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Fundamental Theorems of Averaging for Functional Differential Equations

Brad Lehman[1] and Steven P. Weibel
Department of Electrical and Computer Engineering
Northeastern University; Boston, MA 02115

Abstract

In this paper, we improve on classical averaging theorems for functional differential equations by proposing new averaged models.

1 Introduction

In the 1960's, authors such as [4], [5], [6], [17], [3], and [13] applied the method of averaging to time-varying functional differential equations (FDE's) that admit a small parameter. The most general of these results is given in [6], where the FDE

$$\dot{x}(t) = \epsilon f(t, x_t) \tag{1}$$

is considered. (For an explanation of this standard notation, see Section 2.)

Suppose that, for any constant vector c, we define $\tilde{c}(\theta) = c$, $\theta \in [-r, 0]$. Then the work of [6] (as well as [4], [5], [17], [3], and [13]) gives conditions under which solutions of (1) can be approximated by the averaged autonomous ODE

$$\dot{\xi}(t) = \epsilon F_{av}(\tilde{\xi}); \quad \tilde{\xi}(\theta) = \xi(t), \quad \theta \in [-r, 0] \tag{2}$$

where

$$F_{av}(\psi) \equiv \lim_{T\to\infty} \frac{1}{T}\int_t^{t+T} f(s, \psi)ds. \tag{3}$$

Additionally, authors have also presented results on averaging of FDE's in the form of (1) with pointwise delays [5], [17], [3], and [13]. If one carefully examines the proofs in these FDE averaging papers, it can be seen that each method proposes two important upper bounds on ϵ. First, there exists some ϵ_1, sufficiently small, such that for $0 \le \epsilon \le \epsilon_1$, the FDE can be approximated by a nonautonomous ODE. Next, there exists an upper bound on ϵ_2 such that for $0 \le \epsilon \le \epsilon_2$, the time dependence can be averaged out. Since averaging theorems are written for "sufficiently small ϵ," the averaging proofs in [4], [5], [6], [17], [3], and [13] do not distinguish between ϵ_1 and ϵ_2. However, if ϵ is not infinitesimely small, this could lead to errors in the averaging approximation.

In this paper we propose to approximate (1) by the alternate averaged system

$$\dot{z}(t) = \epsilon F_{av}(z_t) \tag{4}$$

where F_{av} is given in (3). Notice that (4) is an FDE and not an ODE. No attempt will be made to search for the upper bound on ϵ allowing for a finite dimensional approximation of (1). Numerical simulations indicate that (4) is usually a more accurate appoximation of (1) than (2). This seems to imply that $\epsilon_1 < \epsilon_2$.

It should be mentioned that there has been a recent interest (see [7], [9], [12] and [10]) in studying averaging for FDE's in the form of $\dot{x}(t) = f(t/\epsilon, x_t)$. This FDE has found engineering applications in vibrational control [9], [12] and periodic control design [16]. The work of [10] was the first to suggest that (4) may be an improved averaged approximaion of (1); however, [10] views averaging from a Lyapunov stability point of view and considers only point delays. In this paper, we consider general classes of FDE's and present fundamental averaging results pertaining to the closeness of solutions of (1) and (4) over specified time intervals.

2 Preliminaries

Let $\Re^n$ be the $n-$dimensional Euclidean space. Let $\mathcal{C} = \mathcal{C}([-r, 0], \Re^n)$, $r \ge 0$, denote the space of continuous functions that map $[-r, 0]$ into $\Re^n$. If $x(t)$ is a continuous function defined on $[t_0 - r, L]$, then we define $x_t \in \mathcal{C}$ by setting $x_t(\theta) = x(t + \theta)$, $\theta \in [-r, 0]$ for each $t_0 \le t \le L$, where $L > t_0$. For each $\psi \in \mathcal{C}$, let $||\psi||$ denote $sup\{|\psi(\theta)| : \theta \in [-r, 0]\}$, where $|\cdot|$ is a norm of $\Re^n$. For any $D \subset \Re^n$, let $\mathcal{C}(D) = \mathcal{C}([-r, 0], D)$. The functional $f : \Re \times \mathcal{C} \to \Re^n$ is always assumed to be continuous. Let $\phi(t)$ be a continuous function on $t \in [t_0 - r, t_0]$, and assume in (1) that $x(t) = \phi(t)$ on this interval. Then (1) has a solution which is denoted as $x(t) = x(t; t_0, \phi)$. (We also sometimes write $x_{t_0} = \phi_{t_0} = \phi \in \mathcal{C}$, in a standard mild abuse of notation.) Likewise, the solution of (4) is denoted as $z(t) = z(t; t_0, \phi)$ for $z_{t_0} = \phi$. All derivatives are assumed to be right-hand derivatives.

