Towards an Understanding of Last Encounter Routing in Ad Hoc Networks

by

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A thesis submitted in conformity with the requirements
for the degree of Master of Science
Graduate Department of Computer Science
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Abstract

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2004

A wireless ad hoc network is a collection of nodes that communicate with each other in the absence of a supporting infrastucture. Approximate information protocols are routing protocols that utilize approximate, inaccurate information on nodes in order to make routing decisions. An example of such a protocol is the Last Encounter Routing (LER) protocol. We propose a network model under which we prove fundamental properties of the LER protocol and make progress towards a rigorous analysis of its behaviour.

στον πατέρα μου,
στη μητέρα μου
και στην αδερφή μου

Acknowledgements

I wish to express my gratitude to my supervisor, Professor Peter Marbach, for working closely with me throughout the time that this thesis was written; I would not have been able to go through this without his guidance. Also, I cannot thank enough my second reader, Professor Bálint Virág, for the most fruitful discussions that we had; indeed, this thesis would not have been possible without his help, which he offered to me with such openhandedness. Professor Jeremy Quastel provided me with a proof of Proposition 5.2, for which I am most grateful, and Prof. Wayne Enright referred me to several sources on Volterra equations, which were indeed very useful. I would also like to thank the members my group, especially Xiaoyang Guan and Ehsan Foroughi, for the discussions that we had and for their comments on this thesis. Finally, I would like to thank my family and my friends for supporting me and bearing with me during the past year.

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Chapter 1

Introduction

1.1 Ad hoc networks

A *communication network* is a collection of entities that exchange information with each other. The networks we shall study will be *wireless*, in the sense that the physical medium of communication will be wireless. Furthermore, we will consider *mobile* networks, meaning that the relative positions of the entities that constitute the network change through time.

For the purposes of this thesis, a network can be viewed as a *graph* $G(V, E)$ where V is the set of the aforementioned communicating entities and E is a set of unordered pairs of elements in V. We will call the communicating entities *nodes*, whereas the elements of E shall be referred to as *links*. If a link between two nodes u and v exists, i.e. $(u, v) \in E$, then u and v will be *adjacent* or *neighbouring* nodes, or just *neighbours*.

A link between two nodes indicates that direct communication between them is possible. The networks we will deal with are *packet-switched*, in the sense that the communication between nodes happens through the exchange of information through messages of information, which are called *packets*. Two nodes that are not adjacent cannot exchange packets directly. Indirect communication between them however is possible if nodes *relay* or *forward* packets on behalf of other nodes; this act is called *routing*. A node which generates a packet of information is called a *source* whereas a node that is the final recipient of the packet is a *destination*.

An *infrastructured* or *hierarchical* network is a network in which a distinction exists between nodes that generate or receive packets, called *hosts*, and nodes that route packets, called *relay nodes* or simply *routers*. The algorithm that specifies how routers relay

packets is called a *routing protocol*. A network in which no nodes are designated as routers but all hosts relay packets on behalf of other hosts is called an *infrastructureless* or an *ad hoc* network.

Recent advance in wireless technology has led to an increased interest in ad hoc networks. Applications for ad hoc networks were initially envisioned in a setting where a permanent infrastructure would be impossible to deploy, such as military or emergency search-and-rescue operations in an inhospitable terrain, or in situations were temporary networks, easy to set-up and dismantle, would be desirable, such as a meeting or a conference. However, the proliferation of cheap, ubiquitous wireless devices has provided further incentive for using ad hoc techniques, since they allow for large, self-organizing networks comprising of autonomous mobile nodes.

Many interesting problems arise while designing an ad hoc network. The focus of this thesis will be routing. There are two aspects of ad hoc networks that make routing a challenging task. First of all, due to the absence of a hierarchy or any central authority, routing must be done in a distributed manner; that is, nodes should decide where to forward an incoming packet based only on local information, like who their neighbours are, their geographical position e.t.c. and information contained in the packet, e.g. identifiers (addresses) of the destination and the source. Furthermore, the fact that nodes are mobile leads to frequent and abrupt changes in the topology of the network. An ad hoc routing protocol should be able to function under such circumstances.

