Geometry of Reactive Power: A Divertimento

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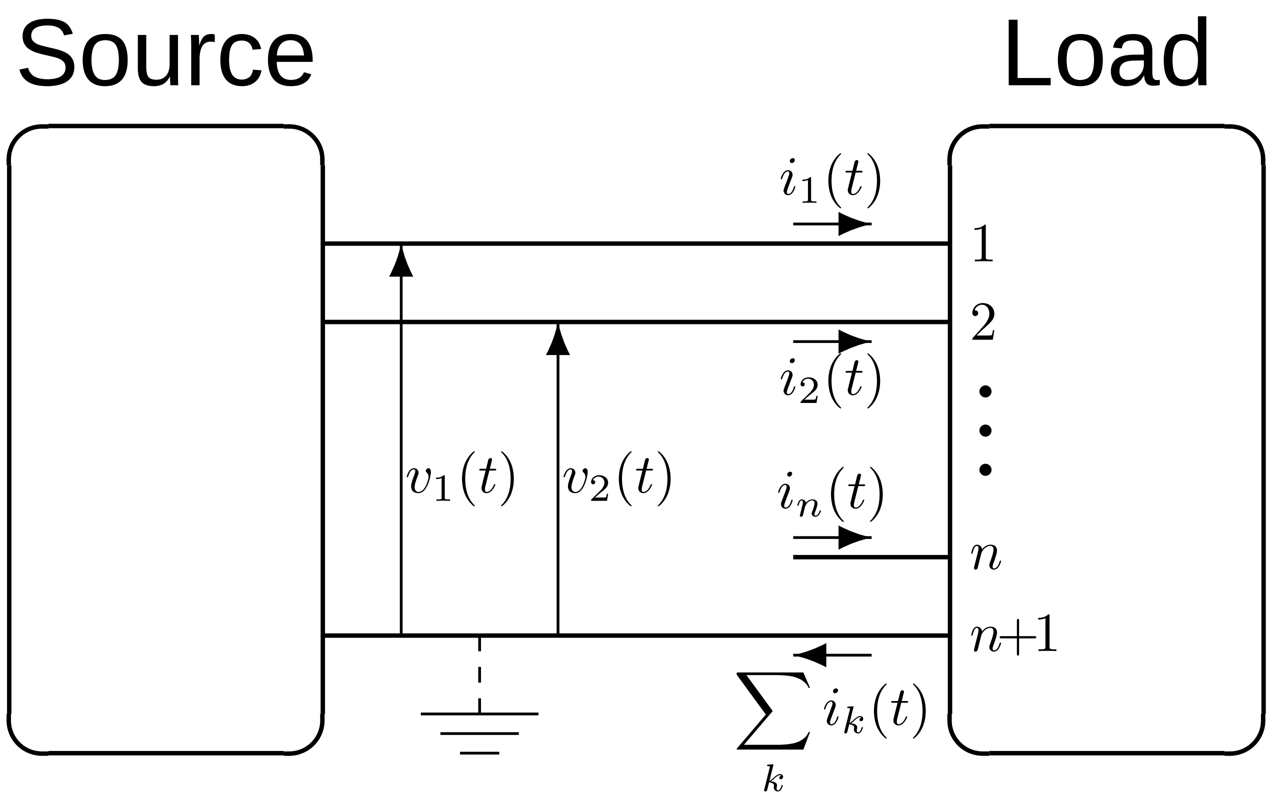
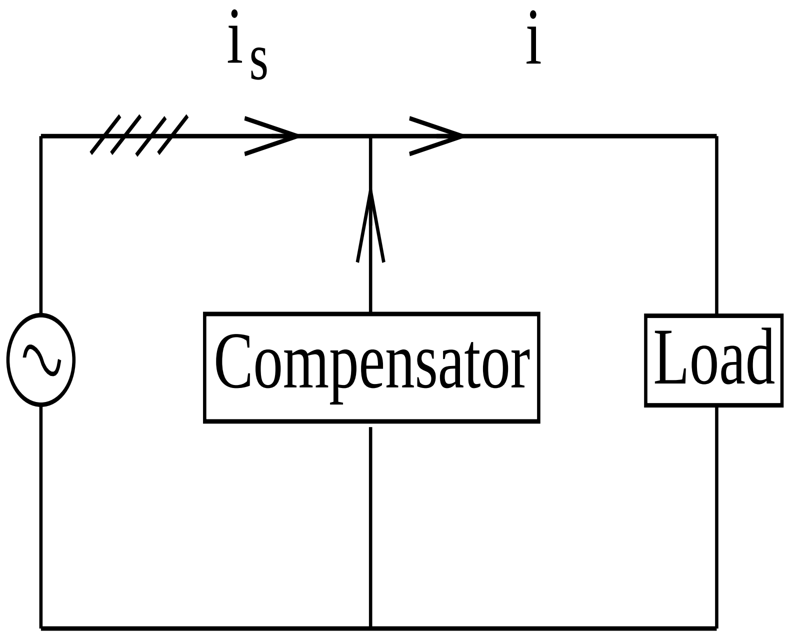
Problem Formulation

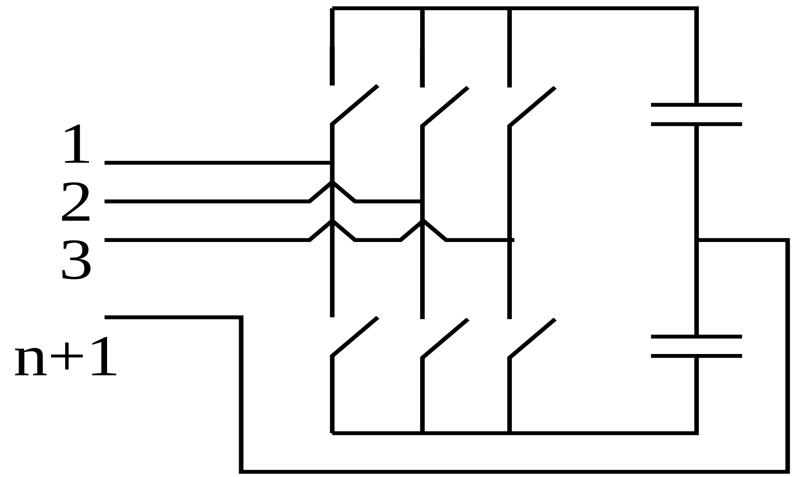
Consider a possibly unbalanced system with n phases, $n+1$ conductors (with possibly different resistances), and arbitrary many harmonics:



Problem Formulation (2)

Compensation framework





Compensator

For now, all time domain (row-)vector waveforms $x(t)$ are periodic, with period T; let $\omega = {}^{2\pi}\!/_{T}$ so the Fourier coefficients [Steinmetz] are

$$X_\ell = \frac{1}{T}\int_T x(t)e^{-j\ell\omega t}dt \stackrel{\text{def}}{=} < x(t) >_\ell \qquad (1)$$

Agreement on Reactive Power

Standard definitions [Buchholz, Schallenberger, Stanley]:

$$\|V\|^2 =< v\, v^\top >_0, \quad \|I\|^2 =< i\, i^\top >_0, \quad S = \|V\| \|I\|$$

$$P =< i^\top\, v >_0, \quad Q^2 = S^2 - P^2, \quad k_{PF} = \frac{P}{S}$$

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For single frequency and single phase/ balanced polyphase in steady state:

$$P = \|V\| \|I\| \cos \phi_1, \quad Q = \|V\| \|I\| \sin \phi_1 = Q_{B1}$$

$$k_{PF} = \cos \phi_1, \quad S_c \triangleq P + \jmath Q$$

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In multi-frequency $k_{PF} \neq \cos\phi_1$, [Steinmetz 1898]; similarly for unbalanced.