Definition 1 Suppose that $f : \Re \times \mathcal{C} \to \Re^n$ is continuous and is uniformly bounded such that $|f(t, \psi)| \le M$ for all (t, ψ) on $\Re \times \mathcal{C}(D)$. Assume further that f is locally Lipschitz, i.e., for any (t, ψ^1, ψ^2) in $(\Re \times \mathcal{C}(D) \times \mathcal{C}(D))$ there exists a $K > 0$ such that $|f(t, \psi^1) - f(t, \psi^2)| \le K||\psi^1 - \psi^2||$. Furthermore, suppose that the average in (3) exist uniformly in t for all $\psi \in \mathcal{C}(D)$. Then f is said to be a **KBM-functional.**

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Definition 2 Suppose that $x(t) = x(t; t_0, \phi)$ is the solution to (1) with initial function $\phi \in \mathcal{C}$. The **moving average** of $x(t)$ is denoted by $\overline{x}(t)$ and is defined as

$$\overline{x}(t) \equiv \begin{cases} \phi(t), & \text{for } t \in [t_0 - r, t_0) \\ \frac{1}{T}\int_t^{t+T} x(s)ds, & \text{for } t \geq t_0, \end{cases}$$

where $T > 0$.

Definition 3
Consider a functional $f : \Re \times \mathcal{C}([-r, 0], \Re^p) \to \Re^n$. The **local average** of f, denoted by f_T is defined by

$$f_T(t, \psi) \equiv \frac{1}{T}\int_0^T f(t+s, \psi)ds$$

where $T > 0$ and p is a non-negative integer.

Remark 1 The notion of a KBM-functional is an extension of the definition of a classical KBM-vectorfield for ODE's (Krylov-Bogolyubov-Mitropolsky as discussed in [8] and [1]). The use of moving averages and local averages for ODE's has been introduced in [14] and [15], respectfully.

Now in addition to (1) and (4), consider the locally averaged FDE

$$\dot{y}(t) = \epsilon f_T(t, y_t). \tag{5}$$

As usual, denote the solution of (5) with the initial function $y_{t_0} = \phi \in \mathcal{C}$ as $y(t) = y(t; t_0, \phi)$.

The goal of this paper is to derive conditions in which $|x(t; t_0, \phi) - z(t; t_0, \phi)| = \beta(\epsilon)$ on time intervals of length $O(1/\epsilon)$, where $\beta(\epsilon) \to 0$ as $\epsilon \to 0$. This is accomplished by first showing $|x(t) - \overline{x}(t)| = O(\epsilon T)$. Next, the moving and local average are shown to be $|\overline{x}(t) - y(t)| = O(\epsilon T) + O(\epsilon r)$. Then, as a final step, it is shown that $|y(t) - z(t)| = O(\frac{B(\epsilon)}{\epsilon T})$, where $B(\epsilon) \to 0$ as $\epsilon \to 0$. Hence, it is possible to select $\epsilon T = \sqrt{B(\epsilon)}$ and prove the final results. In order to achieve these steps, we will use the following lemmas.

Lemma 1 *[11] Assume that the solution to (1) satisfies $x(t) \in D$ for $t \in [t_0 - r, t_0 + L_1 + T]$ where $L_1 > 0$ and $T > 0$. Assume further that $|f(t, \psi)| \leq M$ for all (t, ψ) on $([t_0 - r, t_0 + L_1] \times \mathcal{C}(D))$. Then $|x(t) - \overline{x}(t)| \leq \epsilon MT/2 = O(\epsilon T)$ for all $t \in [t_0 - r, t_0 + L_1]$.*