1.2 Routing protocols for mobile ad hoc networks

Early approaches on routing in ad hoc networks were adaptations of routing protocols for traditional, infrastructured networks in a mobile setting. These protocols are called *table-driven* or *proactive* [37, 8, 34, 7, 24]. In such protocols, each node maintains consistent information about the topology of the network. Usually this means maintaining a *routing table* with *next-hop* information. This table has an entry for each destination in the network, which contains the neighbour to which a packet should be forwarded in order to reach that destination. The creation and maintenance of routing tables are based on the well known Bellman-Ford algorithm [26]. Consistency is preserved by broadcasting maintenance packets either at fixed time intervals or when a topology change occurs (e.g. when a link ceases to exist), or both. The main drawback of this approach is that the maintenance packets generated constitute a very large portion of the traffic of the

system, especially if nodes are mobile and the topology changes frequently. In other words, proactive protocols suffer from a large *overhead*, i.e. traffic consisting of packets that do not contribute to the exchange of information between nodes.

In order to deal with this overhead, *demand-driven* (or *source-driven*) or *reactive* protocols were developed [38, 25, 36]. In reactive protocols, routing tables do not exist per se; whenever a source node needs to send a packet to a destination, it initiates a query called a *routing request* in order to locate it. This query happens in the form of *flooding*: every node that receives a routing request packet for the first time and is not the destination forwards it to all its neighbours. Assuming that the network is connected, the destination will eventually be reached. The routing request packet contains information about the nodes that it has passed from, and the destination can use this to send a *route reply* to the source. With this reply the source learns a path to the destination (namely, the one that the routing request packet followed). This entire process is called the *route discovery* phase. Once route discovery is accomplished, the source node can send subsequent packets over the established route. Some reactive protocols also define route maintenance mechanisms, so that, if a link breaks while packets are being sent over a route, this route can be repaired locally. However, even if this is not possible, the protocol can always fall back to the route discovery phase to seek for a new route and resume transmission.

The advantage of reactive protocols is obviously that, in contrast to table-driven protocols, no overhead is incurred by routing table maintenance. This makes them fare better with high mobility environments (see [10, 5, 42]). On the other hand, the overhead generated during the routing discovery phase can be quite significant. However, as the route discovered can be used for transmitting subsequent packets and nodes very rarely exchange a single packet, the cost of the overhead is amortized during long sessions of communication.

The above protocols can be characterized with the general term of *topology-based* routing, since the information used to route packets is topology related, e.g. next-hops. In contrast to topology based routing, *position-based* or *location-based* routing uses location information in order to route a packet.

Position-based routing was suggested as an alternative for topology-based routing because of the benefits gained by working with positions instead of paths. It relies on the following three assumptions: first, all nodes lie on a plane. Second, their connectivity is modeled by a *unit graph*, i.e. two nodes share a link if their Euclidean distance is less

than a radius r, the *transmission radius*, usually taken to be one. These are realistic assumptions since they apply to real-life wireless networks to some extent, the second one perhaps less than the first one. Their predominance can be seen by the fact that they are quite often used in the analysis of topology-based networks, although they are not necessary. The third assumption is that each node is either aware of its own location or it can compute it quickly and efficiently. The most frequently encountered argument in bibliography for justifying this assumption as realistic is that nodes can make use of GPS, the Global Positioning System [27]. This is a satellite service widely used by marine vessels, expeditions e.t.c. to obtain their current geographical longitude and latitude.

Under these assumptions, the position of a node, i.e. its coordinates on the plane, can be used to route a packet towards the destination. The locality of links in unit graphs intuitively suggests that forwarding the packet towards the direction of the destination should lead to a successful routing. This idea is very enticing, given that location information is much more concise (it is merely two coordinates) than entire path information. Thus, instead of dealing with topology, in which entire paths need to be maintained, one can only record position information.