Presentation Map

- Multivectors and Geometric Algebra,
- Polyphase Sinusoidal,
- Non-sinusoidal,
- Transients.

Geometric (Clifford) Algebra

$v(t)$ and $i(t)$ are row vectors, elements in the Hilbert space [Quade 1939] of all n-phase, square-integrable, T-periodic waveforms, with the inner product

$$\langle\, x \,,\, y \,\rangle \;\triangleq\; \frac{1}{T}\int_T x(s)\, y^\top(s)\, ds \tag{2}$$

The geometric algebra we consider is generated by a voltage-derived finite-dimensional subspace of the Hilbert space. A vector $\mathbf{x}$ int he given Euclidean space satisfies

$$\mathbf{x}^2 = \langle\, \mathbf{x} \,,\, \mathbf{x} \,\rangle \equiv \|\, \mathbf{x}\|^2$$

For two distinct vectors

$$\mathbf{x}\mathbf{y} - \langle\, \mathbf{x} \,,\, \mathbf{y} \,\rangle \;\triangleq\; \mathbf{x} \wedge \mathbf{y}$$

where $\mathbf{y} \wedge \mathbf{x} = -\,\mathbf{x} \wedge \mathbf{y}$.

Geometric Algebra (2)

Consider a single phase sinusoidal circuit and let $\mathbf{e}_1 = \sqrt{2}\cos\omega t$, $\mathbf{e}_2 = \sqrt{2}\sin\omega t$. Then

$$\begin{aligned} v(t) &\equiv \mathbf{v} = \alpha_1\mathbf{e}_1 + \alpha_2\mathbf{e}_2 \\ i(t) &\equiv \mathbf{i} = \beta_1\mathbf{e}_1 + \beta_2\mathbf{e}_2 \end{aligned}$$

The apparent power multivector is then

$$\mathbb{S} \stackrel{\triangle}{=} \mathbf{v}\,\mathbf{i} = \underbrace{\alpha_1\beta_1 + \alpha_2\beta_2}_{\langle\, \mathbf{v}\,,\,\mathbf{i}\,\rangle} + \underbrace{(\alpha_1\beta_2 - \alpha_2\beta_1)\,\mathbf{e}_1\mathbf{e}_2}_{\mathbf{v}\wedge\mathbf{i}}$$

where we used the rules ${\mathbf{e}_i}^2 = 1, \;\; \mathbf{e}_2\mathbf{e}_1 = -\mathbf{e}_1\mathbf{e}_2$.

Geometric Algebra (3)

If we use the voltage-derived choice of basis waveforms

$$\mathbf{e}_1 = \sqrt{2}\,\Re\left\{\frac{V}{|V|}\,e^{\jmath\omega t}\right\} \;\equiv\; \frac{v(t)}{|V|} \tag{3a}$$

$$\mathbf{e}_2 = \sqrt{2}\,\Im\left\{\frac{V}{|V|}\,e^{\jmath\omega t}\right\} \;\equiv\; \sqrt{2}\,\Re\left\{-\jmath\frac{V}{|V|}\,e^{\jmath\omega t}\right\} \tag{3b}$$

where we recognize the Hilbert transform [Nowomiejski 1980], then $v(t) \equiv \mathbf{v} = |V|\,\mathbf{e}_1$, and $\mathbf{i} \;=\; \frac{P}{|V|}\,\mathbf{e}_1 + \frac{Q_B}{|V|}\,\mathbf{e}_2$, so that

$$\mathbb{S} \;\equiv\; \mathbf{v}\,\mathbf{i} \;=\; P + Q_B\,\mathbf{e}_1\mathbf{e}_2 \tag{4}$$

where $P = \Re\{VI^*\}$, and $Q_B = \alpha_1\beta_2 - \alpha_2\beta_1 = \Im\{VI^*\}$ Budeanu's reactive power [Budeanu 1927].

Geometric Algebra (4)

The *norm* of a multivector $\mathbb{A}$ is defined as

$$\|\mathbb{A}\|^2 \triangleq \left\langle \mathbb{A}\mathbb{A}^\dagger \right\rangle_0 = \sum_{r=0}^{N} \|\langle \mathbb{A} \rangle_r\|^2$$

so

$$\|\mathbb{S}\|^2 = \|\mathbf{v}\|^2 \; \|\mathbf{i}\|^2 = P^2 + Q_B^2$$

The grade structure of a geometric algebra is symmetric, so that grade-r and grade-$(N-r)$ subspaces always have the same dimensions - correspondence via duality relation $\mathbb{A} \leftrightarrow \mathbb{A}\,\mathcal{J}$, pseudoscalar $\mathcal{J}$ is the geometric product of a complete set of basis vectors, viz., $\mathcal{J} \triangleq \mathbf{e}_1\mathbf{e}_2 \ldots \mathbf{e}_N$.

$\mathbb{S} = P + Q_B\,\mathcal{J}$ shares key properties with the classical complex apparent power $S_c = P + \jmath Q_B$.

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Three Phase Sinusoidal

The apparent power multivector $\mathbb{S} = \mathbf{v}\,\mathbf{i}$ now consists of 16 components: the scalar product $\langle\, \mathbf{v}\,,\, \mathbf{i}\,\rangle = P$ and 15 components of the wedge product $\mathbf{v} \wedge \mathbf{i}$; we redefine it to get a coarser (but physically meaningful) decomposition.

A voltage-derived basis - let $\eta_p = \left[\underbrace{0 \cdots 0\, 1}_{p}\; 0 \cdots 0\right]$, so

$$\mathbf{e}_p \triangleq \sqrt{2}\,\Re\left\{\frac{V_p}{|V_p|}\, e^{\jmath\omega t}\right\} \eta_p \tag{5a}$$

and

$$\begin{aligned} \mathcal{H}(\mathbf{e}_p) &= \sqrt{2}\,\Re\left\{-\jmath\,\frac{V_p}{|V_p|}\, e^{\jmath\omega t}\right\} \eta_p \\ &= \sqrt{2}\,\Im\left\{\frac{V_p}{|V_p|}\, e^{\jmath\omega t}\right\} \eta_p \end{aligned} \tag{5b}$$

Three Phase Sinusoidal (2)

The 3-phase waveform $v(t)$ and $i(t)$

$$v(t) = |V_1|\,\mathbf{e}_1 \; + \; |V_2|\,\mathbf{e}_2 \; + \; |V_3|\,\mathbf{e}_3$$

$$i(t) \; = \; i_v(t) + \mathcal{H}\big(i_w(t)\big)$$

where

$$\begin{aligned} \mathbf{i}_v \; \equiv \; i_v(t) \; & \triangleq \; \sum_{p=1}^{3} \frac{\Re\{V_p I_p^*\}}{|V_p|}\,\mathbf{e}_p \\ \mathbf{i}_w \; \equiv \; i_w(t) \; & \triangleq \; \sum_{p=1}^{3} \frac{\Im\{V_p I_p^*\}}{|V_p|}\,\mathbf{e}_p \end{aligned} \qquad (6)$$

Three Phase Sinusoidal (3)