Lemma 2 *[9] Let L_1 and T be as defined in Lemma 1. Assume that for any (t, ψ^1, ψ^2) in $([t_0, L_1 + T] \times \mathcal{C}(D) \times \mathcal{C}(D))$ there exists a $K > 0$ such that $\left|f(t, \psi^1) - f(t, \psi^2)\right| \leq K||\psi^1 - \psi^2||$. Then $\left|f_T(t, \psi^1) - f_T(t, \psi^2)\right| \leq K||\psi^1 - \psi^2||$, for all (t, ψ^1, ψ^2) in $([t_0, L_1] \times \mathcal{C}(D) \times \mathcal{C}(D))$.*

3 Fundamental Averaging Theorems

Lemma 3 *Let the assumptions of Lemma 1 and Lemma 2 hold true for $L_1 = L/\epsilon$, where L and ϵ are positive constants. Assume that $x(t) = y(t) = \phi(t)$ on $[t_0 - r, t_0]$, where $\phi \in \mathcal{C}(D)$, and assume that both $\overline{x}(t)$ and $y(t) \in D$ for all $t \in [t_0, t_0 + L/\epsilon + T]$. Then $|y(t) - \overline{x}(t)| = O(\epsilon T) + O(\epsilon r)$ on $t \in [t_0 - r, t_0 + L/\epsilon]$.*

Proof: On $t \in [t_0 - r, t_0)$, $|y(t) - \overline{x}(t)| = 0$. For $t \geq t_0$,

$$|y(t) - \overline{x}(t)| = \left|y(t_0) + \epsilon\int_{t_0}^t f_T(s, y_s)ds - \overline{x}(t)\right|.$$

Taking the derivative of $\overline{x}(t)$, we have for $t > t_0$

$$\begin{aligned} \dot{\overline{x}}(t) &= \frac{1}{T}[x(t+T) - x(t)] \\ &= \frac{\epsilon}{T}\int_t^{t+T} f(s, x_s)ds = \frac{\epsilon}{T}\int_0^T f(t+\tau, x_{t+\tau})d\tau. \end{aligned}$$

Therefore, for $t \geq t_0$

$$\begin{aligned} |y(t) \quad - \quad \overline{x}(t)| = \bigg|y(t_0) - \overline{x}(t_0) + \epsilon\int_{t_0}^t [f_T(s, y_s) \\ -\frac{1}{T}\int_0^T f(s+\tau, x_{s+\tau})d\tau]ds\bigg| \end{aligned}$$

We note that $\overline{x}(t)$ usually has a discontinuity at $t = t_0^-$, and therefore, $\overline{x}(t-r)$ has a discontinuity at $t = t_0^- + r$. This requires us to be especially careful at $t = t_0$ and at $t = t_0 + r$. Let $\delta > 0$ be an arbitrarily small constant, and consider $|y(t) - \overline{x}(t)|$ on $t \in [t_0, t_0 + r + \delta]$. On this interval

$$|f_T(s, y_s))| \leq \frac{1}{T}\int_0^T |f(\tau + s, y_s)|\, d\tau \leq \frac{1}{T}\int_0^T M d\tau = M.$$

Likewise, $|f(\tau, x_\tau)| \leq M$ on this interval since it has been assumed that $x \in D$. Therefore, for $t \in [t_0, t_0 + r + \delta]$

$$\begin{aligned} |y(t) \quad - \quad \overline{x}(t)| \leq |\overline{x}(t_0) - y(t_0)| \\ +\epsilon\int_{t_0}^{t_0+r+\delta}\left(\frac{1}{T}\int_s^{s+T} M d\tau + M\right)ds. \end{aligned}$$

From the proof of Lemma 1 and the assumption that $y(t_0) = x(t_0)$, we have that $|y(t_0) - \overline{x}(t_0)| \leq \epsilon MT/2$. Therefore, for $t \in [t_0, t_0 + r + \delta]$, we have $|y(t) - \overline{x}(t)| \leq \epsilon M(T/2 + 2r + 2\delta)$.

