Proceedings of the 38th
Conference on Decision & Control
Phoenix, Arizona USA • December 1999

Partial Averaging of Functional Differential Equations

Brad Lehman[1]
Dept. of Electrical&Computer Engineering
Northeastern University
Boston, MA 02115

Steven P. Weibel
Guidance, Navigation, and Control Center
Raytheon Systems Company
Tucson, AZ 85747

Abstract

This paper develops a framework for averaging functional differential equations (FDEs) with two time scales. Averaging is performed on the fast time system, while slow time is 'frozen.' This creates an averaged equation which is slowly time-varying, hence the terminology of partial averaging. We show that solutions of the original FDE and its corresponding partially averaged equation remain close on arbitrarily long but finite time intervals. Next, assuming that the partially averaged system has an exponentially stable equilibrium point and that we restrict our interest to initial conditions that lie in the domain of exponential stability, the finite-time averaging results are extended to infinite time. In the special case of pointwise delays, exponential stability of the averaged system can be related to the frozen-time eigenvalues of its linearization.

1 Introduction

The method of averaging has found widespread applications in fields such as adaptive identification and control, vibration control, stability of oscillators(see [1, 4] for general discussions), and open-loop control[2, 16, 15]. The goal of "classical averaging" is to determine conditions in which the solution of a time-varying ordinary differential equation which admits a small parameter can be approximated by the solution of autonomous ordinary differential equation. Classical approaches use near identity changes of variables to transform the time-varying system into a system with an autonomous component and a small time-varying component. Once this transformation has been made, stability and transient properties of the original time-varying system can be inferred from conventional analysis of the autonomous component.

The theory of averaging for FDE's is currently less developed than for ODE's. In the 1960's, authors such as [5], [6], [7], [14], [3], and [10] approximated classes of FDE's by autonomous, averaged ODE's. In [13], [9] and [8] averaged models that are FDE's (instead of the classical ODE approximation by [5], [6], [7], [14], [3], and [10]) are presented. Apparently, maintaining the infinite dimensional nature of the averaged system improves its accuracy. The trade-off is that analysis of the averaged system becomes more difficult.

This paper extends the results of [13] and [9] to include systems with two time scales. Unlike the one time scale case, the averaged equation remains time-varying. However, only the slow time scale appears in the averaged equation, simplifying its analysis.

Specifically, we consider the FDE

$$\dot{x}(t) = \epsilon f(t, x_t, \epsilon t). \tag{1}$$

Here, $x_t \in \mathcal{C} = \mathcal{C}([-r, 0], \mathbb{R}^n)$, where $r \geq 0$ is a constant and $x_t = x_t(\theta), -r \leq \theta \leq 0$. The parameter ϵ satisfies $0 \leq \epsilon << 1$ and $f : \mathbb{R} \times \mathcal{C} \times \mathbb{R} \to \mathbb{R}^n$ is a smooth functional.

Define the corresponding *partial average* of (1) by

$$\dot{z}(t) = \epsilon F_{av}(z_t, \epsilon t) \tag{2}$$

where

$$F_{av}(\psi, \epsilon t) \equiv \lim_{T\to\infty} \frac{1}{T} \int_t^{t+T} f(s, \psi, \epsilon t) ds. \tag{3}$$

In Theorem 1, we give conditions that guarantee solutions of (1) can be approximated by solutions of the partially averaged system (2), with F_{av} given in (3), on arbitrarily large but finite time intervals, i.e. $|x(t) - z(t)| = \mathcal{O}(\epsilon)$ on time intervals $t \sim \mathcal{O}(1/\epsilon)$.

Next, we extend the averaging results to infinite time intervals under the assumption that the partial averaged equation has an exponentially stable equilibrium point, given by Theorem 2. The proof of Theorem 2 follows almost directly from previously known results. Essentially, the infinite time interval can be viewed as two different time intervals. First, there is the finite time interval in which solutions of the averaged equation uniformly approach the equilibrium point to arbitrarily small accuracy. On this time interval, Theorem

[1]The first author gratefully acknowledges the support of an NSF Presidential Faculty Fellowship through grant CMS 9596268.

1 can be used to guarantee closeness of the partial averaged and original time-varying FDE's. In some sense, the partial averaged solution has 'dragged' $x(t)$ into an arbitrarily small ball around a point. Using properties of uniformity and exponential stability, it is then straightforward to show that neither $x(t)$ nor $z(t)$ can ever leave this ball, and therefore, $|x(t) - z(t)| = \mathcal{O}(\epsilon)$ for all $t \geq t_0$. This can be proved in several ways, e.g. induction, proof by contradiction, etc.

