Problem 2.34

Define the following random variables as follows: "D_i" represents the event that i defective scopes were found in the 4 that were tested; $0 \leq i \leq 4$. "A" represents the event that the entire batch of 200 scopes are accepted following the random test. By independence the following two conditions must hold in order to accept the entire batch:

- Every defect-free scope must be declared defect-free when inspected; the probability of this is simply $1 - 0.01 = 0.99$.
- Every defective scope that is inspected must be undetected, and, hence, declared defect-free; the probability of this is simple $1 - 0.98 = 0.02$.

The events D_i partition the sample space—they are mutually exclusive and collectively exhaustive with respect to the random testing procedure. Specifically, by way of the random inspection it is possible to find either $0, 1, 2, 3$ or 4 defective scopes. Hence, we can use total probability to solve for the probability of accepting the batch as follows:

$$\begin{aligned} Pr\{A\} &= Pr\{A|D_0\}Pr\{D_0\} + Pr\{A|D_1\}Pr\{D_1\} + Pr\{A|D_2\}Pr\{D_2\} + Pr\{A|D_3\}Pr\{D_3\} \\ &+ Pr\{A|D_4\}Pr\{D_4\} = \sum_{i=0}^{4} Pr\{A|D_i\}Pr\{D_i\} \end{aligned}$$

The conditional probabilities of $A|D_i$ are evaluated are found as follows:

$$Pr\{A|D_i\} = (0.99)^{4-i}(0.02)^i$$

$$\begin{aligned} Pr\{A|D_0\} = (0.99)^{4-0}(0.02)^0 &= 0.96059601 \\ Pr\{A|D_1\} = (0.99)^{4-1}(0.02)^1 &= 0.01949598 \\ Pr\{A|D_2\} = (0.99)^{4-2}(0.02)^2 &= 0.00039204 \\ Pr\{A|D_3\} = (0.99)^{4-3}(0.02)^3 &= 0.00000792 \\ Pr\{A|D_4\} = (0.99)^{4-4}(0.02)^4 &= 0.00000016 \end{aligned}$$

The next step is to determine the probabilities of the D_is. This is a counting problem—we begin by observing that the random indpection involves taking 4 scopes at random from a collection of 200 without replacement. We also note that in the batch of 200 it is known that exactly 5 are truly defective. There are 200 choose 4 different ways to randomly select 4 scopes from the 200. The question is, how many different ways can we choose i defective scopes? The answer is—the product of the number of ways of choosing $4 - i$ non-defective scopes *and* i defective scopes. Hence, the following expression can be evaluated to determine the number of ways to randomly select $0, 1, 2, 3$ or 4 defective scopes respectively: (Note: there are 5 defective scopes, hence, 195 non-defective scopes; selection of i defective scopes implies that $4 - i$ non-defective scopes were selected—hence, the product form.)

$$\binom{200-5}{4}\binom{5}{0}, \binom{200-5}{3}\binom{5}{1}, \binom{200-5}{2}\binom{5}{2}, \binom{200-5}{1}\binom{5}{3}, \binom{200-5}{0}\binom{5}{4}$$

The probabilities are simply evaluated by dividing the number of ways that i defective scopes can be choosen by the total number of possible ways to choose 4 scopes—since these probabilites must be mutually exclusive and collectively exhaustive they must sum to unity, hence, a good check.

$$Pr\{D_0\} = \frac{\binom{200-5}{4}\binom{5}{0}}{\binom{200}{4}} = 0.902984697$$

$$Pr\{D_1\} = \frac{\binom{200-5}{3}\binom{5}{1}}{\binom{200}{4}} = 0.094960905$$

$$Pr\{D_2\} = \frac{\binom{200-5}{2}\binom{5}{2}}{\binom{200}{4}} = 0.002924173$$

$$Pr\{D_3\} = \frac{\binom{200-5}{1}\binom{5}{3}}{\binom{200}{4}} = 0.000030146$$

$$Pr\{D_4\} = \frac{\binom{200-5}{0}\binom{5}{4}}{\binom{200}{4}} = 0.00000077$$

This provides us with all the necessary information to solve the total probability expression given above.

$$\begin{aligned} Pr\{A\} &= Pr\{A|D_0\}Pr\{D_0\} + Pr\{A|D_1\}Pr\{D_1\} + Pr\{A|D_2\}Pr\{D_2\} \\ &+ Pr\{A|D_3\}Pr\{D_3\} + Pr\{A|D_4\}Pr\{D_4\} \\ &= (0.96059601)(0.902984697) + (0.01940598)(0.094060905) + ... = 0.8692 \end{aligned}$$