Next, assume $L/\epsilon \geq r + \delta \equiv t_1$. On this interval we write

$$\begin{aligned} &|y(t) - \overline{x}(t)| \\ &\leq \ |y(t_1) - \overline{x}(t_1)| + \epsilon\int_{t_1}^t |f_T(s, y_s) - f_T(s, \overline{x}_s)|\, ds \\ &\quad +\epsilon\int_{t_1}^t |f_T(s, \overline{x}_s) - f_T(s, x_s)| \\ &\quad +\epsilon\int_{t_1}^t \left|f_T(s, x_s) - \frac{1}{T}\int_0^T f(s+\tau, x_{s+\tau})d\tau\right| ds. \end{aligned} \tag{6}$$

From above, we have that $|y(t_1) - \overline{x}(t_1)| \leq \epsilon M(T/2 + 2r + 2\delta)$. By Lemma 2 and the assumption that x, $\overline{x}$ and y remain in D, we have $|f_T(s, \overline{x}_s) - f_T(s, x_s)| \leq K\epsilon MT/2$ and

$|f_T(s,y_s) - f_T(s,x_s)| \le K||y_s - x_s||$ for $t \in [t_0, t_0 + L/\epsilon]$. Likewise, for $s \in [t_0, t_0 + L/\epsilon]$ and $\tau \in [0,T]$

$$\begin{aligned}|f_T(s,x_s) \quad &- \quad \frac{1}{T}\int_0^T f(s+\tau, x_{s+\tau})d\tau| \\ &= \frac{1}{T}|\int_0^T [f(s+\tau, x_s) - f(s+\tau, x_{s+\tau})]d\tau| \\ &\le \frac{1}{T}\int_0^T K||x_s - x_{s+\tau}||d\tau.\end{aligned}$$

For $t_0 \le t_2 \le t_3$, it is known that $x(t_3) - x(t_2) = \epsilon\int_{t_2}^{t_3} f(\lambda, x_\lambda)d\lambda$. This implies for $s \ge t_1$

$$\begin{aligned}x_s - x_{s+\tau} &= x(s+\theta) - x(s+\tau+\theta) \\ &= \epsilon\int_{s+\theta}^{s+\tau+\theta} f(s,x_s)ds.\end{aligned}$$

Therefore, for $t \in [t_0, t_0 + L/\epsilon]$ and $\tau \in [0,T]$

$$||x_s - x_{s+\tau}|| = \epsilon||\int_{t+\theta}^{t+\tau+\theta} f(s,x_s)ds|| \le \epsilon M\tau.$$

Using the above inequalities, for $t \in [t_0, t_0 + L/\epsilon]$, (6) becomes

$$\begin{aligned}|y(t) - \overline{x}(t)| &\le \epsilon M(T/2 + 2r + 2\delta) + \epsilon K\int_{t_1}^t ||y_s - \overline{x}_s||ds \\ &\quad +\epsilon^2\int_{t_1}^t (KMT/2 + \frac{1}{T}\int_0^T KM\tau d\tau)ds \\ &\le \epsilon M(T/2 + 2r + 2\delta) + \epsilon 2KMTL \\ &\quad +\epsilon K\int_{t_0}^t \sup_{\sigma\in[t_0,s]} |y(\sigma) - \overline{x}(\sigma)|ds.\end{aligned}$$

The right-hand side of the above inequality is increasing, and therefore, for $t \in [t_0, t_0 + L/\epsilon]$

$$\begin{aligned}\sup_{s\in[t_0,t]} |y(s) - \overline{x}(s)| &\le \epsilon M(T/2 + 2r + 2\delta) + \epsilon 2KMTL \\ &\quad +\epsilon K\int_{t_0}^t \sup_{\sigma\in[t_0,s]} |y(\sigma) - \overline{x}(\sigma)|ds.\end{aligned}$$

By Gronwall's inequality, this implies for $t \in (t_0, t_0 + L/\epsilon]$ that

$$\begin{aligned}\sup_{s\in[t_0,t]} |y(s) - \overline{x}(s)| &\le [\epsilon M(T/2 + 2r + 2\delta) \\ &\quad +\epsilon 2KMTL]exp\{\epsilon K(t - t_0)\}.\end{aligned}$$

The constant δ is arbitrarily small (e.g. select $\delta = \epsilon r$), and therefore, the above inequality implies that $|\overline{x}(t) - z(t)| = O(\epsilon T) + O(\epsilon r)$ on $t \in [t_0, t_0 + L/\epsilon]$. □.