Theorem 3 considers the special case when there are pointwise delays in (1). In this case, it is possible to relate the conditions of Theorem 2 to the corresponding frozen-time eigenvalues of the Jacobian of (2), and hence, guarantee $|x(t) - z(t)| = \mathcal{O}(\epsilon)$ for all $t \geq t_0$.

The outline of the paper is as follows. Section 2 gives preliminary definitions and lemmas. Section 3 presents the averaging results. Section 4 gives examples, followed by conclusions in Section 5.

2 Preliminaries

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space. Let $\mathcal{C} = \mathcal{C}([-r, 0], \mathbb{R}^n)$, $r \geq 0$, denote the space of continuous functions that map $[-r, 0]$ into $\mathbb{R}^n$. If $x(t)$ is a continuous function defined on $[t_0 - r, L]$, then we define $x_t \in \mathcal{C}$ by setting $x_t(\theta) = x(t + \theta)$, $\theta \in [-r, 0]$ for each $t_0 \leq t \leq L$, where $L > t_0$. For each $\psi \in \mathcal{C}$, let $||\psi||$ denote $sup\{|\psi(\theta)| : \theta \in [-r, 0]\}$, where $|\cdot|$ is a norm of $\mathbb{R}^n$. For any $D \subset \mathbb{R}^n$, let $\mathcal{C}(D) = \mathcal{C}([-r, 0], D)$. The functional $f : \mathbb{R} \times \mathcal{C} \to \mathbb{R}^n$ is always assumed to be continuous. Let $\phi(t)$ be a continuous function on $t \in [t_0 - r, t_0]$, and assume in (1) that $x(t) = \phi(t)$ on this interval. Then (1) has a solution which is denoted as $x(t) = x(t; t_0, \phi)$. (We also sometimes write $x_{t_0} = \phi_{t_0} = \phi \in \mathcal{C}$, in a standard mild abuse of notation.) Likewise, the solution of (2) is denoted as $z(t) = z(t; t_0, \phi)$ for $z_{t_0} = \phi$. All derivatives are assumed to be right-hand derivatives. The parameter ϵ will always be assumed to be non-negative.

Definition 1 Suppose that $f : \mathbb{R} \times \mathcal{C} \times \mathbb{R} \to \mathbb{R}^n$ is continuous and is uniformly bounded such that $|f(t, \psi, \lambda)| \leq M$ for all (t, ψ, λ) on $\mathbb{R} \times \mathcal{C}(D) \times \mathbb{R}$. Assume further for any (t, ψ^i, λ_i) in $\mathbb{R} \times \mathcal{C}(D) \times \mathbb{R}$, $i = 1, 2$, there exist $K_1 > 0$ and $K_2 > 0$ such that $|f(t, \psi^1, \lambda_1) - f(t, \psi^2, \lambda_2)| \leq K_1||\psi^1 - \psi^2|| + K_2|\lambda_1 - \lambda_2|$. Furthermore, suppose that the average in (3) exists uniformly in t for all (ψ, λ) in $\mathcal{C}(D) \times \mathbb{R}$. Then f is said to be a **partial KBM-functional** (Krylov-Bogolyubov-Mitropolsky).

Definition 2 Suppose that $x(t) = x(t; t_0, \phi)$ is the solution to (1) with initial function $\phi \in \mathcal{C}$. The **moving average** of $x(t)$ is denoted by $\bar{x}(t)$ and is defined as

$$\overline{x}(t) \equiv \begin{cases} \phi(t), & \text{for } t \in [t_0 - r, t_0) \\ \frac{1}{T}\int_t^{t+T} x(s)ds, & \text{for } t \geq t_0, \end{cases}$$

where $T > 0$.

Definition 3 Consider a functional

$$f : \mathbb{R} \times \mathcal{C}([-r, 0], \mathbb{R}^p) \times \mathbb{R} \to \mathbb{R}^n,$$

where $p \in \mathbb{Z}^+$. Then the **local partial average** of f, denoted by f_T is defined as

$$f_T(t, \psi, \lambda) \equiv \frac{1}{T}\int_0^T f(t + s, \psi, \lambda)ds$$

where $T > 0$.

