Proceedings of the 32nd Conference
on Decision and Control
San Antonio, Texas • December 1993

STABILITY OF FAST ALMOST PERIODIC SYSTEMS WITH SPECIAL CLASSES OF TIME VARYING DELAY

Brad Lehman,[1] John R. Graef,[2] Deepak Sahay,[2] and Khalil Shujaee[1]

[1]Department of Electrical and Computer Engineering,
[2]Department of Mathematics and Statistics
Mississippi State University, Mississippi State, MS 39762

Abstract

This paper develops stability criterion for fast oscillating linear systems with time varying state delay. The results are compared to known results for constant delay and for zero delay.

1. Introduction

Recent papers [1, 2] have presented stability criteria for classes of the time varying dynamical system

$$x'(t) = [A + \tfrac{1}{\epsilon}D(t/\epsilon)]x(t) + Bx(t-r) \qquad (1)$$

where $x \in \mathbb{R}^n, A$ and B are constant $n \times n$ matrices, t is dimensionless time, $D(t/\epsilon)$ is a zero average periodic matrix with period of order $\epsilon > 0$, and r is a nonnegative constant.

In [1], it was shown that if $r = \epsilon b$, where b is a nonnegative constant, there exists an ϵ_0 such that for $0 < \epsilon \leq \epsilon_0$, the hyperbolic stability properties of (1) are identical to the hyperbolic stability properties of the o.d.e.

$$z'(t) = [\overline{A} + \overline{B}(b)]z(t) \qquad (2)$$

where

$$\overline{A} = \frac{1}{T}\int_0^T \Phi^{-1}(t)A\Phi(t)\,dt$$

$$\overline{B}(b) = \frac{1}{T}\int_0^T \Phi^{-1}(t)B\Phi(t-b)\,dt$$

and $\Phi(t)$ is a T-periodic fundamental solution of

$$y'(t) = D(t)y(t). \qquad (3)$$

These results show that if the delay in (1) is sufficiently small, i.e. $r = O(\epsilon)$, the hyperbolic stability properties of (1) correspond to the hyperbolic stability properties of a time invariant o.d.e. for sufficiently small ϵ. However, the delay still influences the spectrum of (2) via the elements of $\overline{B}(\cdot)$.

Later, in [2], the above results were extended to systems with arbitrarily large but bounded constant delay. In [2] it was shown that for a sufficiently small fixed $\epsilon = \epsilon_1$, the hyperbolic stability properties of (1) correspond to the hyperbolic stability properties of the time invariant delay differential equation

$$z'(t) = \overline{A}z(t) + \overline{B}(r/\epsilon_1)z(t-r) \qquad (4)$$

where $\overline{A}$, $\overline{B}(\cdot)$, and $\Phi(\cdot)$ are as defined in (2) and (3). Also, in [2] the authors show how these stability theorems can be used to synthesize vibrational controllers for general classes of time lag systems. Likewise, the results of [1] were used to develop the techniques of vibrational control for systems with small delay [3]. In fact, the stability of (1) was originally studied for the special case when $r = 0$ in [4]. Later, these results were used to design both vibrational and periodic controllers for finite dimensional systems [5].

It is the purpose of this paper to extend the stability results in [1, 2, 4] to classes of systems with time varying, bounded delay. Since the previous results in [1, 2, 4] have been useful in the design of vibrational and periodic controllers, it seems reasonable to expect that the results of this paper could also be used to design vibrational controllers for systems with time varying delay. In fact, we will show that the stability results in [1, 2, 4] are special cases of the results in this paper. In particular, this paper removes the restrictions in [1, 2] that the delays in a system must be precisely known and constant.

2. Preliminary lemmas

Consider an almost periodic system

$$y'(t) = F_0(t/\epsilon)y(t) + F_1(t/\epsilon)y(t-g(t)) \qquad (5)$$

where $y \in \mathbb{R}^n$, $F_i(\cdot)$, $i = 0, 1$, are almost periodic on $[0,\infty)$, $\epsilon > 0$, and $g(t)$ is a time varying delay satisfying $0 \leq g(t) \leq r$ for $t \in [0,\infty)$ and some nonnegative constant r. Define the averaged equation corresponding to (5) as

$$z'(t) = \overline{F}_0 z(t) + \overline{F}_1 z(t-g(t)) \qquad (6)$$

where

$$\overline{F}_i = \lim_{T\to\infty}\frac{1}{T}\int_0^T F_i(t)\,dt \qquad i = 0, 1.$$

Lemma 1: There exists a sufficiently small ϵ_0 such that for $0 < \epsilon \leq \epsilon_0$, the hyperbolic stability properties of (6) are identical to the hyperbolic stability properties of (5).