We redefine the apparent power multivector $\mathbb{S}$ as a sum of two multivectors - first, the current multivector is

$$\mathbf{i} \triangleq \mathbf{i}_v + \mathbf{i}_w \mathcal{J} \tag{7}$$

and then

$$\mathbb{S} \triangleq \mathbf{v}\,\mathbf{i} = \mathbb{S}_v + \mathbb{S}_w \mathcal{J} \tag{8}$$

where

$$\mathbb{S}_v = \mathbf{v}\,\mathbf{i}_v = \underbrace{\langle \mathbf{v}, \mathbf{i}_v \rangle}_{\text{grade 0}} + \underbrace{\mathbf{v} \wedge \mathbf{i}_v}_{\text{grade 2}} \tag{9}$$

and

$$\mathbb{S}_w \mathcal{J} = \mathbf{v}\,\mathbf{i}_w \mathcal{J} = \underbrace{\langle \mathbf{v}, \mathbf{i}_w \rangle \mathcal{J}}_{\text{grade 3}} + \underbrace{(\mathbf{v} \wedge \mathbf{i}_w) \mathcal{J}}_{\text{grade 1}} \tag{10}$$

Three Phase Sinusoidal (4)

We will use the shorthand notations

$$P \triangleq \langle\, \mathbf{v}\,,\, \mathbf{i}_v \,\rangle\,,\; \mathbb{S}_g \triangleq \mathbf{v} \wedge \mathbf{i}_v$$
$$Q_B \triangleq \langle\, \mathbf{v}\,,\, \mathbf{i}_w \,\rangle\,,\; \mathbb{S}_b \triangleq \mathbf{v} \wedge \mathbf{i}_w \tag{11}$$

so that $\mathbb{S}_g$ and $\mathbb{S}_b$ are bi-vectors with 3 components each.

$$\mathbb{S} = \underbrace{P + \mathbb{S}_g}_{\mathbb{S}_v} + \underbrace{(Q_B + \mathbb{S}_b)}_{\mathbb{S}_w}\,\mathcal{J} \tag{12a}$$

$$\|\mathbb{S}\|^2 = \|\mathbf{v}\|^2\,\|\mathbf{i}\|^2 = P^2 + \|\mathbb{S}_g\|^2 + Q_B^2 + \|\mathbb{S}_b\|^2 \tag{12b}$$

using all four grades (0,1,2,3) of the geometric algebra generated by the three-dimensional vector space spanned by $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$. In fact, $\mathbb{S}_v = P + \mathbb{S}_g$ is a quaternion, and so is $\mathbb{S}_w = Q_B + \mathbb{S}_b$.

Polyphase Sinusoidal

This extension follows directly

$$\mathbb{S} = \underbrace{P}_{\text{grade } 0} + \underbrace{\mathbb{S}_g}_{\text{grade } 2} + \underbrace{Q_B \mathcal{J}}_{\text{grade } n} + \underbrace{\mathbb{S}_b \mathcal{J}}_{\text{grade } n-2}$$

The statistical interpretation of the four components of $\mathbb{S}$ in terms of the distributions of equivalent conductances $\{g_p\}$ and susceptances $\{b_p\}$ is unchanged from our previous, projection-based work.

Presentation Map

- Multivectors and Geometric Algebra,
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- Non-sinusoidal,
- Transients.

Non-Sinusoidal

Previous results largely unchanged if we can derive a voltage-based orthonormal basis - decide on which harmonics are important

$$\Omega_v \triangleq \{k;\ |V_k| > \varepsilon\}$$

we define a set of orthonormal basis waveforms

$$\mathbf{e}_k \triangleq \sqrt{2}\,\Re\left\{\frac{V_k}{|V_k|}\,e^{\jmath k\omega t}\right\}\ ,\ k \in \Omega_v \tag{13}$$

and use the Euclidean space span$\{\mathbf{e}_k;\ k \in \Omega_v\}$ to generate the corresponding geometric algebra.

$$\begin{aligned} i(t) &= \sum_{k\in\Omega_v} \sqrt{2}\,\Re\{I_k\,e^{\jmath k\omega t}\} + \underbrace{\sum_{k\notin\Omega_v} \sqrt{2}\,\Re\{I_k\,e^{\jmath k\omega t}\}}_{i_\perp(t)} \\ &= i_v(t) + \mathcal{H}\big(i_w(t)\big) + i_\perp(t) \end{aligned} \tag{14}$$

Non-Sinusoidal (2)

We have a 5-component decomposition of the apparent power, consisting of the out-of-band (unsigned) component $\|\mathbf{v}\|\,\|i_\perp\| \triangleq S_\perp$ and a four-component multivector

$$\mathbb{S} = P + \mathbb{S}_g + (Q_B + \mathbb{S}_b)\,\mathcal{J} \tag{15a}$$

such that

$$\|\mathbf{v}\|^2\|\mathbf{i}\|^2 = \underbrace{P^2 + \|\mathbb{S}_g\|^2 + Q_B^2 + \|\mathbb{S}_b\|^2}_{\|\mathbb{S}\|^2} + S_\perp^2 \tag{15b}$$

The polyphase version is the same, except that the set of voltage-derived basis vectors is doubly indexed (phase and harmonic)

Decussis Mirabilis: 1927-1937

$$S^2 = P^2 + Q_B^2 + D^2$$

Constantin Budeanu (1886-1959)

$$i_a(t) = \frac{< i^\top v >}{< v^\top v >} v(t)$$

Stanislaw Fryze (1885-1964)

Funktionenraums

Wilhelm Quade (1898-1975)

Reactive Power - Schools

- Romania: A.D. Iliovici (1925), C. Busila, C. Budeanu, I. Antoniu (1950), F. Manea, V. Nedelcu (1960) M. Milic (1970), A. Tugulea, A. Emanuel,
- Poland: S. Fryze, M. Erlicki (1950), Z. Nowomiejski (1980), L. Czarnecki,
- Germany: F. Buchholz (1920), W. Quade (1930), R. Richter (1930), M. Depenbrock (1970), H. Fisher (1980).

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Instantaneous Reactive Power

The instantaneous apparent power is

$$s(t) = \sqrt{[v(t)\ v^\top(t)][i(t)\ i^\top(t)]}$$

$$s^2(t) - p^2(t) = \sum_{r<m} q_{rm}^2(t) \tag{16}$$

where $q_{rm}(t)$ are the (strictly above diagonal) elements of the skew-symmetric matrix

$$Q(t) \triangleq i^\top(t)v(t) - v^\top(t)i(t) \tag{17}$$

usually referred to as the “instantaneous reactive power”.

Instantaneous Reactive Power (2)

In a three-phase $a - b - c$ system there are three instantaneous reactive power quantities, viz., $q_{ab}(t),\ q_{ac}(t)$ and $q_{bc}(t)$.

Some authors proposed interpreting $q_{rm}(t)$ as the three components of a vector (cross) product between the current vector $i(t)$ and the voltage vector $v(t)$. However, this interpretation does not carry over to $n > 3$.

The entries of $Q(t)$ depend on coordinates chosen (e.g., $a - b - c$ vs. symmetrical components vs. Park transformed).

The instantaneous apparent power multivector is again a quaternion

$$\mathbb{S} \triangleq \boldsymbol{i}\,\boldsymbol{v} = \boldsymbol{i} \cdot \boldsymbol{v} + \boldsymbol{i} \wedge \boldsymbol{v} \tag{18}$$

Instantaneous Reactive Power (3)

Instantaneous Akagi-Nabae compensation achieves $p(t) = s(t)$ by forcing the compensated load current to equal

$$i_a(t) = \frac{i(t)\ v^\top(t)}{v(t)\ v^\top(t)} v(t)$$

In the periodic case we can establish (unfortunately quite tenuous) links with average power quantities:

$$P = \frac{1}{T}\int_T p(t)\ dt =< p(t) >_0$$

$$S = \sqrt{< i(t)\ i^\top(t) >_0 < v(t)\ v^\top(t) >_0}$$

so that, in general, $S^2 \neq < s(t) >_0^2 \neq < s^2(t) >_0$, as well as $P^2 \neq < p^2(t) >_0$. All one can say with certainty is $< s(t) >_0 \leq\ S$.