A position-based protocol consists of two parts: the *location service* and the *forwarding strategy*. Assuming that nodes are positioned on a plane and that all are aware of their locations and the destination's location, a forwarding strategy specifies how a packet is to reach the destination. For example, the obvious greedy strategy is to forward the packet to the neighbour which is closest to the destination. However, it is easy to see that this strategy does not guarantee delivery even in static networks. As a matter of fact, defining a forwarding strategy that guarantees delivery in a mobile network is still an open problem [20].

On the other hand, forwarding strategies assume that the source node knows the destinations position at all time. This, in short, is a transformation from the problem of maintaining path information (in topology-based routing) to maintaining position information, and is not easy to solve. In most location based protocols, a location service is defined in order to address the problem, most of the times independent from the forwarding strategy. A location service specifies how a source determines the location of the destination. Due to the nature of ad hoc networks, a location service has to be distributed; in other words, the information of the current positions of nodes in the network cannot be stored in a centralized manner. For example, the solution in which one node maintains all the location information of all other nodes is not considered valid.

Consequently, designing a location service is quite involved and has been the subject of extensive research [21, 32].

1.3 Approximate information protocols

The protocols that we will describe in this thesis abide by a general scheme. The intuition behind devising such a scheme lies in using useful features from both proactive and reactive protocols. This combination should, ideally, be done in such a way that the overhead incurred is less than the respective overhead for both of the above categories.

Proactive protocols ensure that route discovery is achieved with no overhead whatsoever, since routing tables maintain exact routing information about all destinations. However, a large overhead is demanded for the maintenance of information in each node's routing table. On the other hand, the exact opposite holds for reactive protocols; although routing tables are not used and thus no overhead is necessary for the maintenance of information, route discovery is quite costly.

The protocols we will analyze can be seen as lying in between proactive and reactive approaches. In these protocols, although nodes maintain routing tables, the information stored in them will not have to be updated as often as in table-driven protocols. This will result in inconsistency - entries in routing tables will not provide accurate information about the destination and will inevitably lead to routing failures. Upon a failure, the protocols resort to flooding, just like demand-driven protocols. However, instead of flooding in order to find the destination, as in the route discovery phase of demand-driven protocols, flooding will be directed towards locating nodes with more accurate information on the destination. We shall refer to these protocols as *approximate information protocols*, due to the inherent inaccuracy of the information maintained by nodes in the network.

Before giving a general definition of routing in approximate information protocols, we will try to illustrate the basic concepts involved with an example. Consequently, we will see how these concepts can be extended to a more general scheme. Finally, we will discuss possible features of approximate information protocols.

1.3.1 An example

The protocol we will describe was devised by Grossglauser et al. and is based on last encounter routing (LER), a concept also first presented by the same authors [19]. It is also the protocol we will attempt to analyze in this thesis. As it has slight modifications from EASE, the original protocol presented in [19], we shall refer to it under the more generic term "the LER protocol".

Assume that the nodes forming the network reside on a plane. Each node i maintains a routing table RT_i containing the coordinates of every other node j at some time in the past. Furthermore, for every entry of its routing table, each node i maintains in a table M_i the elapsed time since the destination was at that position. In other words, if the entry RT_{ij} at the routing table of node i is the position of node j at time t ago, then its associated entry in table M_i will be $M_{ij} = t$.

The information RT_{ij} in the routing table can be considered as an approximate, inaccurate *estimation* of the current position of node j, i.e. the position of node j some time in the past is, in some sense, an estimation of j's current position. In this context, time t can also be perceived as a *metric of accuracy*, i.e. as a measure of how close to the actual value this estimation is.

A natural question to ask is how the entries of these routing tables get updated. The approach followed by Grossglauser et al. is a minimalistic one, in terms of the overhead involved. Nodes update the j-th entry of their routing table every time they are within transmission radius of node j. Grossglauser et al. called such an event an encounter, hence the name last encounter routing (LER). Note that this method of updates implies that the "metric of accuracy" $M_{ij} = M_{ji}$ is none other than the elapsed time since the last encounter of i and j.