Proof: Noting that $y(t) = 0$ is the unique *a priori* known almost periodic solution of (5) in the neighborhood of $z(t) = 0$, the proof of the lemma directly follows from [6, Theorem 5.3].

Remark 1: Lemma 1 states the following: for sufficiently small ϵ, if $z = 0$ is a uniformly asymptotically stable (unstable) equilibrium point of (6), then $y = 0$ is a uniformly asymptotically stable (unstable) equilibrium point of (5). The value of ϵ_0 in Lemma 1 is best described by numerical simulations.

Lemma 2: Assume that $g(t)$ is a continuous bounded scalar function with $\lim_{t\to\infty} g(t) = M$, $M \geq 0$. Suppose further that $f(t)$ is a continuous scalar almost periodic function for all $t \in (-\infty, \infty)$. Then there exists some $t_1 > 0$ such that $f(t - g(t))$ is almost periodic for $t \geq t_1$.

Lemma 3: Let $f(t)$ be as in Lemma 2. Assume that $g(t)$ is continuous and almost periodic for $t \geq t_2$. Then $f(t - g(t))$ is almost periodic for $t \geq t_2$.

The proofs of Lemmas 2 and 3 are minor exercises using the properties of almost periodic functions (see [7]).

3. Main results

Consider the fast oscillating delay differential equation

$$x'(t) = [A + \tfrac{1}{\epsilon}D(t/\epsilon)]x(t) + Bx(t - g(t)) \qquad (7)$$

where x, A, $D(\cdot)$, and B are as in (1). We will always assume:

> *$g(t)$ is a continuous scalar function with $0 \leq g(t) \leq r$ for all $t \geq 0$ and for some $r \geq 0$. Further, $g(t)$ is either (i) almost periodic for $t \geq t_2$ or (ii) $\lim_{t\to\infty} g(t) = M < r$.*

The above statement will be referred to as hypothesis (H).

Remark 2: Clearly, not all time varying delays satisfy hypothesis (H). However, in many cases it may be more accurate to model a system with a delay that periodically drifts between certain values, or with a delay that eventually approaches a constant, than it is to model a system with purely constant delay.

Along with (7), consider the following autonomous delay differential equation

$$z'(t) = \overline{A}z(t) + \overline{C}(g, \epsilon_1)z(t - g(t)) \qquad (8)$$

where $\overline{A}$, $\Phi(\cdot)$, z, $g(\cdot)$ and t are as previously defined, $\epsilon_1 > 0$ is a fixed constant, and $\overline{C}(g, \epsilon_1)$ is given by

$$\overline{C}(g, \epsilon_1) = \lim_{T\to\infty} \frac{1}{T} \int_0^T \Phi^{-1}(\tfrac{t}{\epsilon_1})B\Phi(\tfrac{t-g(t)}{\epsilon_1})\, dt.$$

Theorem 1: Assume that $D(t)$ in (7) is a piecewise continuous almost periodic matrix with zero average on $t \in (-\infty, \infty)$. Suppose, also, that $g(\cdot)$ satisfies hypothesis (H). Then there exists an ϵ_0 sufficiently small such that for each fixed $\epsilon = \epsilon_1$, $0 < \epsilon_1 \leq \epsilon_0$, the trivial solution of (7) is:

(i) asymptotically stable if $z = 0$ in (8) is asymptotically stable;

(ii) if $z = 0$ in (8) is unstable.

Proof: Let $\Phi(t)$ be a fundamental solution to the differential equation

$$\zeta'(t) = D(t)\zeta(t). \qquad (9)$$

Since $D(\cdot)$ is piecewise continuous and almost periodic, $\Phi(t)$ will be both (uniformly) continuous and almost periodic. Introducing the Lyapunov transformation $x(t) = \Phi(t/\epsilon)y(t)$ into (7), we obtain

$$y'(t) = \Phi^{-1}(t/\epsilon)A\Phi(t/\epsilon)y(t) + \Phi^{-1}(t/\epsilon)B\Phi(\tfrac{t-g(t)}{\epsilon})y(t - g(t)). \qquad (10)$$