Doing it Right - Dynamic Phasors

$x(t-\Delta)$ on the interval $[0 \le \Delta < T_0)$ via (short-time) Fourier series:

$$x(t-\Delta) = \sum_{k=-\infty}^{\infty} X_k(t) e^{jk\omega_0(t-\Delta)} = \sum_{k=-\infty}^{\infty} e^{-jk\omega_0\Delta} \mathcal{X}_k(t)$$

$X_k(t)$ are the complex, time-varying Fourier coefficients, or dynamic phasors.

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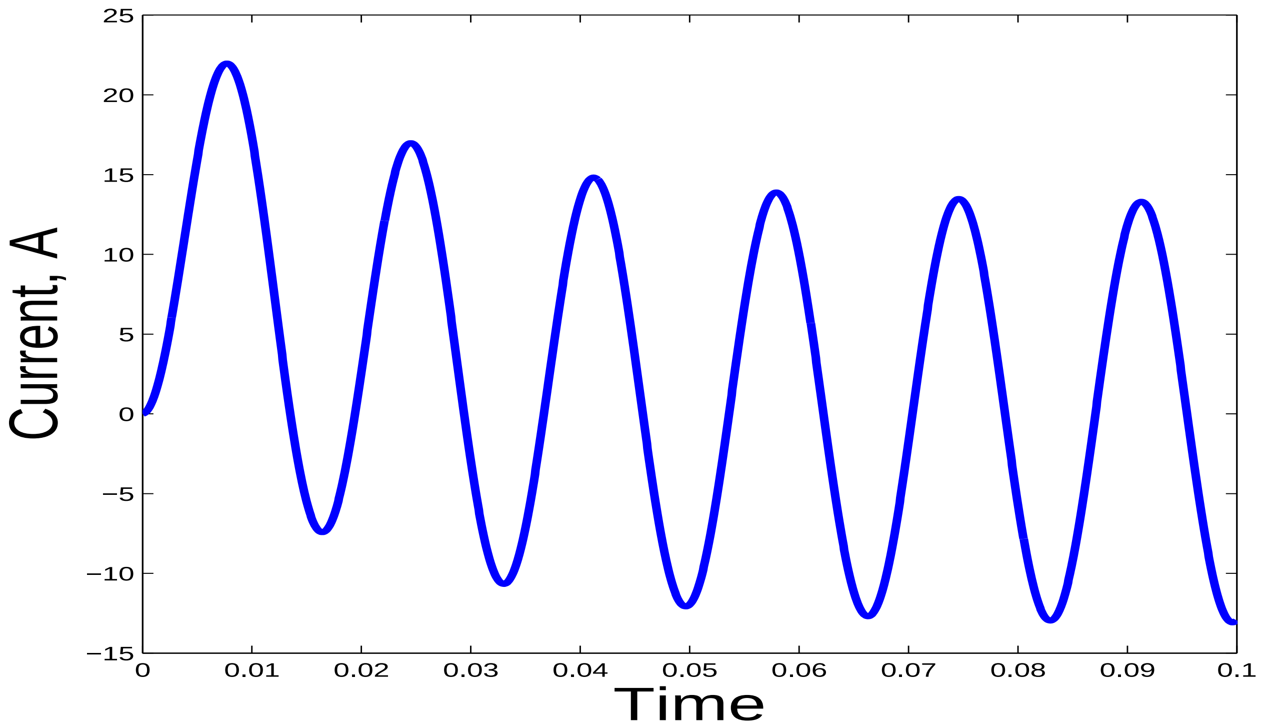
$$X_k(t) = \frac{1}{T_0} \int_{t-T_0}^{t} x(\tau) e^{-jk\omega_0 \tau} d\tau = <x>_k(t)$$

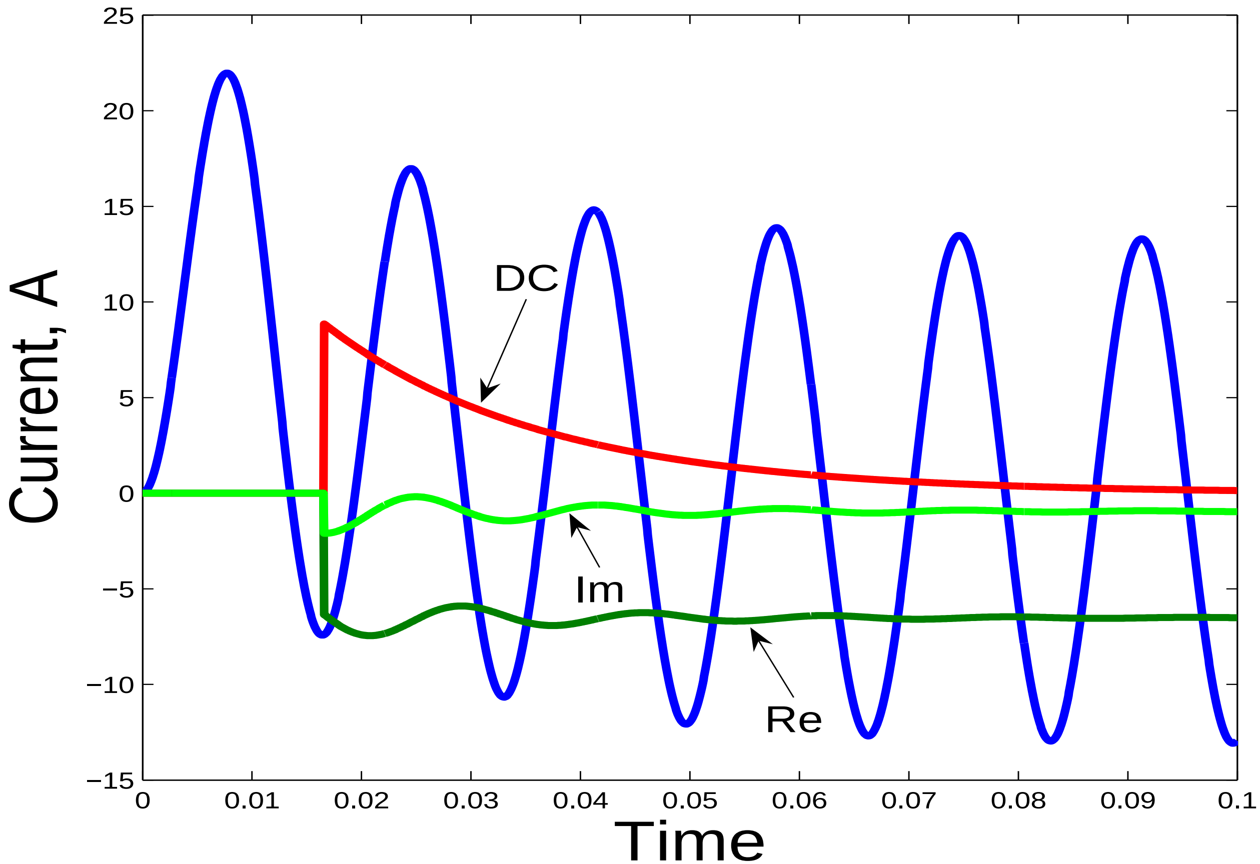
A Simple Example

Consider a simple RL circuit (R=0.1, L=0.002) with a $\sin$ excitation (V=10), at 60Hz



A Simple Example (2)

The dynamic phasors (according to our definition)



