Inverse Source Problem with Source Energy and Field Energy Constraints

A.J. Devaney

January 15, 2001

1 Problem Formulation

We consider a source ρ confined to a finite source region $\tau = \{\mathbf{r} : r \le a\}$ radiating a scalar field ψ in infinite free space:

$$[\nabla^2 + k^2]\psi(\mathbf{r}) = \rho(\mathbf{r}). \tag{1}$$

We will select the source ρ to radiate a prescribed field ψ everywhere *outside* the source region τ and also to minimize a weighted sum of the source "energy"

$$\mathcal{E}_\rho = \int_\tau d^3r\, |\rho(\mathbf{r})|^2 \tag{2}$$

and the field energy

$$\mathcal{E}_\psi = k^4 \int_\tau d^3r\, |\psi(\mathbf{r})|^2 \tag{3}$$

within the source region. It is not difficult to show that the field energy evaluated over all of space is given by the expression

$$\mathcal{E}_\infty = k^4 \int_\infty d^3r\, |\psi(\mathbf{r})|^2 = \Re k^2 \int_\tau d^3r\, \rho^*(\mathbf{r})\psi(\mathbf{r}). \tag{4}$$

Moreover, the field energy evaluated outside the source region τ is fixed (since the source is selected to radiate a prescribed field outside τ). Thus, minimizing the field energy within τ is equivalent to minimizing the source field interaction term

$$\mathcal{E}_{\rho,\psi} = \Re k^2 \int_\tau d^3r\, \rho^*(\mathbf{r})\psi(\mathbf{r}). \tag{5}$$

We can thus cast our inverse problem as being that of minimizing the sum

$$\mathcal{E} = \mathcal{E}_\rho + \alpha\mathcal{E}_{\rho,\psi} \tag{6}$$

subject to the constraint that the field is prescribed everywhere outside the source region τ where α is a non negative parameter.

As is well known the field everywhere outside τ is completely and uniquely determined by the radiation pattern $f(\mathbf{s})$. Indeed, the radiation pattern uniquely determines the field multipole coefficients a_l^m which, in turn, are related to the source via the formulae

$$a_l^m = \int d^3r\, \rho(\mathbf{r}) j_l(kr) Y_l^{m*}(\hat{\mathbf{r}}). \tag{7}$$

It then follows that the source that minimizes the weighted sum of field and source energies defined in Eq.(6) subject to the constraint of radiating a prescribed field outside τ must minimize the generalized Lagrangian

$$\mathcal{L} = \mathcal{E}_\rho + \alpha \mathcal{E}_{\rho,\psi} + \Re \sum_{l,m} C_l^m [a_l^m - \int d^3r\, \rho(\mathbf{r}) j_l(kr) Y_l^{m*}(\hat{\mathbf{r}})] \tag{8}$$

where C_l^m are a set of Lagrange multipliers with $|C_l^m| > 0$, $\forall l, m$.

1.1 Minimizing the Generalized Lagrangian

The source field interaction energy $\mathcal{E}_{\rho,\psi}$ can be expressed in terms of the source and Green function via the equation

$$\mathcal{E}_{\rho,\psi} = k^2 \int_\tau d^3r \int_\tau d^3r'\, \rho^*(\mathbf{r}) G_D(\mathbf{r}, \mathbf{r}') \rho(\mathbf{r}') \tag{9}$$

where

$$G_D = \frac{1}{2}[G + G^*] \tag{10}$$

is the "Dirac" Green function, with

$$G(\mathbf{r}, \mathbf{r}') = -\frac{1}{4\pi} \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r} - \mathbf{r}'|} \tag{11}$$

being the infinite free space Green function that satisfies the radiation condition. On making use of the expression Eq.(9) for the interaction energy and the definition Eq.(2) for the source energy we find that

$$\begin{aligned} \mathcal{L} &= \int_\tau d^3r\, |\rho(\mathbf{r})|^2 + \alpha k^2 \int_\tau d^3r \int_\tau d^3r'\, \rho^*(\mathbf{r}) G_D(\mathbf{r}, \mathbf{r}') \rho(\mathbf{r}') \\ &\quad + \Re \sum_{l,m} C_l^m [a_l^m - \int d^3r\, \rho(\mathbf{r}) j_l(kr) Y_l^{m*}(\hat{\mathbf{r}})] \end{aligned} \tag{12}$$

On taking the first variation of the above Lagrangian we find that

$$\delta\mathcal{L} = \int_\tau d^3r\, \delta\rho^*(\mathbf{r}) \{\rho(\mathbf{r}) + \alpha k^2 \int_\tau d^3r'\, G_D(\mathbf{r}, \mathbf{r}') \rho(\mathbf{r}') - \sum_{l,m} C_l^{m*} j_l(kr) Y_l^m(\hat{\mathbf{r}})\} + \text{c.c.}$$

where c.c. stands for the complex conjugate of the first term. On setting the first variation equal to zero we obtain

$$\rho(\mathbf{r}) = -\alpha k^2 \int_\tau d^3r'\, G_D(\mathbf{r}, \mathbf{r}')\rho(\mathbf{r}') + \sum_{l,m} C_l^{m*} j_l(kr) Y_l^m(\hat{\mathbf{r}}), \tag{13}$$

which must hold at all space points $\mathbf{r}$ contained in the source region. The above equation constitutes an integral equation that must be satisfied by the desired source.

