Proceedings of the 35th
Conference on Decision and Control
Kobe, Japan • December 1996

RECENT ADVANCES IN AVERAGING THEORY FOR PWM DC–DC CONVERTERS

Brad Lehman[1]
Department of Electrical and Computer Engineering
Northeastern University
Boston, MA 02115

Richard M. Bass
School of Electrical & Computer Engineering
Georgia Institute of Technology
Atlanta, GA 30332–0250

Abstract – – This paper presents a recent extension of averaging theory for power electronic systems to include feedback controlled converters. A ripple estimate is presented. New switching frequency dependent averaged models for PWM dc–dc converters are introduced. Two important benefits of these new averaged models are the correction of dc offset error in steady–state, and the modeling of switching frequency effects on closed loop performance and stability.

I. BACKGROUND

Averaging techniques provide the analytical foundation for most power electronic design procedures of the system level. These techniques are widely used in the analysis and control design of pulse width modulated (PWM) power electronic systems [1]. It was not until recently, however, that rigorous mathematical justification [2,3,4] has been given to theoretically explain the application of these averaging techniques.

In [2], classical Russian averaging methods [5] are shown to be applicable to open–loop PWM dc–dc converters. However, the results of [2] are limited to systems with time discontinuity. In fact, classical averaging theory is not applicable when there are state discontinuities. This is significant because all feedback controlled converters are state discontinuous systems. (See [6] for explanation of time and state discontinuity.)

This paper summarizes recent advances in averaging theory using formal mathematical methods for periodic differential equations. In Section II, the results of [2] are extended to include feedback controlled converters. Section III then makes advances on the techniques of [2] by introducing improved averaged models which are frequency dependent.

II. ADVANCES IN THEORY

The difficulty in mathematically justifying averaging approximation techniques for state discontinuous systems can be best explained by considering the following example,

$$\frac{dx}{dt} = f(x) + bu(d(x) - tri[t,T]) \qquad (1)$$

where $x \in \Re^n, b \in \Re^n, f: \Re^n \rightarrow$ and $d: \Re^n \rightarrow [0,1]$ are both continuous functions, and $u(\cdot)$ is the Heaviside unit step function, i.e. $u(s) = 1$ for $x \geq 0$ and $u(s) = 0$ for $s < 0$. The function $tri[t,T] = t/T - floor(t,T)$. The PWM feedback controlled buck converter [2] is typically described by an equation in the form of (1).

The right–hand side of (1) is not Lipschitz because it is discontinuous with respect to x. In general, "standard" solutions to (1) are known to exist, under the conditions that there only a finite number of jumps in the right–hand side of (1) on any finite time interval, and that each jump is norm bounded. Under these conditions, the solution of the integral equation,

$$x(t;t_0,x(t_0)) = x(t) \equiv x(t_0) + \int_{t_0}^{t} [f(x(s)) + bu(d(x(s)) - tri[s,T])ds, \qquad (2)$$

is unique and satisfies the state differential equation (1) almost everywhere. The underlying assumption is that the mathematical models do not exhibit chattering (switching infinitely often, i.e., power electronic switches turn on and off only once each PWM switching period). Conditions for avoiding chattering are discussed in [7]. Under the assumption of no chattering, this paper applies new averaging results [8] to the integral equation (2).

A. Averaging Theorems

In general form, feedback controlled PWM dc–dc converters in continuous conduction mode can be modeled by the differential equation,

$$\frac{dx}{dt} = A_0x + b_0 + [A_1x + b_1]u(d(x) - tri(t,T)) . \quad (3)$$

where $x \in \Re^n$, $d(x) = V_{ref} - [k_1, k_2, ...k_n]x = V_{ref} - Kx$, and $tri(\cdot,\cdot)$ and $u(\cdot)$ are as previously defined. Along with (2), consider the corresponding averaged equation

$$\frac{dy}{dt} = A_0y + b_0 + [A_1y + b_1]d(y) \qquad (4)$$

where $y \in \Re^n$. Under the assumption of no chattering, the following theorem relates solutions of the averaged system (4) to the original system (3).

THEOREM 1: *Let $x(t)$ and $y(t)$ denote solutions to (3) and (4) respectively. For any constant $L > 0$ and for any constant $\eta > 0$, there exists a $T_0 = T_0(\eta, L) > 0$ and a constant $K > 0$ such that for $0 < T \leq T_0$,*

$$\| x(t) - y(t) \| \leq (\| x(0) - y(0) \| + \eta) exp\{Kt\} \qquad (5)$$

for all $t \in [0, L]$.

[1]Work Supported by an NSF PFF, CMS9596268.