Extensions - Classical

Dynamic symmetrical components - recall $\alpha = e^{j\frac{2\pi}{3}}$,

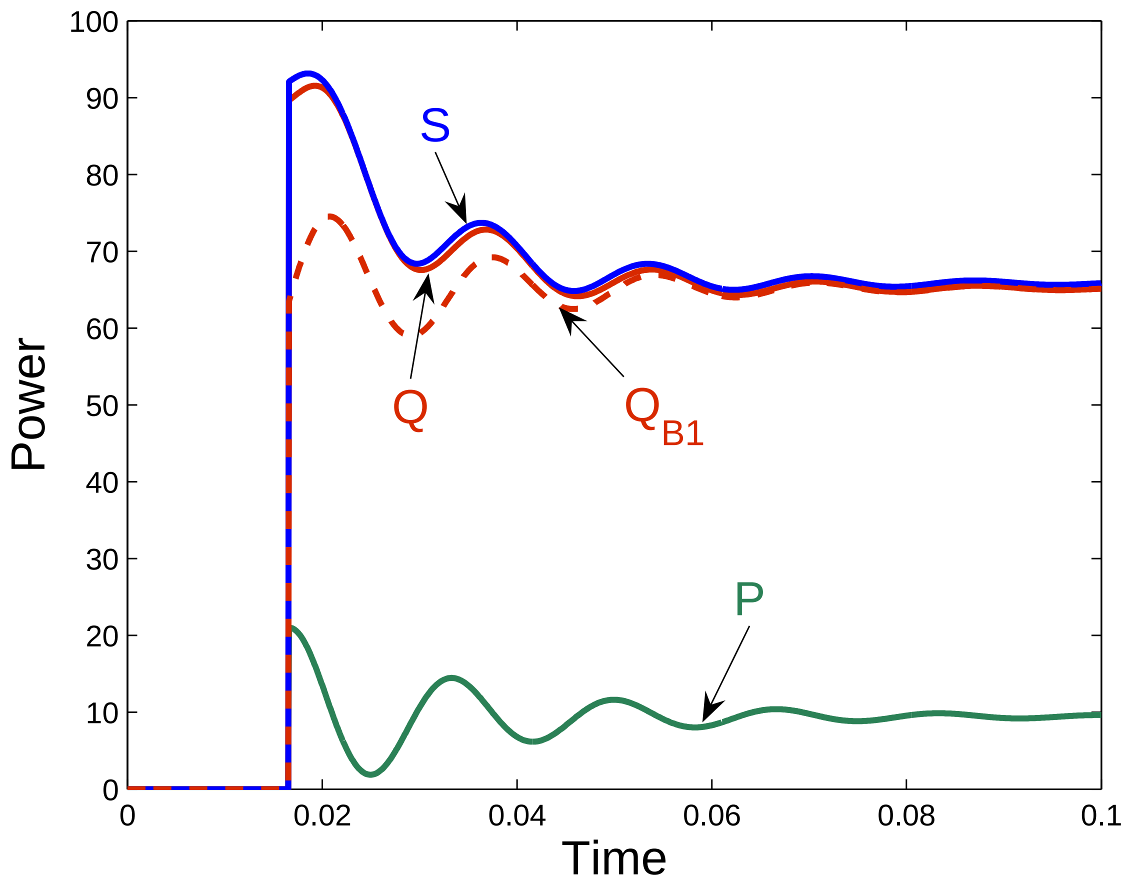
$$\begin{bmatrix} x_a \\ x_b \\ x_c \end{bmatrix}(\tau) = \sum_{k=-\infty}^{\infty} e^{jk\omega_0\tau} \underbrace{\frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 & 1 \\ \alpha^* & \alpha & 1 \\ \alpha & \alpha^* & 1 \end{bmatrix}}_{F} \begin{bmatrix} X_{p,k} \\ X_{n,k} \\ X_{z,k} \end{bmatrix}(t)$$

$$\begin{bmatrix} X_{p,k} \\ X_{n,k} \\ X_{z,k} \end{bmatrix}(t) = \frac{1}{T_0} \int_{t-T_0}^{t} e^{-jk\omega_0\tau} \; F^H \begin{bmatrix} x_a \\ x_b \\ x_c \end{bmatrix}(\tau) d\tau = \begin{bmatrix} < x >_{p,k} \\ < x >_{n,k} \\ < x >_{z,k} \end{bmatrix}(t).$$

$$\frac{d}{dt} \begin{bmatrix} X_{p,k} \\ X_{n,k} \\ X_{z,k} \end{bmatrix}(t) = F^H \begin{bmatrix} < \frac{d}{d\tau} x_a(\tau) >_k \\ < \frac{d}{d\tau} x_b(\tau) >_k \\ < \frac{d}{d\tau} x_c(\tau) >_k \end{bmatrix}(t) - jk\omega_0 \begin{bmatrix} X_{p,k} \\ X_{n,k} \\ X_{z,k} \end{bmatrix}(t)$$