Since $\Phi(t)$ is a fundamental matrix, its inverse exists and will also be almost periodic (see [7, page 35]). Likewise, by Lemmas 2 and 3, there will always exist some $L \geq 0$ such that $\Phi(t - g(t))$ will be almost periodic for $t \geq L$. Hence, each matrix in (10) is almost periodic, and therefore, so are their products. This implies that Lemma 1 may be applied provided that the solution to (10) is bounded for $t \in [0, L]$. This is the case under the assumptions of the theorem since the right hand side of (10) is Lipschitz. Therefore, by Lemma 1, there will always exist an ϵ_0 sufficiently small such that for each fixed $\epsilon = \epsilon_1$, $0 < \epsilon_1 \leq \epsilon_0$, the hyperbolic stability properties of (10) are identical to the hyperbolic stability properties of (8). Q.E.D.

Remark 3: Although (8) is a time invariant delay differential equation, the coefficients of the matrix $\overline{C}(g, \epsilon_1)$ depend explicitly on ϵ_1. Therefore, it may be possible for (7) to have an asymptotically stable trivial solution for $\epsilon = \epsilon_1 > 0$, but an unstable trivial solution for $\epsilon = \epsilon_2 > 0$, even though both ϵ_1 and ϵ_2 are less than ϵ_0. This is in direct contrast to the no-delay case discussed in [4]. In [4], the matrix $B = 0$ and the corresponding average of (7) becomes

$$z'(t) = \overline{A}z(t).$$

It is shown that for any $0 < \epsilon \leq \epsilon_0$, if $\overline{A}$ is Hurwitz, then (7) (with $B = 0$) has an asymptotically stable trivial solution. Here, the frequency of vibration is permitted to vary over a range of ϵ, without affecting the hyperbolic stability properties of the averaged equation. For the delay case, ϵ must be fixed and cannot vary since for each different ϵ, the coefficients of $\overline{C}(g, \cdot)$ will change.

Remark 4: It is trivial to show that the stability results in [1,2,4] are special cases of Theorem 1.

Of course, the main restriction on the usefulness of Theorem 1 is hypothesis (H). In general, we can prove the following result.

Theorem 2: Let $D(t)$, $\Phi(t)$, and ϵ_1 be as in Theorem 1. Suppose that $g(t)$ in (7) satisfies $0 \leq g(t) < r$ and, for all $t \geq L \geq 0$,

$$\lim_{T\to\infty} \frac{1}{T} \int_t^{t+T} \Phi^{-1}(\tfrac{s}{\epsilon_1})B\Phi(\tfrac{s-g(s)}{\epsilon_1})\, ds$$

exists uniformly with respect to t. Then there exists an ϵ_0 sufficiently small such that for each fixed $\epsilon = \epsilon_1$, $0 < \epsilon_1 \le \epsilon_0$ the trivial solution of (7) is:

(i) asymptotically stable if $z = 0$ in (8) is asymptotically stable;

(ii) unstable if $z = 0$ in (8) is unstable.

Proof: The proof follows from the results in [6, Chapter 5] and [8, Chapter II].

Remark 5: To this date, the only delays $g(t)$ known to satisfy the conditions of Theorem 2 are those which satisfy hypothesis (H).

Example: Consider the delay differential equation

$$x'(t) = \left(\begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} + \begin{bmatrix} 0 & (\alpha/\epsilon)\cos(t/\epsilon) \\ 0 & 0 \end{bmatrix}\right) x(t) + \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix} x(t - g(t)), \quad (11)$$

and assume that $g(t)$ satisfies hypothesis (H). In this example a fundamental matrix is

$$\Phi(t) = \begin{bmatrix} 1 & \alpha \sin t \\ 0 & 1 \end{bmatrix},$$

and the equation corresponding to (10) is

$$y'(t) = \begin{bmatrix} -1 & \alpha \sin(t/\epsilon) \\ 0 & -2 \end{bmatrix} y(t) + \begin{bmatrix} \alpha \sin(t/\epsilon) & \alpha^2 \sin(\frac{t}{\epsilon}) \sin(\frac{t-g(t)}{\epsilon}) \\ -1 & -\alpha \sin(\frac{t-g(t)}{\epsilon}) \end{bmatrix} y(t - g(t)). \quad (12)$$