0-7803-3590-2/96 $5.00 © 1996 IEEE

When $x(0) = y(0)$, Theorem 1 guarantees that there will always exist a sufficiently small switching period such that for any $\eta > 0$, however small, $\| x(t) - y(t) \| < \eta$ on any finite time interval. This bound is true, even when (3) or (4) is unstable. The extension to the infinite time interval is given in the following theorem.

THEOREM 2: *Let $x(t)$ and $y(t)$ denote solutions to (3) and (4), respectively. Suppose that y_s is a uniformly asymptotically stable equilibrium point, and that $y(t) \to y_s$ as $t \to \infty$. Then there exist constants $\beta_0(\eta)$ and $T_0(\eta)$ such that, for any $\eta > 0$, any $\| x(0) - y(0) \| < \beta$, $0 \le \beta < \beta_0 < \eta$ and any $0 < T \le T_0$,*

$$\| x(t) - y(t) \| \le \eta \tag{6}$$

for all $t \ge 0$.

The proofs of Theorems 1 and 2 are lengthy and therefore are omitted ([9]).

III. ADVANCES IN PRACTICE

In this section, the application of the averaging theory presented in the previous section is shown to provide two practical benefits: (1) periodic ripple functions which provide a post–simulation estimate of ripple waveforms and (2) new averaged models which more accurately track the one–cycle average value of state variables.

A. Ripple Estimate

It is often desirable to obtain an estimate on the ripple of the system, which will be denoted in this paper as $\psi(t, T, \cdot)$. Practical applications of averaging tell us that a better approximation of the solution to (3) will be given by

$$x(t) = y(t) + \psi(t, T, y(t)), \tag{7}$$

where $x(t)$ and $y(t)$ are the solutions of (3) and (4) respectively, T is the switching period, and $\psi(t, T, \cdot)$ is the ripple estimate which is obtained by the following algorithm:

Consider only the right–hand sides (R.H.S.) of (3) and (4). Let $x(0) = y(0)$, and replace every $x(t)$ and $y(t)$ in the R.H.S. of (3) and (4) by the constant $c \in \Re^n$. Now take the difference between the R.H.S. of (3) and (4) and integrate to obtain

$$\Gamma(t, T, c) = [A_1 c + b_1] \int_t [u(d_1(c) - tri(t, T)) - d_1(c)] dt, \tag{8}$$

where $\int_t h(t) dt$ denotes the indefinite integral or primitive of $h(t)$. The ripple estimate is given as

$$\psi(t, c) = \Gamma(t, T, c) - \frac{1}{T} \int_0^T \Gamma(t, T, c) dt. \tag{9}$$

Replacing c by $y(t)$ yields $\psi(t, T, \cdot)$.

Performing integrations (8) and (9), using (3) and (4), an estimate on the ripple is computed to be

$$\psi(t, T, y(t)) = TA_1 y(t) \{ [u(d(y(t)) - tri(t, T)) - d(y(t))] tri(t, T) + [1 - u(d(y(t)) - tri(t, T))] \cdot d(y(t)) + \frac{1}{2} d(y(t)) [d(y(t)) - 1] . \tag{10}$$

As the switching period becomes smaller, the amplitude of $\psi(t, T, \cdot)$ will also become smaller, and ripple of the system will almost become negligible. Additionally, an adjustment on the initial condition can be made by solving the equation $x(0) = y(0) + \psi(t, T, y(0))$, for $y(0)$ in terms of $x(0)$.

<u>Example 1</u>

Consider the dc–dc boost converter with PWM feedback control shown in Fig. 1. The equation for this converter is of the form of (3) with $b_0 = [E/L \quad 0]^T$, $b_1 = 0$,

$$A_0 = \begin{bmatrix} 0 & \frac{1}{L} \\ \frac{1}{C} & -\frac{1}{RC} \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & \frac{1}{L} \\ \frac{1}{C} & 0 \end{bmatrix}.$$

For this example, the component values are $E = 5V$, $L = 50\ \mu H$, $C = 4.4\ \mu F$, and $R = 28\ \Omega$. The controller parameters are $V_{ref} = 0.13\ V$, $k_1 = 0.174$, and $k_2 = -0.0435$.

Figure 2 shows capacitor voltage and inductor current of the original system for 100 kHz switching. A comparison of these plots can be made with Fig. 3, which shows the improvement of the averaging technique using the new ripple estimate $x(t) = y(t) + \psi(t, T, y(t))$.

B. Switching Frequency Dependent Averaged Models

In order to illustrate limitations of the conventional averaged model [1] given by (4), consider the simulated start–up transient for Example 1 shown in Fig. 4.