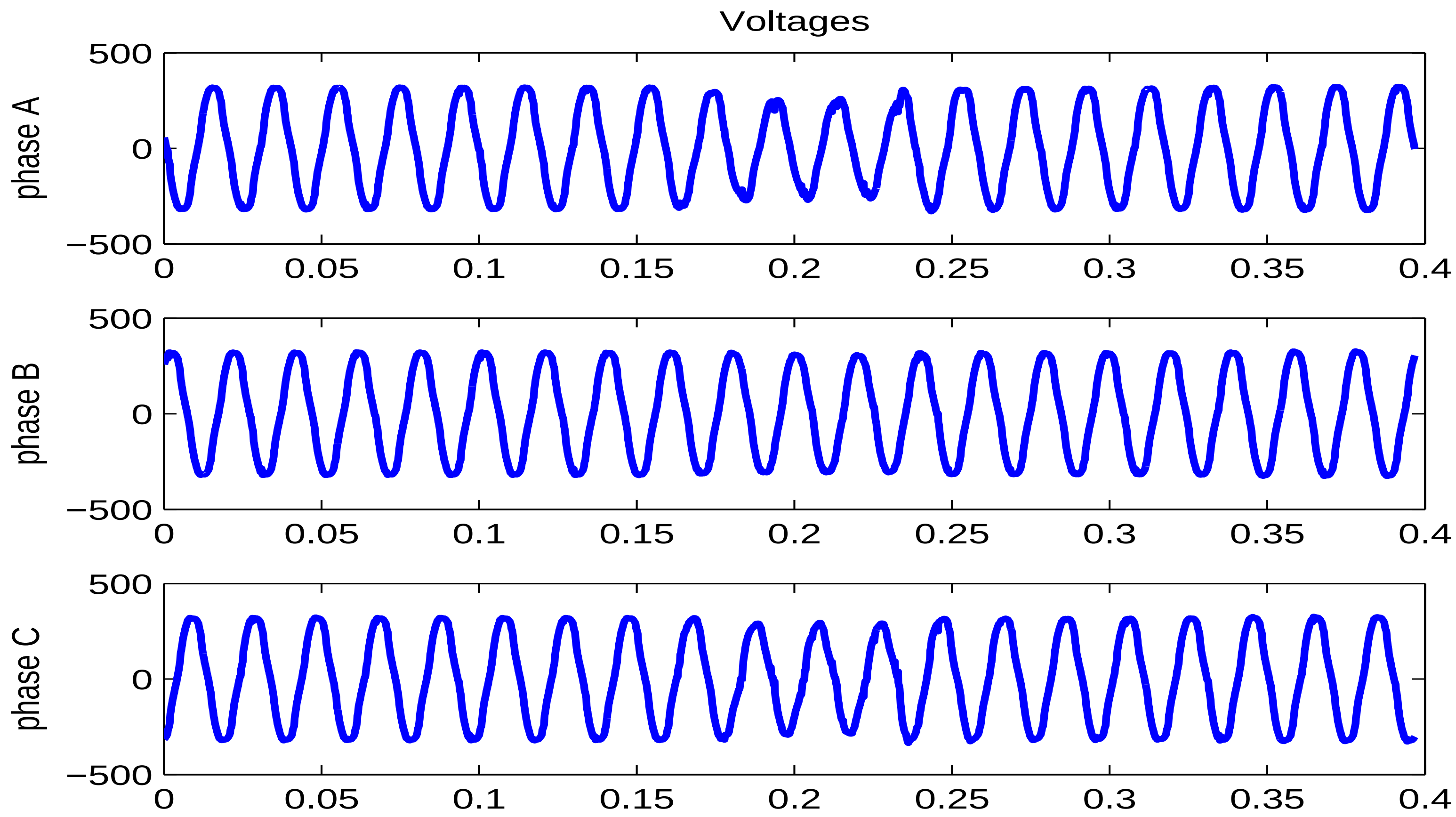
Reactive Power in the Example

Our inductor example

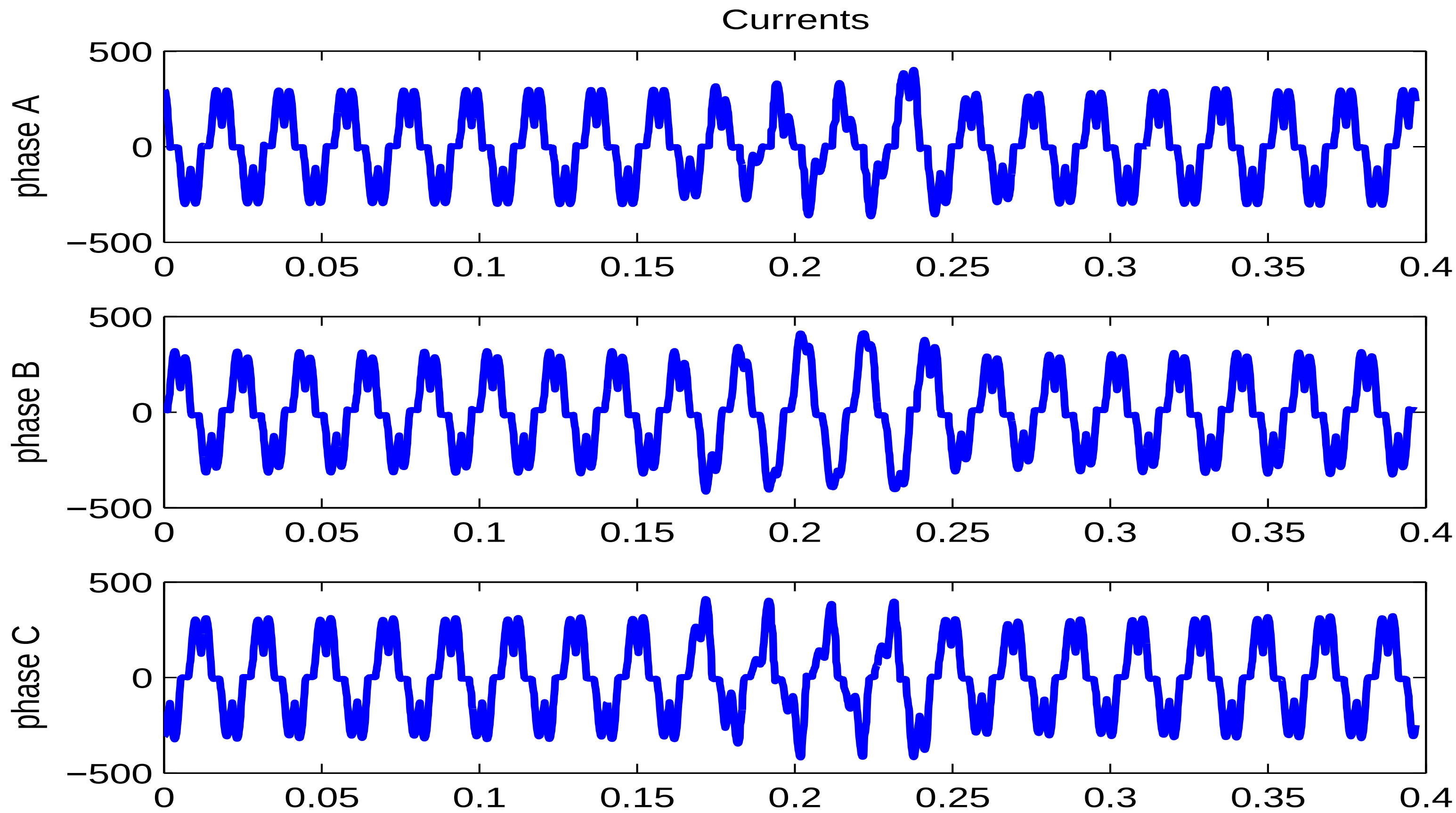


An Industrial Example

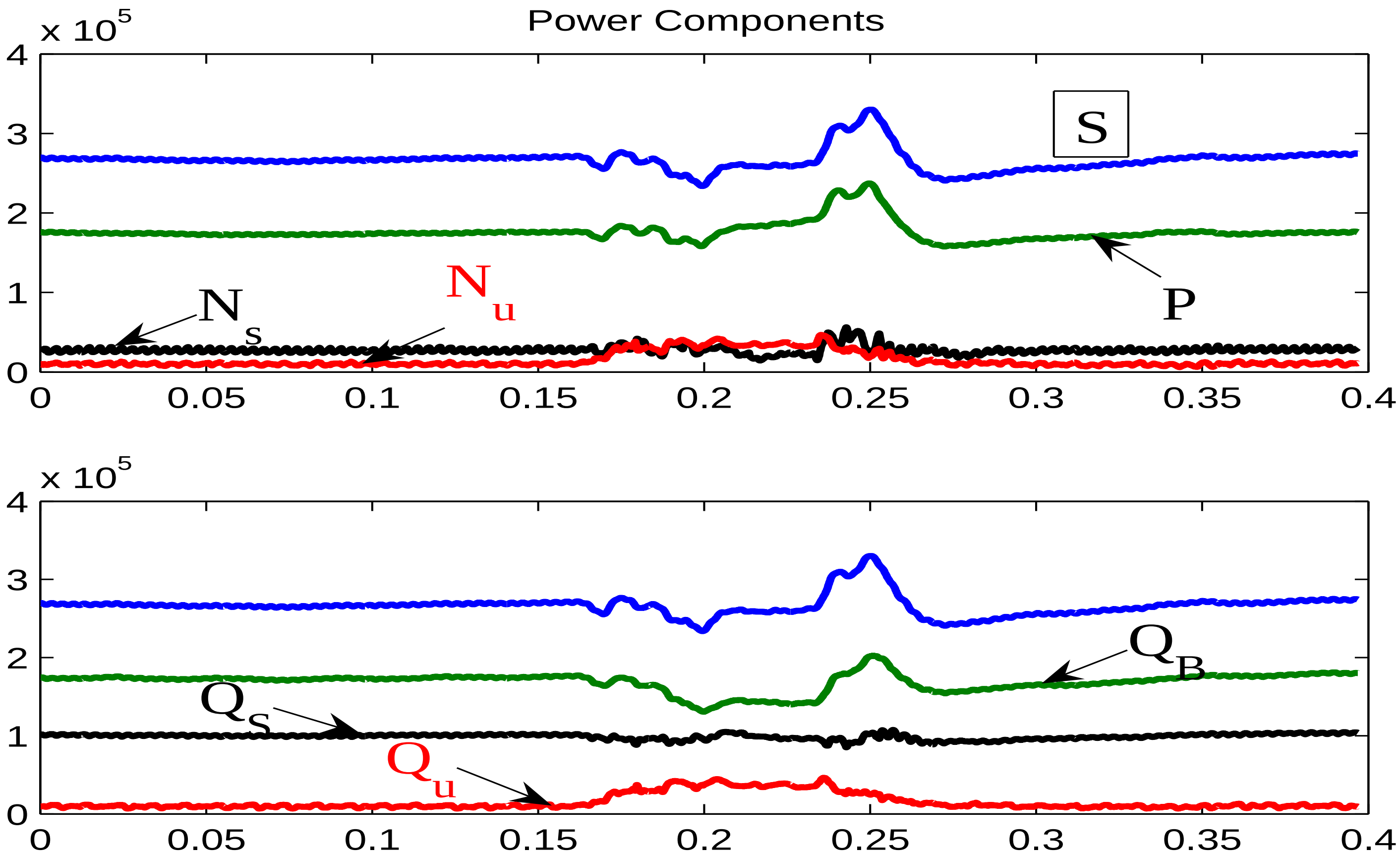
Paper plant transient



Industrial Example (2)