Remark 2 If f is T−periodic, i.e., $f(t+T,\cdot) = f(t,\cdot)$, then we have 'almost' proven averaging. The only remaining task is to give conditions which guarantee that x, $\bar{x}$, and y all remain in D.

Lemma 4 *Assume that f is a KBM-functional, and consider (4) and (5) with continuous initial function $z(t) = y(t) = \phi(t)$ on $t \in [t_0 - r, t_0]$. Assume that, for any $L > 0$ and $\epsilon > 0$, both $z(t)$ and $y(t)$ remain in D for $t \in [t_0 - r, t_0 + L/\epsilon + T]$. Then $|y(t) - z(t)| = O(\frac{B(\epsilon)}{\epsilon T})$ on $t \in [t_0 - r, t_0 + L/\epsilon]$, where*

$$B(\epsilon) \equiv \sup_{\psi\in C(D)}\sup_{t\ge t_0} \epsilon|\int_0^T [f(s+t,\psi) - F_{av}(\psi)]ds|.$$

Proof: Since z and y have the same initial functions, it is only necessary to consider $t_0 \le t \le t_0 + L/\epsilon$. On this time interval

$$\begin{aligned}|y(t) - z(t)| &= \left|y(t_0) - z(t_0) \right. \\ &\quad \left. +\epsilon\int_{t_0}^t (f_T(s,y_s) - F_{av}(z_s))\,ds\right| \\ &\le \epsilon\int_{t_0}^t |f_T(s,z_s) - F_{av}(z_s)|\,ds \\ &\quad +\epsilon\int_{t_0}^t |f_T(s,y_s) - f_T(s,z_s)|\,ds \\ &\le \frac{LB(\epsilon)}{\epsilon T} + \epsilon K\int_{t_0}^t ||y_s - z_s||ds.\end{aligned}$$

Using the same arguments found at the end of Lemma 3, by Gronwall's inequality the proof is complete. □

Remark 3 Since f is assumed to be a KBM-functional, the function $B(\epsilon) \to 0$ as $\epsilon \to 0$. Using this fact, it is now possible to prove fundamental averaging results for FDE's. In fact, if $x(t)$, $y(t)$, $\overline{x}(t)$, and $z(t)$ all lie in D, then there is very little left to prove.

Theorem 1 *Suppose the f is a KBM-functional and that (1), (4) and (5) have the same continuous initial function, $\phi \in C(D)$, on $t \in [t_0 - r, t_0]$. Let $L > 0$ be a constant that is independent of ϵ, and define $B(\epsilon)$ as in Lemma 4. Assume that $x(t)$, $y(t)$, $\overline{x}(t)$ and $z(t)$ all lie in D for $t \in [t_0 - r, t_0 + L/\epsilon + \frac{\sqrt{B(\epsilon)}}{\epsilon}]$. Then $|x(t) - z(t)| = O(\sqrt{B(\epsilon)}) + O(\epsilon r)$ for all $t \in [t_0 - r, t_0 + L/\epsilon]$.*

Proof: From Lemmas 1, 2, and 4 for $t \in [t_0 - r, t_0 + L/\epsilon + T]$

$$\begin{aligned}|x(t) - z(t)| &\le |x(t) - \overline{x}(t)| + |\overline{x}(t) - y(t)| \\ &\quad +|y(t) - z(t)| \\ &\le O(\epsilon T) + O(\epsilon T) + O(\epsilon r) + O\left(\frac{B(\epsilon)}{\epsilon T}\right).\end{aligned}$$

Setting $\epsilon T = \sqrt{B(\epsilon)}$ completes the proof. □

Remark 4 It is common (for ODE's) to write Theorem 1 assuming that $z(t) \in D_0 \subset D$ and to remove the assumption that $x(t) \in D$ and $y(t) \in D$. One can always assume that D_0 and D have sufficient distance between them that $x(t)$ and $y(t)$ remain in D.