Therefore, the averaged equation corresponding to (8) is given by

$$z'(t) = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} z(t) + \begin{bmatrix} 0 & \alpha^2 \rho \\ -1 & 0 \end{bmatrix} z(t - g(t)) \quad (13)$$

where $\rho = \lim_{T\to\infty} \frac{1}{T} \int_0^T \sin(\frac{t}{\epsilon}) \sin(\frac{t-g(t)}{\epsilon})\, dt$, which is guaranteed to exist since $g(\cdot)$ satisfies hypothesis (H). By Theorem 1, for each sufficiently small fixed $\epsilon = \epsilon_1$, the hyperbolic stability properties of (13) are identical to those of (11). From (13) it is seen that as ϵ_1 changes, the coefficients in the matrices of $\overline{C}(g, \epsilon_1)$ change. Clearly, if $g(t) = 0$ (no-delay), this would not be the case.

For example, let $g(t) = 1 + \sin^2 t$ and $\epsilon = \epsilon_1 = 0.053$. Then, computer simulations show that $\rho = 0.091385$ and that the trivial solution of (13) is asymptotically stable when $0 \le \alpha \le 5.4$ and unstable for $\alpha > 5.4$. If ϵ is changed to $\epsilon = \epsilon_2 = 0.05$, then $\rho = -0.01896$ and the trivial solution is asymptotically stable for $0 \le \alpha \le 10.3$ and unstable for $\alpha > 10.3$. Even though there was only a small change between ϵ_1 and ϵ_2, the range of α in which the trivial solution of (13) is asymptotically stable almost doubled.

It is tempting to think that (13) will behave similar to a system with constant delay equal to the average value of $g(t)$. This, however, is not the case. For example, let $g(t) = r$ in (13), where $r = \overline{1 + \sin^2 t} = 1.5$. Then, for $\epsilon = \epsilon_1 = 0.053$, the trivial solution of (13) is asymptotically stable for $0 \le \alpha \le 2.1$ and unstable for $\alpha > 2.1$. Similarly, for $\epsilon = \epsilon_2 = 0.05$ the trivial solution of (13) is asymptotically stable for $0 \le \alpha \le 5.8$ and unstable for $\alpha > 5.8$. These are quite different from the ranges of α found when $g(t) = 1 + \sin^2 t$. This suggests that if a delay is varying between certain bounds, as is often the case, it may be inaccurate to model the system with a constant delay.

4. Conclusions

This paper presents constructive tools for the stability analysis of fast almost periodic linear differential equations with classes of time varying state delay. It is shown that stability properties of these systems are sensitive to both the frequency of vibration and the time varying delay.

REFERENCES

[1] J. Bentsman, J. Fakhfakh, H. Hvostov, and B. Lehman, "Stability of fast periodic systems with time lags," *IEEE Trans. Automat. Contr.*, vol. AC-34, pp. 462-465, Apr. 1989.

[2] B. Lehman and J. Bentsman, "Vibrational control of linear time lag systems with arbitrarily large but bounded delays," *IEEE Trans. Automat. Contr.*, vol. 37 AC-10, pp. 1576-1582, Oct. 1992.

[3] J. Bentsman, J. Fakhfakh, and B. Lehman, "Vibrational stabilization of linear time delay systems and its robustness with respect to the delay size," *Syst. Contr. Lett.*, vol. 12, pp. 267-272, 1989.

[4] R. Bellman, J. Bentsman, and S.M. Meerkov, "Stability of fast periodic systems," *IEEE Trans. Automat. Contr.*, vol. AC-30, pp. 289-291, Mar. 1985.

[5] S. Lee, S. M. Meerkov, and T. Runolfsson, "Vibrational feedback control: Zeroes placement capabilities," *IEEE Trans. Automat. Contr.*, vol. AC-32, pp. 717-724, Aug. 1986.

[6] J. K. Hale, S. M. Verduyn Lunel, "Averaging in infinite dimensions," *J. Integ. Eq. Appl.*, vol. 2 no. 4, pp. 463-493, Fall 1990.

[7] H. Bohr, *Almost Periodic Functions.* New York: Chelsea Publishing Co., 1951.

[8] B. Lehman, J. Bentsman, S. M. Verduyn-Lunel and E. I. Verriest, "Vibrational control of nonlinear time lag systems with arbitrarily large but bounded delays: Averaging theory, stabilizability, and transient behavior," *Proc. IEEE 31st CDC*, vol. 2, pp. 1287-1294, 1992.