The above integral equation can be reduced to a differential equation by applying the D'Alembertian operator to both sides of the equation to obtain

$$[\nabla^2 + k^2]\rho(\mathbf{r}) = -\alpha k^2 \rho(\mathbf{r})$$

a result that follows from the fact that the last term in Eq.(13) satisfies the homogeneous Helmholtz equation. The general solution to the above equation can be expressed in the form

$$\rho(\mathbf{r}) = \sum_{l,m} \rho_l^m j_l(Kr) Y_l^m(\hat{\mathbf{r}}) \tag{14}$$

where $K^2 = (1 + \alpha)k^2$ and where the coefficients ρ_l^m are selected to satisfy the constraint; i.e., must generate a prescribed field outside τ. The above equation defines the source within the source region τ. The source must, of course, vanish outside τ.

2 Optimum Source and Radiated Field

2.1 Optimum Source

On making use of the constraint Eq.(7) we find that

$$\begin{aligned} a_l^m &= \int_\tau d^3r\, \rho(\mathbf{r}) j_l(kr) Y_l^{m*}(\hat{\mathbf{r}}) \\ &= \int_\tau d^3r \sum_{l',m'} \rho_{l'}^{m'} j_{l'}(Kr) Y_{l'}^{m'}(\hat{\mathbf{r}}) j_l(kr) Y_l^{m*}(\hat{\mathbf{r}}) \\ &= [\int_0^a r^2 dr\, j_l(Kr) j_l(kr)]\rho_l^m. \end{aligned}$$

On defining the "eigenvalues"

$$\sigma_l^2 = \int_0^a r^2 dr\, j_l(Kr) j_l(kr) \tag{15}$$

we then obtain the desired source

$$\rho(\mathbf{r}) = \sum_{l,m} \frac{a_l^m}{\rho_l^2} j_l(Kr) Y_l^m(\hat{\mathbf{r}}) \tag{16}$$

if $\mathbf{r} \in \tau$ and zero otherwise.

2.2 Radiated Field

To find the field radiated by the optimum source given in Eq.(16) we return to Eq.(13) which we re-write in the form

$$
\begin{aligned}
\rho(\mathbf{r}) &= -\alpha k^2 \int_\tau d^3r'\, G(\mathbf{r},\mathbf{r}')\rho(\mathbf{r}') + \alpha k^2 \int_\tau d^3r'\, G_S(\mathbf{r},\mathbf{r}')\rho(\mathbf{r}') \\
&\quad + \sum_{l,m} C_l^{m*} j_l(kr) Y_l^m(\hat{\mathbf{r}}),
\end{aligned} \tag{17}
$$

where

$$
G_S = \frac{1}{2}[G - G^*] \tag{18}
$$

is the "Schwinger" function. The Schwinger function satisfies the *homogeneous* Helmholtz equation so that we conclude from Eq.(17) that

$$
\psi(\mathbf{r}) = -\frac{1}{\alpha k^2}\rho(\mathbf{r}) + \sum_{l,m} \lambda_l^m j_l(kr) Y_l^m(\hat{\mathbf{r}}) \tag{19}
$$

where the coefficients λ_l^m are linear combinations of the coefficients C_l^{m*} and the expansion coefficients of the Schwinger field (second to last term in Eq.(17)). If we now make use of the expression for the optimum source Eq.(16) we conclude that

$$
\psi(\mathbf{r}) = \sum_{l,m} [-\frac{a_l^m}{\alpha k^2 \rho_l^2} j_l(Kr) + \lambda_l^m j_l(kr)] Y_l^m(\hat{\mathbf{r}}) \tag{20}
$$

which is valid everywhere within the source region $\tau = \{\mathbf{r} : r \leq a\}$.

To find the coefficients λ_l^m of the "free field" part of the field ψ given in Eq.(20) we use the fact that this field must reduce to the field exterior to the source region when $r = a$. The exterior field admits the multipole expansion

$$
\psi(\mathbf{r}) = \sum_{l,m} a_l^m h_l(kr) Y_l^m(\hat{\mathbf{r}})
$$

from which we then conclude that

$$
-\frac{a_l^m}{\alpha k^2 \rho_l^2} j_l(Ka) + \lambda_l^m j_l(ka) = a_l^m h_l(ka).
$$

On solving for λ_l^m we find that

$$
\lambda_l^m = \frac{1}{j_l(ka)} [\frac{j_l(Ka)}{\alpha k^2 \rho_l^2} + h_l(ka)] a_l^m. \tag{21}
$$