A packet can be routed from a node s to d using the information in the RT tables as follows. Initially, the source node s can send it to the position RT_{sd}, using any of the location-based forwarding strategies that exist in literature. If the destination is not reached with this forwarding, the protocol resorts to flooding, as described previously. However, instead of looking for the destination itself, it looks for any node that contains a more accurate entry regarding the destination. In our definition of accuracy, a node n has a more accurate entry than s if $M_{nd} < M_{sd}$. In other words, while flooding, a node n that encountered d later than s did is sought for.

After locating such a node n, the protocol resumes forwarding as if the packet was

originated at n. This process is repeated and the protocol alternates between forwarding and flooding until the destination is reached. An illustration of this behaviour can be seen in figure 1.1, which is explained in detail in the next section. Note that, at each flooding, the position of the destination known is always better than the one before in terms of accuracy, given that accuracy is defined as the elapsed time since an encounter. Furthermore, such accuracy improvements happen only through flooding.

1.3.2 A generalization

The above example can be generalized to a wider class of protocols, to which we will refer from now on as approximate information protocols. In such protocols, each node i maintains a routing table RT_i, where RT_{ij} is the j-th entry of the table at node i, containing information that can be used to route a packet from node i to node j. To each entry RT_{ij} corresponds a metric of accuracy M_{ij} ($M_{ij} \in [0, +\infty]$), where table M_i is also maintained by node i. By definition, $M_{ii} = 0$. In general, both tables (of routing information and accuracy metrics) will be functions of time, denoted by $RT_{ij}(t)$ and $M_{ij}(t)$. We will say that the j-th entry of node i is more accurate than the j-th entry of node k at time t if $M_{ij}(t) < M_{kj}(t)$. Equivalently, we will say that in such a case node i has better information than k about destination j at time t. When using the above phrases, we shall drop "at time t" whenever t can be derived from context. In the LER example, the information stored in routing table entry $RT_{ij}(t)$ of node i at time t was the coordinates of a node at some time t' in the past, whereas the metric of accuracy was the elapsed time $t - t'$ since t'. Note that the metric of accuracy at time t imposes a partial ordering among entries of nodes referring to the same destination. This ordering is partial because nodes may have equal accuracy.

Although the nature of the accuracy metric M will depend on the actual protocol, it is important that if $M_{id} = 0$ a packet can be forwarded from i to d without failure or any form of flooding. For example, in LER $M_{id} = 0$ indicates that i is a neighbour of d, and hence a packet can be sent to d without failure. This is necessary for the protocol to converge, as we will discuss in the end of this section. Furthermore, a nice property that the metric of accuracy can have (though not necessary for convergence) is that reaching the destination from a node should be "easier", according to some cost function, than from a point with worse accuracy.

The fact that nodes maintain routing tables is reminiscent of proactive protocols.

However, as indicated by the existence of the metric of accuracy M, information stored in a routing table may be inaccurate. Hence, the updating mechanism employed by the protocol can be less expensive than the one used in proactive protocols. In other words, by deliberately allowing information of routing tables to become inaccurate, we can design update mechanisms that are inexpensive in terms of overhead. The one used in last encounter routing for example incurs no overhead whatsoever.