Remark 1 The above definitions of partial KBM-functionals and local partial averages are extensions of definitions of given in [9] for full (non-partial) averaging of FDE's. The term 'partial' is included in the definitions since applications of Definitions 1 and 3 in this paper will eventually replace λ with ϵt. Hence, time dependence has not been completely eliminated in the averaging process.

In addition to (1) and (2), consider the locally averaged FDE

$$\dot{y}(t) = \epsilon f_T(t, y_t, \epsilon t). \tag{4}$$

As usual, denote the solution of (4) with the initial function $y_{t_0} = \phi \in \mathcal{C}$ as $y(t) = y(t; t_0, \phi)$.

The goals of this paper are to derive conditions in which $|x(t; t_0, \phi) - z(t; t_0, \phi)| = Q(\epsilon)$ on time intervals of length $\mathcal{O}(1/\epsilon)$ and on infinite time intervals (when possible) where $Q(\epsilon) \to 0$ as $\epsilon \to 0$. This is accomplished by first showing $|x(t) - \bar{x}(t)| = \mathcal{O}(\epsilon T)$. Next, the moving and local average are shown to be $|\bar{x}(t) - y(t)| = \mathcal{O}(\epsilon T) + \mathcal{O}(\epsilon r)$. Then, as a final step, it is shown that $|y(t) - z(t)| = \mathcal{O}(\gamma(T))$, where $\gamma(T) \to 0$ as $\epsilon \to 0$. Hence, it is possible to select $T = 1/\sqrt{\epsilon}$ and prove the final results. In order to perform these steps, we will use the following lemmas. Their proofs are similar to the lemmas found in [9], and are therefore, omitted.

Lemma 1 *Assume that the solution to (1) satisfies $x(t) \in D$ for $t \in [t_0 - r, t_0 + L_1 + T]$ where $L_1 > 0$ and $T > 0$. Assume further that $|f(t, \psi, \lambda)| \leq M$ for all (t, ψ, λ) on $([t_0 - r, t_0 + L_1] \times \mathcal{C}(D))$. Then $\overline{x}(t) \in D$ and $|x(t) - \overline{x}(t)| \leq \epsilon MT/2 = \mathcal{O}(\epsilon T)$ for all $t \in [t_0 - r, t_0 + L_1]$.*

Lemma 2 *Assume that f is a partial KBM-functional and let K_1, K_2, and M be as given in Definition 1. Then $|f_T(t,\psi^1,\lambda_1) - f_T(t,\psi^2,\lambda_2)| \leq K_1\|\psi^1-\psi^2\| + K_2|\lambda_1-\lambda_2|$ and furthermore $|f_T(t,\psi^1,\lambda_1)| \leq M$ for all (t,ψ^i,λ_i) in $(\mathbb{R}\times\mathcal{C}(D)\times\mathbb{R})$, $i=1,2$.*

3 Partial Averaging Theorems

In this section we present partial averaging theorems for both finite and infinite time intervals. Lemma 3 represents the important foundation of the averaging results. In this lemma it is shown that solutions to the local partial averaged equation remain close to the moving average of $x(t)$. The proof of the lemma extends methods of [9] to the more complex case of partial averaging. Difficulties arise from the fact that $|f_T(t,\psi^1,t_1)-f_T(t,\psi^2,t_2)| \leq K_1\|\psi^1-\psi^2\|+K_2|t_1-t_2|$, implying that the right hand side of this inequality no longer is time dependent (here we suppose t_1 and t_2 are time instants). Because of this, bounds on time growth are utilized to prove Lemma 3. Once Lemma 3 is proven, it is straightforward to show that $y(t)$ and $z(t)$ remain close to each other. Hence, finite time interval averaging is proven.

To prove closeness on infinite time intervals, it is assumed that the partial averaged equation has an exponentially stable equilibrium point and that all initial functions lie in the domain of exponential stability. By properties of exponential stability, the solution to the partial averaged equation is known to uniformly approach the equilibrium point. From the previously proven averaging theorem, it is known that the solutions to the original time-varying FDE will remain close to the averaged solution on finite time intervals. In essence, $x(t)$ is being dragged closer and closer to the equilibrium point of the averaged system. When both the original and partial averaged solutions become very close to the equilibrium point, then $|x(t)-z(t)|$ is very small. Induction can then be used to prove these infinite time results.