The output voltage transient $v_C(t)$ for the averaged model is shown together with switching transients for several switching frequencies: $f_s = 1/T = 50$ *KHz*, *100 KHz* and *1 MHz*. At the slower switching frequencies, the averaged model fails to accurately capture the average value in steady–state. This is a practical concern in applications where semiconductor device capabilities constrain the controller to operate at slower switching frequencies.

<u>Example 2</u>

The same PWM controlled converter will now be used to illustrate the effect of switching frequency on closed loop stability. For this example, the component values are $E = 4V$, $L = 5.2381\ \mu H$, $C = 0.2\ \mu F$, and $R = 16\ \Omega$. The controller parameters are $V_{ref} = 0.48\ V$, $K_1 = 0.1$, and $K_2 = -0.01$.

A simulated transient is given in Fig. 5 for the averaged and switching model for f_s = 500 KHz and 1 MHz. Note that the closed loop system is stable for f_s = 1 MHz, but unstable for f_s = 500 KHz. Additional simulations confirm the closed loop system is stable for frequencies greater than 500 KHz, and unstable for lower frequencies.

The conventional averaged model [1], which is independent of switching frequency, predicts that the closed loop system is stable. The assumption is that the switching frequency is "fast enough". The practical concern here is that the conventional averaged model gives no precise insight or guidance as to "how fast" the switching frequency needs to be for acceptable closed loop performance. As a result, transient performance is sacrificed.

Modified Averaging Technique

Equation (3) is placed in "standard form", by converting to slow time, $\tau = t/T$

$$\frac{dx}{d\tau} = T[A_0x + b_0 + [A_1x + b_1]u(d(x) - tri(\tau, 1))], \qquad (11)$$

where the perturbation parameter is seen to be the switching period $T = 1/f_s$. Using the "near identity" periodic transformation $x = z + T\varphi(\tau, z)$, (1) becomes (neglecting $O(T^2)$ terms)

$$\frac{dz}{d\tau} = T[A_0(z + T\psi) + b_0 + [A_1(z + T\psi) + b_1]u(d(z + T\psi) - tri(\tau, 1)) - \frac{\partial\psi}{\partial\tau}], \qquad (12)$$

where φ is a periodic function which gives the ripple in the state variables (10) in rescaled time. It can be shown that a more accurate averaged model is obtained by simultaneous solution of the two equations

$$g(y, T) = \int_0^1 [A_0(y + T\psi) + [A_1(y + T\psi) + b_1]u(d(y + T\psi) - tri(\tau, 1))]d\tau . \qquad (13)$$

$$\psi(t, y, T) = \int [A_0(y + T\psi) + b_0 + [A_1(y + T\psi) + b_1]u(d(y + T\psi) - tri(\tau, 1)) - g(y, T)]d\tau - h(y, T) \qquad (14)$$

for g and Ψ, where h is chosen so that Ψ has zero average. In order to obtain the new model for PWM feedback controlled dc–dc converters, it will be necessary to estimate solutions of (13,14). Because of special properties of the step function, this is not too difficult and it can be shown that a new averaged model, in fast time, is

$$\frac{dy}{dt} = A_0y + b_0 + [A_1y + b_1]t_s, \qquad (15)$$

where t_s is solved for via the second order polymonial

$$d(y) = t_s + \frac{T}{2}(t_s - t_s^2)K[A_1y + b_1] . \qquad (16)$$

A detailed derivation of this model is lengthy and will be presented in the journal version of this paper [10].

Returning to Example 1. Fig. 6 shows a simulated start–up transient for the original switching system (3) together with the start–up transient for the conventional averaged model (4) and the new averaged model (15,16) for f_s = *100 KHz*. The new averaged model accurately captures the one–cycle average, while the conventional model exhibits a significant offset error.

Returning to Example 2. Fig. 7 shows a simulated transient for the original system (3) together with the conventional (4) and new (15,16) averaged models for f_s = 1 MHz. Note that the new model provides a much more accurate one–cycle average than does the conventional model.

Fig. 8 shows the transients for the new (15,16) and switching (3) models for two different switching frequencies. Additional simulations confirm that the new averaged model accurately predicts the critical switching frequency for closed–loop stability. The discrepancy between steady–state values in the unstable case is because the averaged model permits duty ratios less than zero. If duty ratio saturation were modeled in (16), the two waveforms would coincide in steady–state. *This simulation result demonstrates that the new averaged model captures the control loop instability at low switching frequencies.* The conventional model transient fails to exhibit this phenomenon, since it only models the behavior in the limit as $f_s \rightarrow \infty$.