When f is a KBM functional, $\sqrt{B(\epsilon)} \to 0$ as $\epsilon \to 0$. Hence, $|x(t) - z(t)|$ becomes vanishingly small as $\epsilon \to 0$. Using this fact, it is straightforward to prove averaging results that encompass functionals more general than f. Consider the FDE

$$\dot{\chi}(t) = \epsilon g(t, \chi_t, \epsilon); \quad \chi_{t_0}(t) = \phi. \tag{7}$$

Along with (7) consider the corresponding averaged equation

$$\dot{\nu}(t) = \epsilon G_{av}(\nu_t); \quad \nu_{t_0} = h, \tag{8}$$

where

$$G_{av}(\psi) \equiv \lim_{T \to \infty} \frac{1}{T} \int_t^{t+T} g(s, \psi, 0) ds. \tag{9}$$

Notice that it is no longer required that (7) and (8) have the same initial function and that g can explicitly depend on ϵ. As usual, let $\chi(t; t_0, \phi)$ and $\nu(t; t_0, h)$ denote the solutions to (7) and (8) respectively, and assume that ϕ and h are continuous functions.

Definition 4 Assume that the functional g is uniformly bounded by M and locally Lipschitz with constant K for all $\epsilon \in [0, \mu]$. Assume further that the limit in (9) exists uniformly with respect to t for all $\psi \in \mathcal{C}(D)$. Then the functional g is said to be a **KBM-functional for all** $\epsilon \in [0, \mu]$.

Theorem 2 *Let σ, μ, and L be arbitrary constants independent of ϵ, and define $\beta \equiv sup_{s \in [t_0 - r, t_0]} |\phi(s) - h(s)|$. Assume that g is continuous with respect to all its arguments and $g(t, \psi, \epsilon)$ is a KBM-functional for all $\epsilon \in [0, \mu]$. Suppose that $\nu(t; t_0, h)$ together with its σ vicinity remains in D for all $t \in [t_0 - r, t_0 + L/\epsilon]$. Then for any $\eta > 0$ there exists a $\beta_0 = \beta_0(\eta, L)$ and an $\epsilon_0 = \epsilon_0(\eta, \beta_0, L)$, such that, for $0 \le \beta \le \beta_0$ and $0 \le \epsilon \le \epsilon_0$,*

$$|\chi(t; t_0, \phi) - \nu(t; t_0, h)| \le \eta$$

for all $t \in [t_0 - r, t_0 + L/\epsilon]$.

Proof: A simple exercise of the properties of continuity of solutions (see Chapter 25, Theorem E of [2]) combined with Theorem 1. □

4 Example - An Inverted Cart and Pendulum Problem

In this section, we present a simple application to a variation of cart and pendulum stabilization by proportional feedback. As illustrated in Figure 1, the system consists of a cart and planar pendulum apparatus in a reference frame which is being subjected to a periodic disturbance of amplitude β and frequency ω along the horizontal axis. Such a disturbance might arise as the result of attempting control on an unsteady platform (i.e. a helicopter) or in the presence of high frequency variations in the system's operating environment.

The net motion of the cart is equal to the sum of the disturbance and the cart's motion in the local frame of reference. The pendulum is modelled as a rigid, massless link of length ℓ and a bob of mass m, and its displacement is referenced to the vertical. The cart and pendulum are connected by a single degree of freedom planar joint. Suppose that the cart position may be precisely controlled. Then the differential equation of motion for the pendulum is

$$m\ell^2\ddot{\theta} + c_d\dot{\theta} - m\ell\left(\ddot{u}\cos\theta + g\sin\theta\right) = 0, \tag{10}$$

where $\ddot{u} = -\omega^2\beta\cos\omega t + \ddot{x}$, $\ddot{x}$ is the acceleration of the cart, and c_d is the damping coefficient for the planar joint.

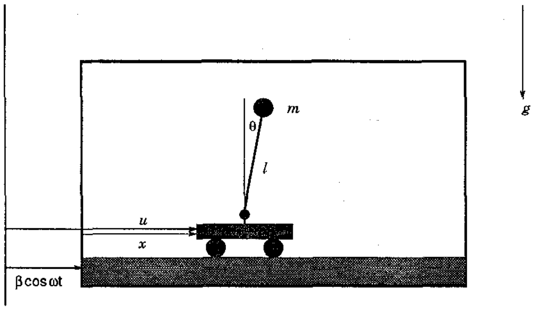

Figure 1: Cart and Pendulum subjected to periodic disturbances.

Remark 5 The assumption that the cart dynamics can be explicity controlled is fairly common and forms a basis for a theory of the control of *velocity controlled mechanical systems*. For an introduction to this topic, see [19] and [18].