Lemma 3 *Let the assumptions of Lemma 1 and Lemma 2 hold true for $L_1 = L/\epsilon$, where L and ϵ are positive constants. Assume that $x(t)=y(t)=\phi(t)$ on $[t_0-r,t_0]$, where $\phi\in\mathcal{C}(D)$, and assume that $y(t)\in D$ for all $t\in[t_0,t_0+L/\epsilon+T]$. Then $|y(t)-\overline{x}(t)| = \mathcal{O}(\epsilon T)+\mathcal{O}(\epsilon r)$ on $t\in[t_0-r,t_0+L/\epsilon]$.*

Proof: On $t\in[t_0-r,t_0)$, $|y(t)-\overline{x}(t)|=0$. For $t\geq t_0$,

$$|y(t)-\overline{x}(t)| = \left|y(t_0)+\epsilon\int_{t_0}^{t} f_T(s,y_s,\epsilon s)ds - \overline{x}(t)\right|.$$

Taking the derivative of $\overline{x}(t)$, we have for $t>t_0$

$$\begin{aligned}\dot{\overline{x}}(t) &= \frac{1}{T}[x(t+T)-x(t)]\\ &= \frac{\epsilon}{T}\int_t^{t+T} f(s,x_s,\epsilon s)ds\\ &= \frac{\epsilon}{T}\int_0^{T} f(t+\tau,x_{t+\tau},\epsilon(t+\tau))d\tau.\end{aligned}$$

Therefore, for $t\geq t_0$

$$\begin{aligned}|y(t) \quad - \quad \overline{x}(t)| = \Big|&y(t_0)-\overline{x}(t_0)+\epsilon\int_{t_0}^{t}[f_T(s,y_s,\epsilon s)\\ &-\frac{1}{T}\int_0^T f(s+\tau,x_{s+\tau},\epsilon(s+\tau))d\tau]ds\Big|\end{aligned}$$

We note that $\overline{x}(t)$ usually has a discontinuity at $t=t_0^-$: however, it has previously been shown in [9] that this does not affect analysis if we divide the interval of interest into two subintervals. Let $\delta>0$ be an arbitrarily small constant, and consider $|y(t)-\overline{x}(t)|$ on $t\in[t_0,t_0+r+\delta]$. By the definition of a partial KBM functional and by Lemma 2, for $t\in[t_0,t_0+r+\delta]$

$$\begin{aligned}|y(t) \quad - \quad \overline{x}(t)| &\leq |\overline{x}(t_0)-y(t_0)|\\ &+\epsilon\int_{t_0}^{t_0+r+\delta}\left(\frac{1}{T}\int_s^{s+T} M d\tau + M\right)ds.\end{aligned}$$

From Lemma 1 and the assumption that $y(t_0)=x(t_0)$, we have that $|y(t_0)-\overline{x}(t_0)|\leq\epsilon MT/2$. Therefore, for $t\in[t_0,t_0+r+\delta]$, we have $|y(t)-\overline{x}(t)|\leq\epsilon M(T/2+2r+2\delta)$.

Next, assume $L/\epsilon\geq r+\delta\equiv t_1$. On this interval we write

$$\begin{aligned}&|y(t)-\overline{x}(t)| \qquad (5)\\ &\leq |y(t_1)-\overline{x}(t_1)|+\epsilon\int_{t_1}^{t}|f_T(s,y_s,\epsilon s)-f_T(s,\overline{x}_s,\epsilon s)|ds\\ &+ \epsilon\int_{t_1}^{t}|f_T(s,\overline{x}_s,\epsilon s)-f_T(s,x_s,\epsilon s)|ds\\ &+ \epsilon\int_{t_1}^{t}|f_T(s,x_s,\epsilon s)-\\ &\qquad\frac{1}{T}\int_0^T f(s+\tau,x_{s+\tau},\epsilon(s+\tau))d\tau|ds.\end{aligned}$$

From above, we have that $|y(t_1)-\overline{x}(t_1)|\leq\epsilon M(T/2+2r+2\delta)$. By Lemma 2 and the assumption that x, $\overline{x}$ and y remain in D, we have $|f_T(s,\overline{x}_s,\epsilon s)-f_T(s,x_s,\epsilon s)|\leq K\epsilon MT/2$ and $|f_T(s,y_s,\epsilon s)-f_T(s,x_s,\epsilon s)|\leq K_1\|y_s-x_s\|$ for $t\in[t_0,t_0+L/\epsilon]$. Likewise, for $s\in[t_0,t_0+L/\epsilon]$ and $\tau\in[0,T]$