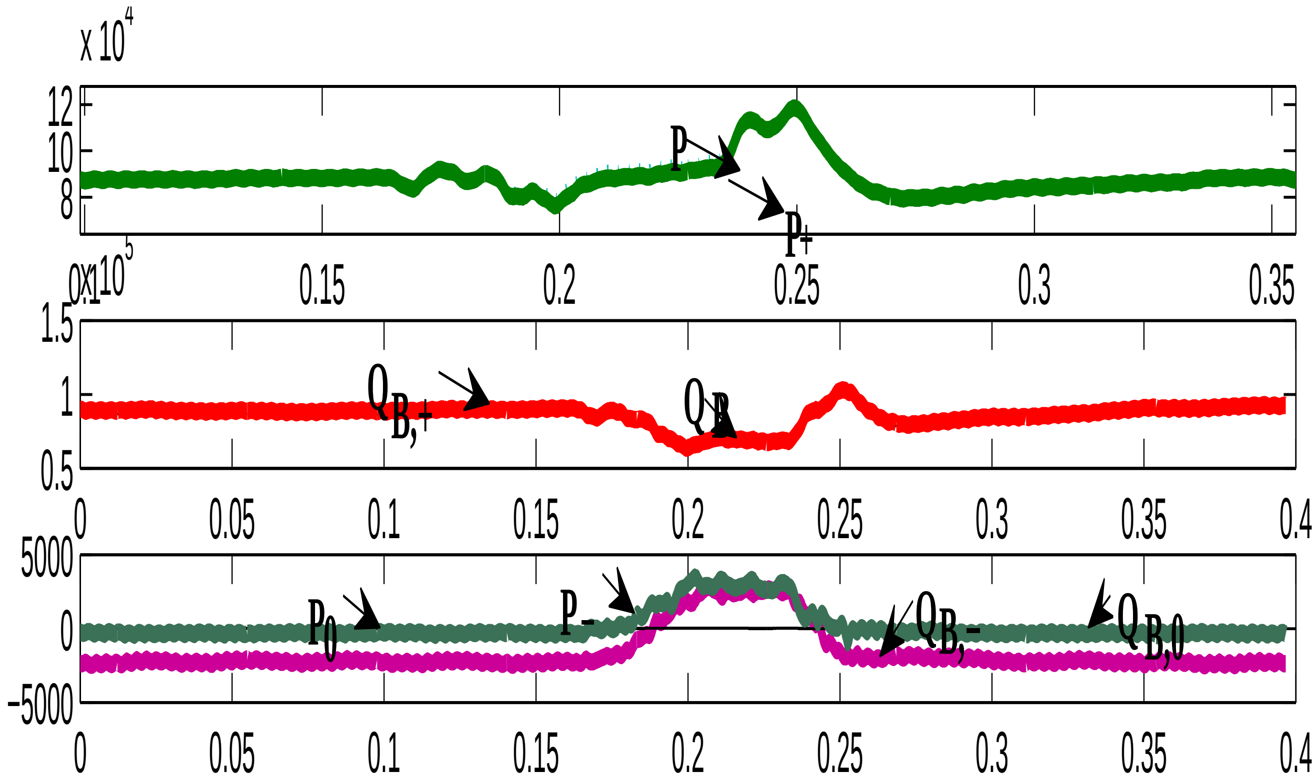
Industrial Example via Phasors (3)



Industrial Example via Phasors (4)

Decomposing signed components - two level decomposition:

$$P = P_+ + P_- + P_0 \qquad Q_B = Q_{B+} + Q_{B-} + Q_{B0}$$



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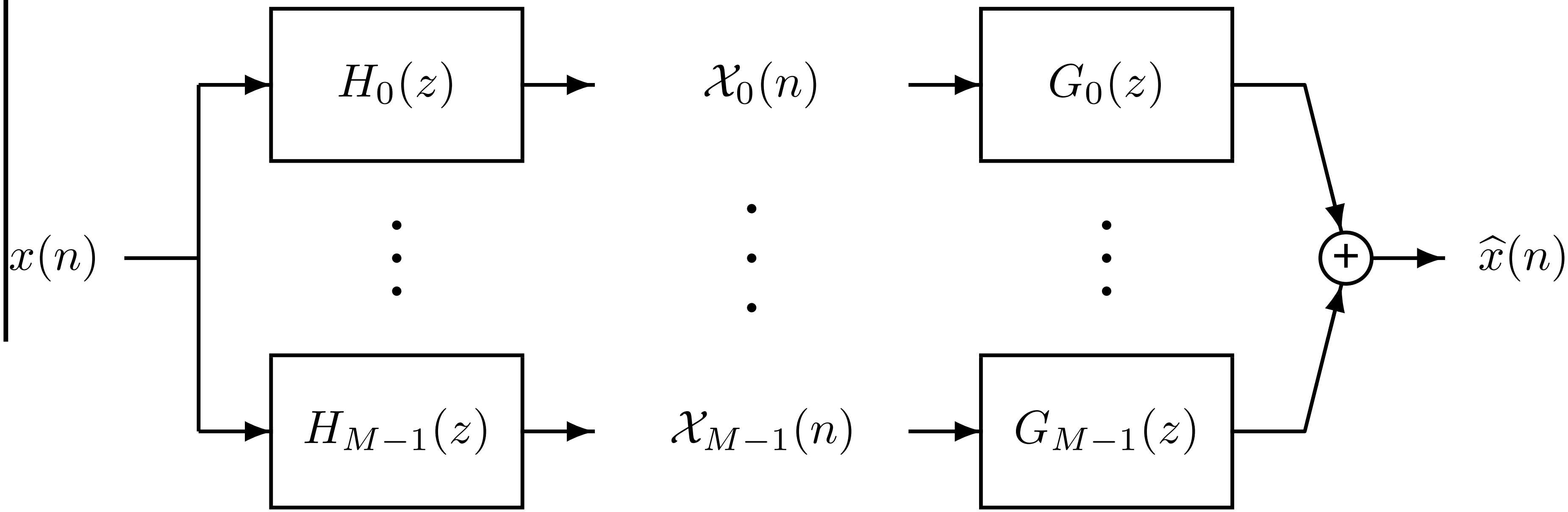
- It has evolved as performance goals and compensation means have changed,
- Reading papers by old masters is a rewarding (and ego-shrinking) experience,
- The electric energy engineering - Ars Longa, Vita Brevis.