To stabilize the inverted equilibrium, we prescribe the proportional control $\ddot{x} = -K_p\theta(t - r)$, where $r > 0$ is a control delay perhaps due to sampling, computation, or even tele-remote operation. Then $\ddot{u}(t) = -\omega^2\beta\cos\omega t - K_p\theta(t - r)$. (10) can be written as a dimensionless delay differential equation

$$\theta'' + \epsilon^2 c\theta' - \epsilon\cos\tau + \epsilon^2\left(k\theta(\tau - r')\cos\theta - \sin\theta\right) = 0,$$

where $\tau = \omega t$, $(\cdot)' = d/d\tau$, $\epsilon = \beta/\ell = \sqrt{g/\ell}/\omega$, $c = \ell c_d/g$, $k = \ell K_p/g$, and $r' = \omega r$. Prescribing the coordinate change $\theta = y_1 - \epsilon\cos\tau\cos y_1$, $\theta' = \epsilon y_2 + \epsilon\sin\tau\cos y_1$ and proceeding as in [1], we eventually have the system of first order equations

$$\begin{aligned} y_1' &= \epsilon y_2 + \mathcal{O}(\epsilon^2) \\ y_2' &= \epsilon\left[-cy_2 + \sin y_1 \cos y_1 \cos^2\tau + y_2 \sin y_1 \sin\tau\right. \\ &\quad \left. -ky_1(\tau - r')\cos y_1 + \sin y_1\right] + \mathcal{O}(\epsilon^2). \end{aligned} \tag{11}$$

By Theorem 2, the average of (11) is given as

$$\begin{aligned} z_1' &= \epsilon z_2 \\ z_2' &= \epsilon\left[-cz_2 + \frac{\sin z_1 \cos z_1}{2} - kz_1(\tau - r')\cos z_1\right. \\ &\quad \left. + \sin z_1\right]. \end{aligned} \tag{12}$$

Linear analysis of the inverted equilibrium for the case $r' = 0$ shows that the system is stabilized if the proportional gain $k > 3/2$.

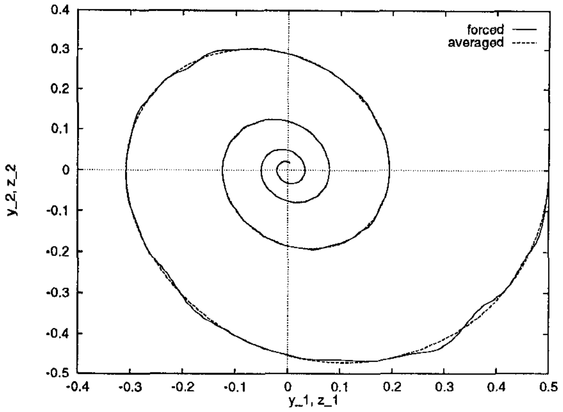

Figure 2: Nonautonomous (solid) and averaged (dashed) trajectories for the controlled cart and pendulum with feedback delay.

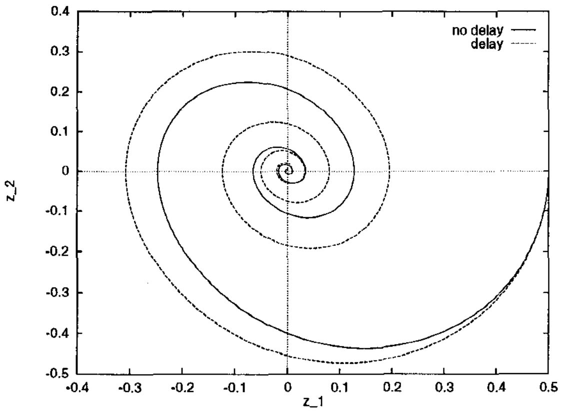

Figure 3: Averaged trajectories with and without feedback delay.

The results of simulations of the nonautonomous and averaged systems are shown in Figure 2. The parameter values used in the simulations are $\epsilon = 0.1$, $k = 3$, and delay value $r' = 0.5$. In the original time scale, these delay values scale back to $r = r'/\omega = \epsilon r' \sqrt{g/\ell}$. Initial condition and function data is given by $(y_1(\tau), y_2(\tau)) = (z_1(\tau), z_2(\tau)) = (0.5, 0)$ for $\tau \in [-r', 0]$.