$$\begin{aligned}&\left|f_T(s,x_s,\epsilon s)-\frac{1}{T}\int_0^T f(s+\tau,x_{s+\tau},\epsilon(s+\tau))d\tau\right|\\ &= \frac{1}{T}\left|\int_0^T[f(s+\tau,x_s,\epsilon s)-f(s+\tau,x_{s+\tau},\epsilon(s+\tau))]d\tau\right|\end{aligned}$$

$$\leq \frac{1}{T}\int_0^T (K_1||x_s - x_{s+\tau}|| + K_2\tau)d\tau.$$

For $t \in [t_0, t_0 + L/\epsilon]$ and $\tau \in [0, T]$

$$||x_s - x_{s+\tau}|| = \epsilon \left\| \int_{t+\theta}^{t+\tau+\theta} f(s, x_s, \epsilon s)ds \right\| \leq \epsilon M\tau.$$

Using the above inequalities, for $t \in [t_0, t_0 + L/\epsilon]$, (5) becomes

$$\begin{aligned}
&|y(t) - \overline{x}(t)| \\
&\leq \epsilon M(T/2 + 2r + 2\delta) \\
&+ \epsilon K_1 \int_{t_1}^t ||y_s - \overline{x}_s||ds \\
&+ \epsilon^2 \int_{t_1}^t \left[K_1 MT/2 + \frac{1}{T}\int_0^T (K_1 M + K_2)\tau d\tau \right] ds \\
&\leq \epsilon M(T/2 + 2r + 2\delta) + \epsilon TL(K_1 M + K_2/2) \\
&\quad + \epsilon K_1 \int_{t_0}^t \sup_{\sigma \in [t_0, s]} |y(\sigma) - \overline{x}(\sigma)|ds.
\end{aligned}$$

The right-hand side of the above inequality is increasing, and therefore, for $t \in [t_0, t_0 + L/\epsilon]$

$$\begin{aligned}
&\sup_{s \in [t_0, t]} |y(s) - \overline{x}(s)| \\
&\leq \epsilon M(T/2 + 2r + 2\delta) + \epsilon TL(K_1 M + K_2/2) \\
&\quad + \epsilon K_1 \int_{t_0}^t \sup_{\sigma \in [t_0, s]} |y(\sigma) - \overline{x}(\sigma)|ds.
\end{aligned}$$

By Gronwall's inequality, this implies for $t \in (t_0, t_0 + L/\epsilon]$ that

$$\begin{aligned}
\sup_{s \in [t_0, t]} |y(s) - \overline{x}(s)| &\leq [\epsilon M(T/2 + 2r + 2\delta)] \\
&+ \epsilon TL(K_1 M + K_2/2)]\, exp\{\epsilon K(t - t_0)\}.
\end{aligned}$$

The constant δ is arbitrarily small (e.g. select $\delta = \epsilon r$), and therefore, the above inequality implies that $|y(t) - \overline{x}(t)| = \mathcal{O}(\epsilon T) + \mathcal{O}(\epsilon r)$ on $t \in [t_0, t_0 + L/\epsilon]$. □

Remark 2 If f is T−periodic in its first argument, i.e., $f(t + T, \cdot) = f(t, \cdot)$, then we have proven partial averaging. For the case when f does not admit periodicity, then it is necessary to relate solutions of the local partial averaged system to the partial averaged system, as given in Lemma 4 below.

Lemma 4 *Assume that f is a partial KBM-functional, and consider (2) and (4) with continuous initial function $z(t) = y(t) = \phi(t)$ on $t \in [t_0 - r, t_0]$. Assume that, for any $L > 0$ and $\epsilon > 0$, both $z(t)$ and $y(t)$ remain in D for $t \in [t_0 - r, t_0 + L/\epsilon]$. Then $|y(t) - z(t)| = \mathcal{O}(\gamma(T))$ on $t \in [t_0 - r, t_0 + L/\epsilon]$, where*

$$\gamma(T) \equiv \sup_{\psi \in \mathcal{C}(D),\ t \geq t_0,\ \lambda \geq 0} |f_T(t, \psi, \lambda) - F_{av}(\psi, \lambda)|.$$

Proof: Since z and y have the same initial functions, it is only necessary to consider $t_0 \leq t \leq t_0 + L/\epsilon$. On this time interval

$$\begin{aligned}
|y(t) - z(t)| &\leq \epsilon \int_{t_0}^t |f_T(s, z_s, \epsilon s) - F_{av}(z_s, \epsilon s)|\, ds \\
&\quad + \epsilon \int_{t_0}^t |f_T(s, y_s, \epsilon s) - f_T(s, z_s, \epsilon s)|\, ds \\
&\leq \epsilon\gamma(T)(L/\epsilon) + \epsilon K_1 \int_{t_0}^t ||y_s - z_s||ds.
\end{aligned}$$