Extensions - Phasor Banks

With H. Lev-Ari – in discrete-time, standard definitions lead to (discrete-time) Gabor Transform – non-decimated DFT analysis and synthesis banks:

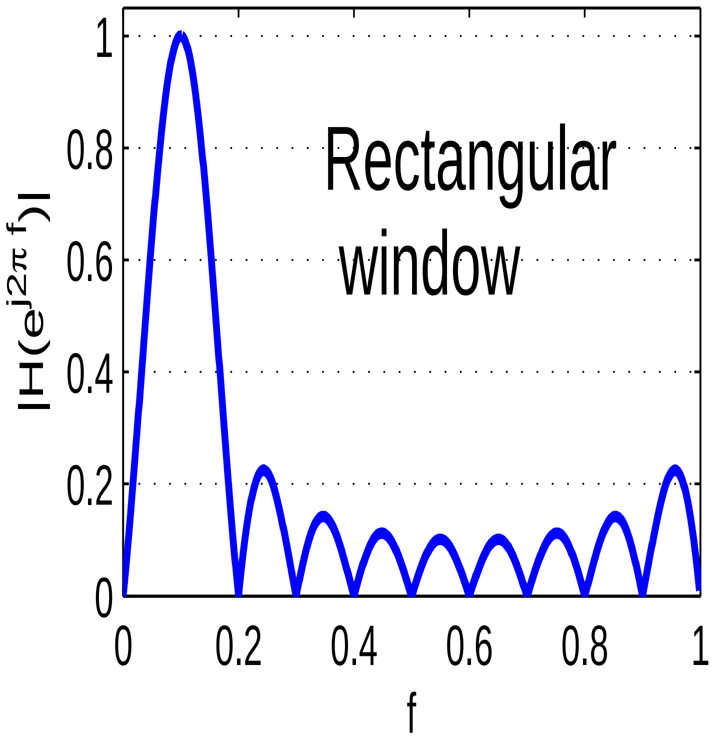


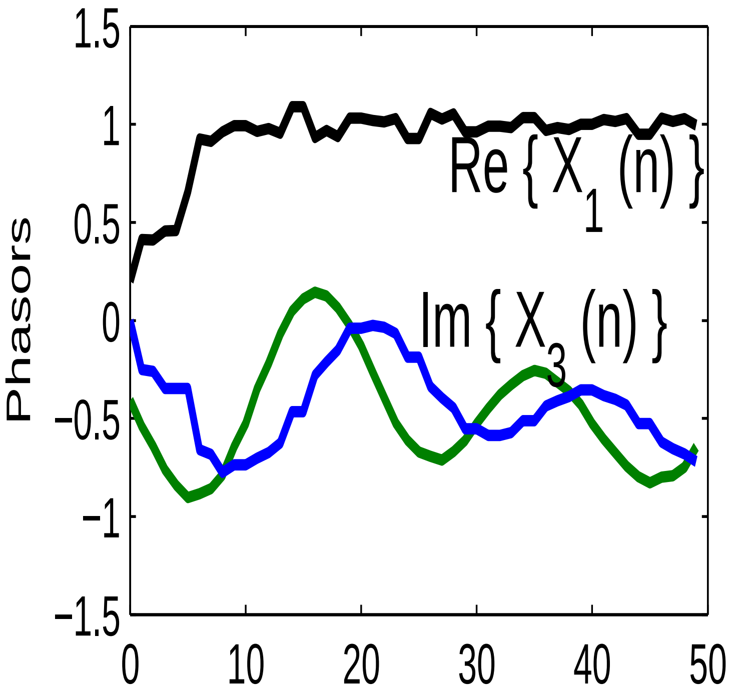
$H_k(z) = H(e^{-jk\omega_0}z)$ $G_k(z) = G(e^{-jk\omega_0}z)$

Phasor Banks (2)

An example - a fundamental and a slowly modulated third harmonic:

$$x(n) = 2\cos(\omega_0 n) + a(n)\sin(3\omega_0 n)$$

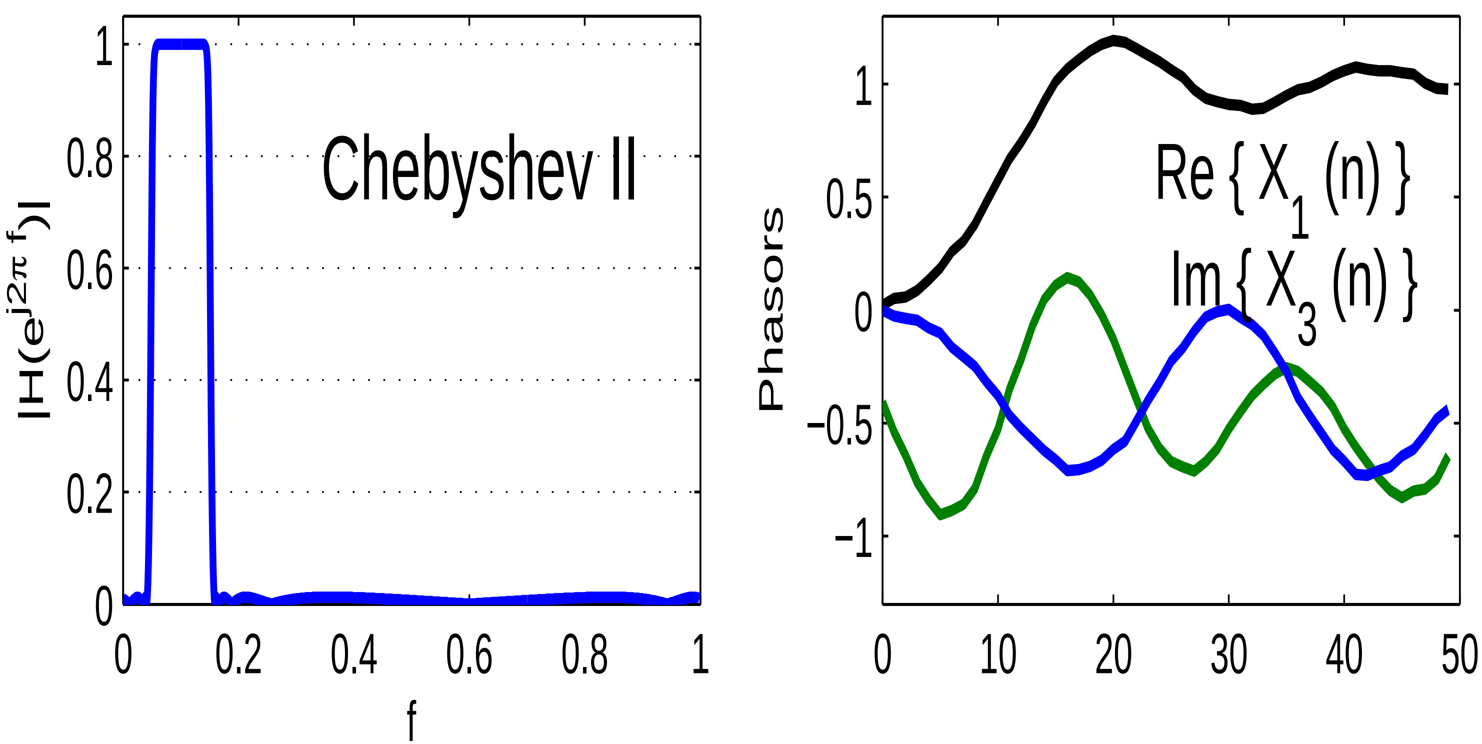




Phasor Banks (3)

The same example:

$$x(n) = 2\cos(\omega_0 n) + a(n)\sin(3\omega_0 n)$$



Other Applications

Power electronics:

- DC/DC converters (model reduction),
- Resonant converters,
- Active filters,
- High-power converters - Unified Power Factor Controller.

Electric Drives:

- Unbalanced electrical machines,
- Torque ripple minimization,
- Position-dependent loads.

Power Systems:

- Unbalanced faults - dynamic symmetrical components,
- Model-based estimation,
- Protection.