IV. CONCLUSIONS

A rigorous averaging theory for power electronic systems has been presented. This new theory extends previous work to include state discontinuous (feedback controlled) PWM systems. The two theorems presented in this paper provide a basis for answering fundamental questions about the averaged model and its relation to the original switching model. First order ripple estimates are computed for a feedback controlled boost converter, and a new averaged model for PWM dc–dc converters is presented. The new averaged model is switching frequency dependent. Simulated transients demonstrate the improvement over the conventional approach. Two benefits of the new averaged model are the correction of dc offset error and the modeling of switching frequency effects on closed–loop stability and performance.

REFERENCES

[1] R.D. Middlebrook and S. Cuk, "A general unified approach to modeling switching–converter power stages," *IEEE Power Electronics Specialists Conf. Rec.*, pp. 18–34, 1976.

[2] P.T. Krein, J. Bentsman, R.M. Bass, and B.C. Lesieutre, "On the use of averaging for the analysis of power electronic systems," *IEEE Transactions on Power Electronics*, vol. 5, pp. 182–190, April 1990.

[3] H. Sira–Ramirez, "A geometric approach to pulse–width modulated systems," *IEEE Trans. Autom. Contr.*, vol. 34, no. 2, pp. 184–187, Feb. 1989.

[4] J. Sun and H. Grotstollen, "Averaged modelling of switching power converters: reformulation and theoretical basis," *IEEE Power Electronics Specialists Conf. Rec.*, pp. 1165–1172, 1992.

[5] N.N. Bogoliubov and Y.A. Mitropolsky, *Asymptotic Methods in the Theory of Nonlinear Oscillations*. NY: Gordon and Breach Science Pub., 1961.

[6] A.F. Filippov, *Differential Equations with Discontinuous Right–hand Sides*. The Netherlands: Kluwer Acad. Pub., 1988.

[7] R.M. Bass and P.T. Krein, "Large–signal design alternatives for switching power onverter control" *IEEE Power Electronics Specialists Conf. Rec.*, pp. 882–887, 1991.

[8] B. Lehman, "Averaging of differential delay equations," *Proc. 1992 IEEE ACC*, vol. 2, pp. 1955–1956, 1992.

[9] B. Lehman and R.M. Bass, "Extensions of averaging theor for power electronic systems" submitted to *IEEE Transactions on Power Electronics*, 1993.

[10] B. Lehman and R.M. Bass, "Switching frequency dependent averaged models for PWM dc–dc converters," submitted to *IEEE Transactions on Power Electronics*, 1994.

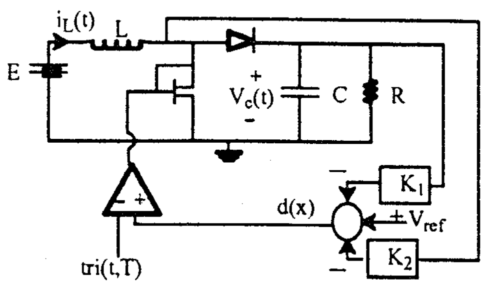

FIGURE 1. Boost converter with PWM feedback

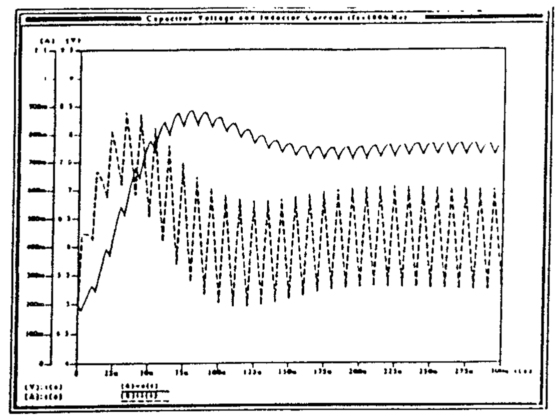

FIGURE 2. Simulated start–up transient response of both capacitor voltage and inductor current for (3) when switching frequency equals 100 KHz.

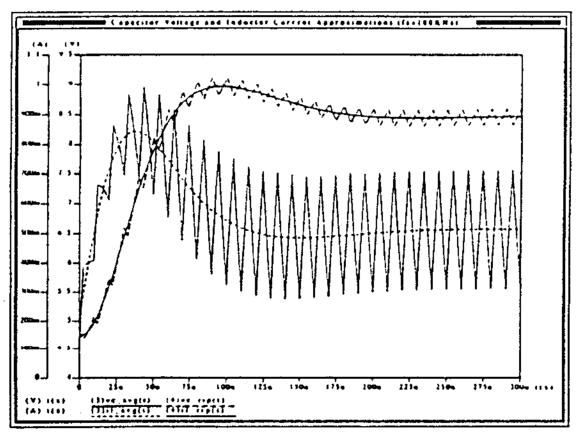

FIGURE 3. Simulated start–up transient response of both capacitor voltage and inductor current for (4) and with ripple correction when switching frequency equals 100 KHz.