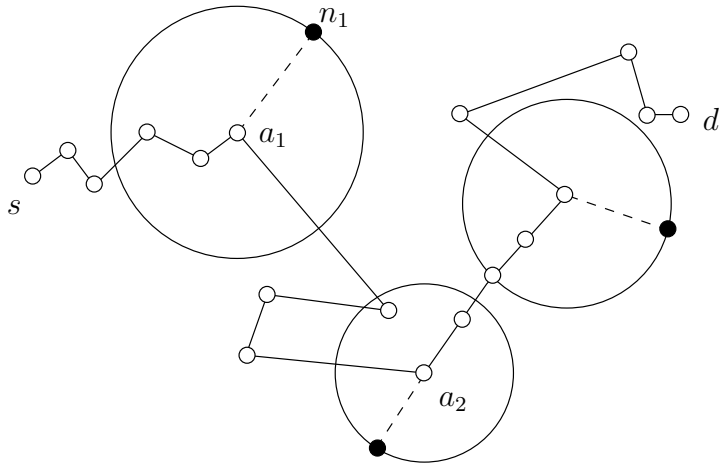

Figure 1.1: An illustration of approximate information routing. Node s sends a packet to d. Initially, it attempts to route the packet using the information (RT_{sd}) it maintains in its routing table. However, because this information is inaccurate, forwarding fails at node a_1. The protocol thus resorts to flooding, looking for a node that has better information about d than the currently available. Such a node is n_1, and based on the information it provides ($RT_{n_1 d}$), the packet is forwarded to a_2. This process is repeated until destination d is reached. Nodes a_1 and a_2 are anchor points and node n_1 is a carrier node.

The existing routing information, although inaccurate, may be used in order to forward a packet. The forwarding method depends very much on the kind of information that is maintained in routing tables. For example, in the LER protocol described above any location-based forwarding algorithm will do. However, since routing information is approximate, it is quite possible that a routing attempt based only on this information might fail. In the LER example, the destination may not physically be at the coordinates provided. In this case, approximate information protocols resort to flooding, just like reactive protocols. However, unlike reactive protocols, instead of the destination itself, more accurate routing information about it is sought for upon flooding. In a more general forwarding scheme, assume that node i forwards a packet to d using its (approximate) information RT_{id}. Because of the inherent inaccuracy of this informa-

tion, forwarding fails and the packet only reaches node a. In that case, node a floods the network querying for a node n that contains more accurate information RT_{nd} -i.e. $M_{nd} < M_{id}$. This information can then be used in a new attempt to forward the packet to the destination until a new failure occurs and the above process is repeated until the destination is reached.

This process is illustrated in figure 1.1. Clearly, the protocol alternates between two phases: a *forwarding* phase, in which inaccurate information is employed to forward the packet to the destination, and a *flooding* or an *information discovery* phase, in which better information about the destination is sought for. Figure 1.2 gives a more formal description of the routing algorithm in approximate information protocols.

```
Route a packet p from s to d
{
        i = s
        forward p towards d using RT_id
        let a be the node reached with this forwarding
        while (a ≠ d)
        {
                query the network through flooding initialized at a
                        until a node n is reached such that
                        it satisfies a halting condition. Node
                        n should be such that M_nd < M_id;
                i = n
                forward p towards d using RT_id
                let a be the node reached with this forwarding
        }
}
```

Figure 1.2: An algorithm for approximate information routing

We give additional definitions that will assist our analysis in the future. Nodes $a_1, a_2, \ldots$, where the protocol fails to forward the packet (because routing information is inaccurate) and consequently resorts to flooding, shall be referred to as *anchor points*[1]. Anchor points are flooding points, i.e. flooding is initiated at them. Nodes that satisfy

[1]The definition is from EASE [19].

the flooding halting condition shall be called *carrier* nodes[2]. Carrier nodes contain information that can be used to resume packet forwarding after a forwarding failure. In the algorithm of figure 1.2, we denote with RT_{id} the approximate information available to the algorithm and with M_{id} the corresponding measure of accuracy for this information. We shall call these the *current routing information* and the *current accuracy* respectively. Note that current information and current accuracy change only through flooding in the aforementioned general scheme. In LER, an anchor point is the position of the destination at t time ago, where t is the current accuracy. Carrier nodes are all the nodes with accuracy better than t, i.e. all the nodes that met the destination after it passed the anchor point.

The routing scheme described in figure 1.2 is left deliberately vague. The nature of information stored in routing tables, the way they can be used to forward packets toward the destination and how such forwarding strategies may fail are not defined. Furthermore, the way flooding is implemented is also not defined and, in particular, nor is the halting condition. We shall address these issues in the next section, where possible attributes of approximate information protocols will be discussed in detail. The only restriction imposed on the halting condition is that the carrier node that will be located by flooding must have better information about the destination than the one currently available. More formally, if the current accuracy is M_{id} and

$$C = \{n : n \text{ satisfies the halting condition}\}$$

is the set of carrier nodes, then $C \subseteq \{n : M_{nd} < M_{id}\}$.

