \to 0$ as $\epsilon \to 0$. However, unlike the ODE model, the zero equilibrium point of delay differential equation (17) is not always asymptotically stable. The characteristic equation of (17) is given by

$$s-\epsilon\left(-2e^{-rs}+1\right)=0,$$

for which all roots have negative real part if and only if

$$\epsilon r < \frac{\pi}{3\sqrt{3}} \sim 0.6046. \tag{18}$$

Clearly, if $x=0$ of (15) is unstable, then the classical averaged model (16) cannot be used as an approximation for time-varying system (16). On the other hand, since the new averaged models proposed in this paper retain information on the delay, (17) will not necessarily lose validity when $\epsilon r > 0.6046$.

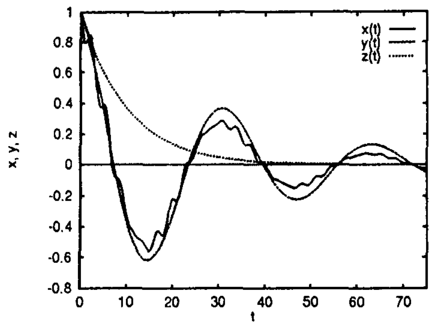

Figure 1: Simulation results for (15), (16), and (17) for $r = 5$ and $\epsilon = 0.1$. Note that $y(t)$ is difficult to resolve because of the vertical scale of the plot.

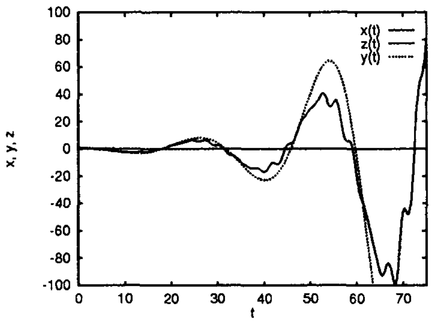

Figure 2: Simulation results for (15), (16), and (17) for $r = 5$ and $\epsilon = 0.18$. Note that $y(t)$ is difficult to resolve because of the vertical scale of the plot.

To illustrate the improvements of the new averaged models, we have simulated (15), (16), and (17) for $r = 5$ and $\epsilon = 0.1$ $\left(\epsilon r = 0.5 < \frac{\pi}{3\sqrt{3}}\right)$ and $\epsilon = 0.18$ $\left(\epsilon = 0.9 > \frac{\pi}{3\sqrt{3}}\right)$. The results of these simulations are shown in Figures 1 and 2.

In Figure 1, we see that all solutions limit on zero as predicted. We note that (17) approximates (15) better than (16). Additionally, the classical averaged equation shows a solution monotonically decreasing. On the other hand, the solution of the averaged equation (17) has local maxima, minima, and several zero crossings. This would not be possible for solutions to (16), but is possible for linear time invariant FDE's.

In Figure 2, we see that the origin of the original system (15) is unstable. The instability of the original system is not reflected in the behavior of (16), where the trajectories still limit on the origin. The instability is, however, reflected in the behavior of (17), which, in addition to being an unstable solution, still approximates the solution of (15) on finite time intervals.

This example exhibits some clear differences between classical averaging and the method of averaging presented here. Under relatively ordinary circumstances, the classical model does not approximate the original system well enough to retain information on the stability of system fixed points.

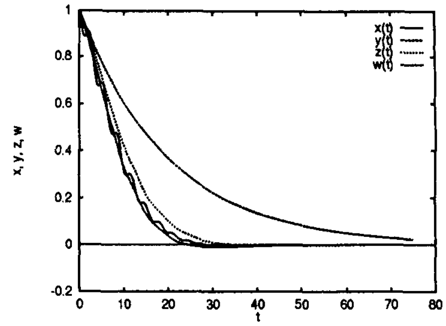

Figure 3: $x(t)$, $y(t)$, $z(t)$, and $w(t)$ vs. t for $\epsilon = 0.05$, $r(t) = 5 + \sin t$.