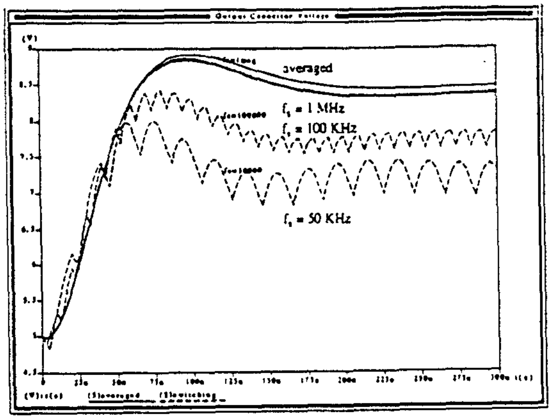

FIGURE 4. Output voltage of simulated start–up transient for averaged and switching models. Example 1.

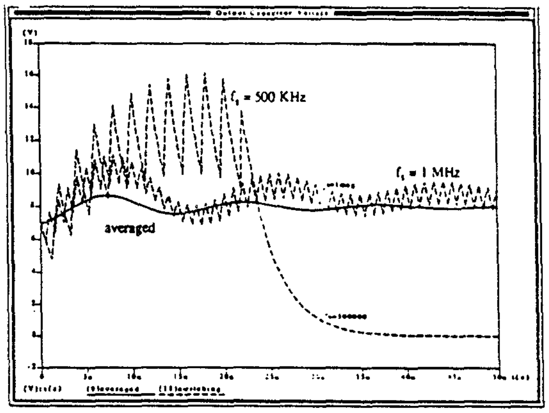

FIGURE 5. Output voltage of simulated start–up transient for averaged and switching models. Example 2.

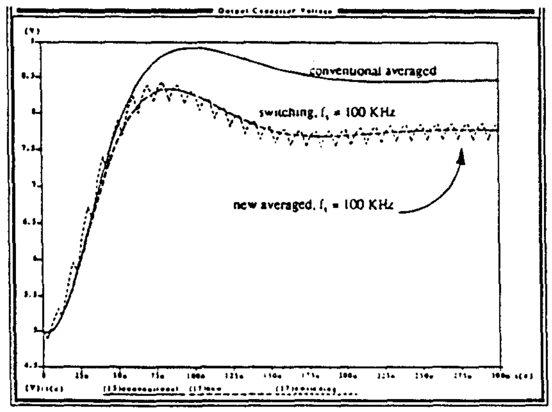

FIGURE 6. Output voltage of simulated start–up transient for conventional, new and switching models.

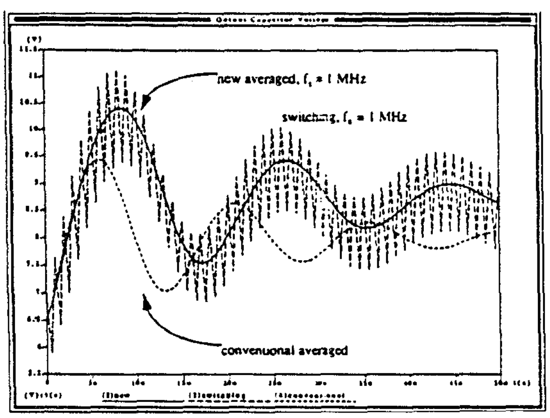

FIGURE 7. Output voltage of simulated start–up transient for conventional, new and switching models.

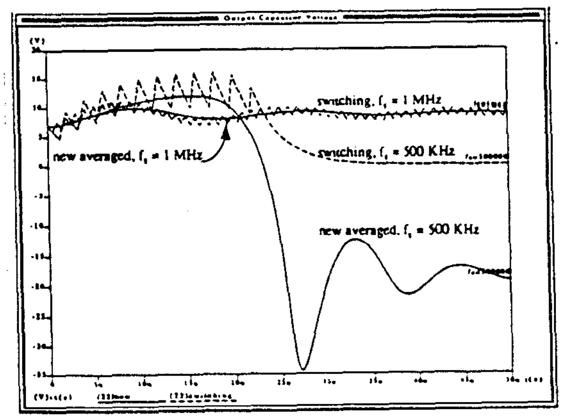

FIGURE 8. Output voltage of simulated start–up transient for, new and switching models.