By Gronwall's inequality and the arguments at the end of Lemma 3, this implies that $|y(t) - z(t)| \leq \gamma(T) Le^{K_1 L} = \mathcal{O}(\gamma(T))$ on $t \in [t_0, t_0 + L/\epsilon]$. □

Remark 3 Since f is assumed to be a partial KBM-functional, the function $\gamma(T) \to 0$ as $T \to \infty$. Using this fact, it is now possible to prove partial averaging for FDE's by letting $T = 1/\sqrt{\epsilon}$.

Theorem 1 (Finite Time Intervals) *Suppose that f is a partial KBM-functional and that (1), (2) and (4) have the same continuous initial function, $\phi \in \mathcal{C}(D)$, on $t \in [t_0 - r, t_0]$. Let $L > 0$ be a constant that is independent of ϵ, and define $\gamma(T)$ as in Lemma 4. Assume that $x(t)$ lies in D for $t \in [t_0 - r, t_0 + L/\epsilon + 1/\sqrt{\epsilon}]$, and that $y(t)$ and $z(t)$ lie in D for $t \in [t_0 - r, t_0 + L/\epsilon]$. Then $|x(t) - z(t)| = \mathcal{O}(Q(\epsilon))$ for all $t \in [t_0 - r, t_0 + L/\epsilon]$, where $Q(\epsilon) \to 0$ as $\epsilon \to 0$.*

Proof: From Lemmas 1, 2, and 4 for $t \in [t_0 - r, t_0 + L/\epsilon + T]$

$$\begin{aligned}
|x(t) - z(t)| &\leq |x(t) - \bar{x}(t)| + |\bar{x}(t) - y(t)| \\
&\quad + |y(t) - z(t)| \\
&\leq \mathcal{O}(\epsilon T) + \mathcal{O}(\epsilon T) + \mathcal{O}(\epsilon r) + \mathcal{O}(\gamma(T)).
\end{aligned}$$

Setting $T = 1/\sqrt{\epsilon}$ completes the proof. □

Remark 4 Theorem 1 shows that solutions to the partial averaged equation remain arbitrarily close to solutions of the original two time scale FDE on finite but arbitrarily large time intervals, provided that ϵ is sufficiently small. Theorem 2, which is presented next, provides additional conditions so that the interval can be extended to infinite time. Solo[12] refers to these types of averaging theorems as *hovering theorems*, since solutions "hover" around equilibria of the averaged system.

Theorem 2 (Hovering Theorem) *Assume the hypotheses of Theorem 1 are true for all $t \geq t_0 - r$. Let z_e be an exponentially stable equilibrium of (2), and* *let the initial function $\phi(t)$ lie in the domain of exponential stability of z_e, where $x(t) = z(t) = \phi(t)$ for $t \in [t_0 - r, t_0]$. Then*

$$\sup_{t \geq t_0} |x(t) - z(t)| = \mathcal{O}(Q(\epsilon)).$$

where $Q(\epsilon) \to 0$ as $\epsilon \to 0$, and is given in Theorem 1.

Proof: Given Theorem 1 and noting that the assumptions of Theorem 2 guarantee uniform decay of solutions of the partial averaged equation to its equilibrium point z_e, the proof follows from Theorem 2 of [9]. □

Suppose now, we are interested in the specific case when we have pointwise delays, i.e., consider (1) with

$$f(t, x_t, \epsilon t) = g(t, x(t), x(t-r), \epsilon t). \tag{6}$$

Then the averaged equation is given by (2) with

$$\begin{aligned} F_{av}(z_t, \epsilon t) &\equiv G_{av}(z(t), z(t-r), \epsilon t) \qquad (7) \\ &\equiv \lim_{T \to \infty} \frac{1}{T} \int_t^{t+T} g(s, z(t), z(t-r), \epsilon t) ds. \end{aligned}$$

We remark that we limit the discussion to systems with one time delay purely for convenience, and the results can easily be extended to systems with multiple time delays.

Shanholt has previously related the stability of classes of slowly time-varying linear FDE's to their frozen time eigenvalues [11]. Using this fact, it is possible to state the following theorem.