6.2 Example - Functional Dependence on ϵ and Time-Varying Delay

Now consider the first order delay differential equation

$$\dot{x}(t) = \epsilon\left[-4\cos^2(t)x(t - r(t)) + (1 - \sqrt{\epsilon})x(t)\right], \quad (19)$$

where $r(t) = r_0 + r_1 \sin \omega t$, $0 \leq r_1 \leq r_0$, and $x(t) = 1$ for $t \in [-(r_0 + r_1), 0]$. Equation (19) is similar to the system studied in the previous example, but now we consider a time-varying delay and additional dependence on ϵ. Following the classical approach, the average of (19) is the same as obtained in the previous example; i.e.

$$\dot{y}(t) = -\epsilon y(t), \quad (20)$$

where $y(0) = x(0) = 1$. Using the averaging method of Theorem 3, the averaged system takes the form

$$\dot{z}(t) = \epsilon\left[-2z(t - r(t)) + z(t)\right], \quad (21)$$

where $z(t) = 1$ for $t \in [-(r_0+r_1), 0]$. Similarly, the averaged system given by Theorem 4 is

$$\dot{w}(t) = \epsilon[-2w(t - r(t)) + (1 - \sqrt{\epsilon})w(t)], \quad (22)$$

where $w(t) = 1$ for $t \in [-(r_0 + r_1), 0]$.

Systems (19), (20), (21), and (22) have been simulated for a time-varying delay $r(t)$ and various values of ϵ. Some results of these simulations are shown in Figures 3, 4, and 5. In all figures, $r(t) = 5 + \sin t$. Figure 3 shows $x(t)$, $y(t)$, $z(t)$, and $w(t)$ for $\epsilon = 0.05$. In the figure, we see that for this small value of ϵ, $y(t)$, $z(t)$, and $w(t)$ all behave qualitatively similar to $x(t)$. Note that $w(t)$ best approximates the solution of the original system $x(t)$. As expected, $y(t)$ gives the worst approximation to $x(t)$, decaying to zero at a much slower rate than $x(t)$.

Figure 4 shows $x(t)$, $y(t)$, $z(t)$, and $w(t)$ for $\epsilon = 0.1$. Once again, $w(t)$ clearly approximates $x(t)$ better than $y(t)$ or $z(t)$, with $w(t)$ and $x(t)$ attaining local maxima and minima at nearly the same values of t on the simulated interval. While neither $z(t)$ or $w(t)$ seem to oscillate at precisely the same frequency as $x(t)$, frequency and phase differences

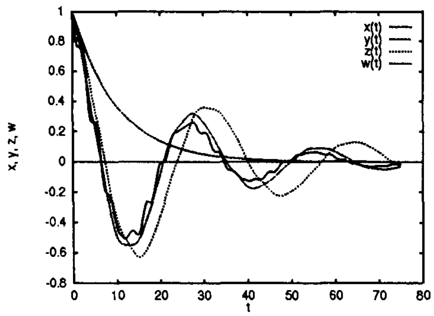

Figure 4: $x(t)$, $y(t)$, $z(t)$, and $w(t)$ vs. t for $\epsilon = 0.1$, $r(t) = 5 + \sin t$.

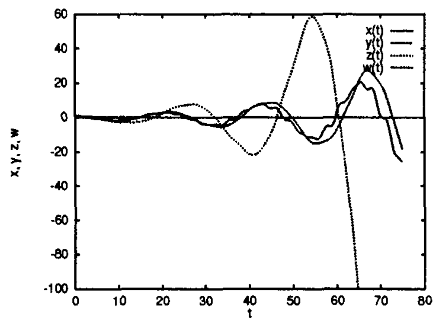

Figure 5: $x(t)$, $y(t)$, $z(t)$, and $w(t)$ vs. t for $\epsilon = 0.18$, $r(t) = 5 + \sin t$. Note that $y(t)$ is difficult to resolve because of the vertical scale of the plot.

are much smaller for $w(t)$ than for $z(t)$. Once again, $y(t)$ yields the worst approximation to $x(t)$, only capturing the qualitative character of $x(t)$.

As in the previous example, the qualitative characteristics of $x(t)$ and $y(t)$ diverge for $\epsilon = 0.18$, as shown in Figure 5. The origin of the original system (19) is unstable for this value of ϵ, but the instability is not captured by $y(t)$. While the origin is unstable for (21) and (22), $z(t)$ diverges quickly from $x(t)$. Conversely, $w(t)$ closely shadows $x(t)$ for the entire simulation interval, with small frequency and phase differences.

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