In Figure 2 we see the phase portrait for the averaged and nonautonomous systems. The averaged phase portrait approximates the nonautonomous trajectory. The significance of this result is that it shows that the appropriate averaged equations retain the delay term, as opposed to earlier results which suggest that the delay term is not important. In Figure 3, we see a comparison of the averaged trajectories for the system without and with delays. It is clear from the figure that these trajectories are distinct, and that trajectory with delay is not a small perturbation of the trajectory without delay. This is true in spite of the fact that the delay is $\mathcal{O}(\epsilon)$ in the original time scale. Hence, the new averaged models seem to be an improved approximation compared to the classical averaged models of [4], [5], [6], [17], [3], and [13].

References

[1] N.N. Bogoliubov and Y.A. Mitropolsky. *Asymptotic Methods in the Theory of Nonlinear Oscillators.* International Monographs on Advanced Mathematics and Physics. Gordon and Breach Science Pub., New York, 1961.

[2] R. Driver. *Ordinary and Delay Differential Equations.* Springer Verlag, New York, 1977.

[3] V.I. Fodcuk. The method of averaging for differential difference equations of the neutral type. *Ukraine Mat. Z.*, 20:203–209, 1968.

[4] A. Halanay. The method of averaging in equations with retardation. *Rev. Mat. Pur. Appl. Acad. R.P.R.*, 4:467–483, 1959.

[5] A. Halanay. On the method of averaging for differential equations with retarded arguments. *Journal of Mathematical Analysis and Applications*, 14:70–76, 1966.

[6] J.K. Hale. Averaging methods for differential equations with retarded arguments with a small parameter. *Journal of Differential Equations*, 2:57–73, 1966.

[7] J.K. Hale and S.M. Verduyn Lunel. Averaging in infinite dimensions. *Journal of Integral Equations and Applications*, 2:463–494, 1990.

[8] N. Krylov and N. Bogoliubov. Introduction to nonlinear mechanics. *Annals Math. Studies*, 11, 1947.

[9] B. Lehman, S.V. Lunel J. Bentsman, and E.I. Verriest. Vibrational control of nonlinear time lag systems with bounded delay: Averaging theory, stabilizability, and transient behavior. *IEEE Trans. Aut. Control*, AC-39:898–912, 1994.

[10] B. Lehman and V.B. Kolmanovskii. Extensions of classical averaging techniques to delay differential equations. In *Proc. 33rd IEEE CDC*, pages 411–416, 1994.

[11] B. Lehman and S. Weibel. *Moving Averages for Periodic Delay Differential and Difference Equations.* Lecture Notes in Control and Information Science. Springer-Verlag, Berlin, 1997.

[12] B. Lehman, I. Widjaya, and K. Shujaee. Vibrational control of chemical reactions in a cstr with delayed recycle. *J. Math. Anal. & Appl.*, 193:28–59, 1995.

[13] G.N. Medvedev. Asymptotic solutions of some systems of differential equations with deviating argument. *Soviet Math Dokl.*, 9:85–87, 1968.

[14] S.M. Meerkov. Averaging of trajectories of slow dynamic systems. *Differential Equations*, 9:1239–1245, 1973.

[15] J.A. Sanders. On the fundamental theory of averaging. *SIAM J. Math. Anal.*, 14:1–10, 1983.

[16] K. Shujaee and B. Lehman. Vibrational feedback control of time delay systems. In *Proc. 34th IEEE CDC*, pages 936–941, 1995.

[17] V.M. Volosov, G.N. Medvedev, and B.I. Morgunov. Mr 32 #7904. *Vestnik Moskov: Univ. Ser. III. Fiz. Astronom*, 6:89, 1965.

[18] S. Weibel and J. Baillieul. Averaging and energy methods for robust open-loop control of mechanical systems. To appear in Proc. of 1993 IMA Wkshp. on Rob., 1997.

[19] S. Weibel, J. Baillieul, and T.J. Kaper. Small-amplitude periodic motions of rapidly forced mechanical systems. In *Proceedings of 34th IEEE Conference on Decision and Control, New Orleans*, pages 533–539, 1995.