Theorem 3 (Point Delays) *Consider (1) with f given by (6) and F_{av} given by given by (7). Assume the hypotheses of Theorem 1 are true for all $t \geq t_0 - r$ and that $G_{av}(c_1, c_2, \lambda)$ has continuous Frechet derivatives with respect to its second and third arguments for all $(c_1, c_2, \lambda) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$. Suppose there is a point z_e such that $lim_{t \to \infty} z(t; t_0, \phi) = z_e$ and that for all $\lambda \geq 0$*

$$Det \left[sI - \frac{\partial G_{av}(z_e, z_e, \lambda)}{\partial z(t)} - \frac{\partial G_{av}(z_e, z_e, \lambda)}{\partial z(t-r)} e^{-rs} \right] = 0$$

has all solutions, $s = s(\lambda)$ with $Re\{s\} \leq -\alpha < 0$.

Then

$$\sup_{t \geq t_0} |x(t) - z(t)| = \mathcal{O}(Q(\epsilon)).$$

where $Q(\epsilon) \to 0$ as $\epsilon \to 0$, and is given in Theorem 2.

Proof: The conditions of the theorem guarantee that the frozen time eigenvalues of the linearization around z_e have negative real part for all time. Therefore, [11] guarantees that the linearization around z_e is exponentially stable. This implies that nonlinear system (2) with F_{av} given by (7) also has exponentially stable equilibrium point z_e. The proof now follows directly from Theorem 2 above. □

4 Example

Consider the nonautonomous scalar differential delay equation given by

$$\dot{x}(t) = \epsilon \left[-4\cos^2(t) x(t-r) + \frac{1}{2}(1 + cos(\epsilon t)) x(t) \right], \tag{8}$$

where $x(t) = \phi(t)$ on the interval $t \in [-r, 0]$. The partial averaged model (2) proposed in this paper is given as

$$\dot{z}(t) = \epsilon \left[-2z(t-r) + \frac{1}{2}(1 + cos(\epsilon t)) z(t) \right], \tag{9}$$

where $z(t) = \phi(t)$ on the interval $t \in [-r, 0]$. By Theorem 1, $|x(t) - z(t)| \leq Q(\epsilon)$ for $t \in [0, \frac{L}{\epsilon}]$ where $Q(\epsilon) \to 0$ as $\epsilon \to 0$. However, the zero equilibrium point of delay differential equation (9) is not always asymptotically stable. The characteristic equation of (9) is given by

$$s - \epsilon \left[-2e^{-rs} + \frac{1}{2}(1 + cos(\epsilon t)) \right] = 0.$$

Since $\frac{1}{2}[1 + cos(\epsilon t)] \leq 1$, the characteristic equation has solutions with negative real part when

$$\epsilon r < \frac{\pi}{3\sqrt{3}} \sim 0.6046. \tag{10}$$

The above condition will guarantee that $z = 0$ is exponentially stable for sufficiently small ϵ. Theorem 3 guarantees that for sufficiently small ϵ, $|x(t) - z(t)| \leq Q(\epsilon)$ for all $t \geq 0$.

This result can be checked numerically, and the results of two simulations are shown in Figures 1 and 2. Figure 1 shows the solutions to (8) and (9) for $\epsilon = 0.2$, $r = 2.5$, and $x(t) = z(t) = 1$ in $t \in [-r, 0]$. In this case, $\epsilon r < 0.6046$ and the solutions approach the origin as predicted. Figure 2 shows solutions to (8) and (9) for $\epsilon = 0.2$, $r = 3.5$, and $x(t) = z(t) = 1$ in $t \in [-r, 0]$. Now, $\epsilon r > 0.6046$ and the solutions wander away from the origin as predicted. Note that in both cases, the solution to (9) closely follows the solution of (8).

5 Conclusions

This paper extends the method of averaging to FDE's with both slow and fast time dependency. The method

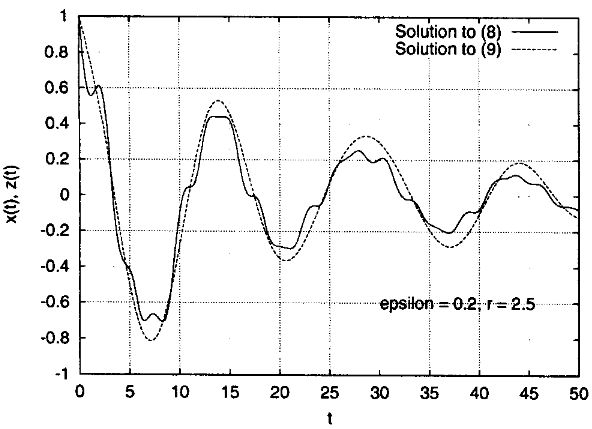

Figure 1: Solutions to (8) and (9) for $\epsilon = 0.2$ and $r = 2.5$.

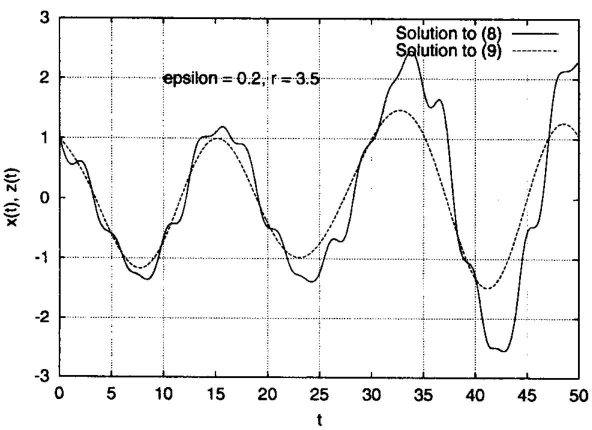

Figure 2: Solutions to (8) and (9) for $\epsilon = 0.2$ and $r = 3.5$.

of taking moving and local averages is used to relate solutions of the original FDE to its corresponding partial average. The partial averaged models are slowly time-varying. However, due to the results of [11] it is sometimes possible to treat the system as if it were time-invariant, by analyzing frozen time eigenvalues.

References

[1] V.I. Arnol'd. *Mathematical Methods of Classical Mechanics*, volume 60 of *Graduate Texts in Mathematics*. Springer Verlag, Berlin, 1989.

[2] J. Baillieul and B. Lehman. Open-loop control using oscillatory inputs. In W.S. Levine, editor, *The Control Handbook*, pages 967–980. CRC and IEEE Press, USA, 1996.

[3] V.I. Fodcuk. The method of averaging for differential difference equations of the neutral type. *Ukraine Mat. Z.*, 20:203–209, 1968.

[4] J. Guckenheimer and P. Holmes. *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields*, volume 42 of *Applied Mathematical Sciences*. Springer Verlag, Berlin, 1983.

[5] A. Halanay. The method of averaging in equations with retardation. *Rev. Mat. Pur. Appl. Acad. R.P.R.*, 4:467–483, 1959.

[6] A. Halanay. On the method of averaging for differential equations with retarded arguments. *J. Math. Anal. Appl.*, 14:70–76, 1966.

[7] J.K. Hale. Averaging methods for differential equations with retarded arguments with a small parameter. *J. Diff. Eq.*, 2:57–73, 1966.

[8] B. Lehman and S. Weibel. Averaging theory for delay difference equations with time-varying delays. *SIAM J. Appl. Math.*, 59 (4):1487–1506, 1999.

[9] B. Lehman and S. Weibel. Fundamental theorems of averaging for functional differential equations. *J. Diff. Eq.*, 152:160–190, 1999.

[10] G.N. Medvedev. Asymptotic solutions of some systems of differential equations with deviating argument. *Soviet Math Dokl.*, 9:85–87, 1968.

[11] G.A. Shanholt. Slowly varying linear functional differential equations. *IEEE Trans. Aut. Cont.*, AC-17:166–167, feb 1972.

[12] V. Solo and X. Kong. *Adaptive Signal Processing: Algorithms, Stability, and Performance*, volume 6 of *Prentice Hall Information and Systems Sciences Series*. Prentice Hall, Englewood Cliffs, NJ, 1995. Thomas Kailath, series editor.

[13] V. Strygin. The averaging principle for equations with heredity. *Ukranian Mathematics Journal (English translation)*, 22(4):430–439, 1971.

[14] V.M. Volosov, G.N. Medvedev, and B.I. Morgunov. Mr 32 #7904. *Vestnik Moskov: Univ. Ser. III. Fiz. Astronom*, 6:89, 1965.

[15] S. Weibel and J. Baillieul. Averaging and energy methods for robust open-loop control of mechanical systems. In J. Baillieul, S. Sastry, and H. Sussmann, editors, *Essays in Mathematical Robotics*, pages 203–270. Springer-Verlag, Berlin, 1998. To appear.

[16] S. Weibel, T.J. Kaper, and J. Baillieul. Global dynamics of a rapidly forced cart and pendulum. *Nonl. Dyn.*, 13:131–170, 1997.