This restriction implies that, if the area flooded in each flooding step is bounded, the protocol converges to an accuracy of zero. Assume that routing happens very fast relatively to the movement of the nodes in the network, and that we can thus consider nodes static while routing takes place. If we define as a distance metric from the destination the metric of accuracy, every flooding results, according to the above restriction, to locating a node closer to the destination in the accuracy space. More intuitively, the protocol is closing in the destination with every flooding step, where the metric of proximity is, in fact, the metric of accuracy. Since the accuracy is lower-bounded by zero, this indicates that the protocol converges to zero in the accuracy space. Note that, by definition, nodes that have accuracy equal to zero can reach the destination without flooding. Furthermore, the restriction that the area flooded is bounded can be easily achieved if the halting

[2]In EASE [19], these nodes are referred to as *messenger* nodes.

condition is such that the destination also satisfies it and thus there always exists a node at which flooding will stop (namely, the destination).

1.3.3 Features of approximate information protocols

Approximate information protocols can be constructed from the aforementioned general scheme by specifying various aspects of the scheme, like the nature of information stored, the forwarding strategy, the update policy e.t.c. These aspects are described here in detail.

Information Stored. Routing tables must contain information that can assist in forwarding a packet to a destination. Two examples are *next-hop* and *location* information.

Next-hop information is used in traditional routing protocols for static networks. In such a scenario, each node maintains the neighbour to which a packet must be forwarded in order to reach the destination. Alternatively, a node can store the entire path from it to the destination; Such redundant information can be used to compute optimal paths.

As in the LER example, location information, i.e. the coordinates of the destination on the plane, in the spirit of location based protocols, is an alternative. Assuming that the metric of accuracy is *elapsed time* (see below), storing location information can be defined as $RT_{ij}(t) = P_j(t - M_{ij}(t))$, where $P_j(t)$ is the position of node j at time t. Although forwarding a packet towards the destination using only location information is more elaborate than forwarding when using next-hops, and guaranteeing delivery is still an open problem even when the exact position of the destination is known [20], location information seems to degrade less rapidly than next-hop information. Topology may change rapidly even in a non highly mobile network, whereas there exist intuitive arguments, such as the *distance effect* (see in DREAM [1] in section 2.3.1), indicating that location information useful even if it is considerably out of date.

Using location information is interesting also for a different reason. As we have seen, location based protocols consist of a forwarding strategy and a location service, which maintains and provides information on the positions of nodes in a distributed manner. Designing a location service that is decentralized is a challenging task. However, approximate information protocols that use location information circumvent this problem. In short, each node has its own, inaccurate location service, and the location information become more refined as the protocol proceeds.

Among the protocols that exist in literature and that will be presented in the next chapter, EASE [19] and DREAM [1] use location information, GREP [13] uses next hop information and FSR [24] uses full path information. FRESH [12], oddly enough, stores no information in a routing table, because it assumes that an estimation of the position of the destination that may assist to routing is none other than the position of the node itself (i.e. $\forall i, j \quad RT_{ij}(t) = P_i(t)$, where $P_i(t)$ is the position of node i at time t). The only table FRESH maintains is the metric of accuracy.

Forwarding Strategy. Closely related to the nature of information stored is the forwarding strategy employed and the corresponding failing condition. For example, if next-hop information is available, the obvious forwarding strategy is to send the packet towards the neighbour indicated by the routing table. Failure to forward then consists of a hop having become obsolete, i.e. the node that is indicated by the routing table as a next-hop is not a neighbor of the current node[3].

Forwarding using location information is much more complicated, and in some sense, it is still an open problem. There is an abundance of strategies [33, 45], most of which guarantee delivery only under certain assumptions. This is not a problem in our case however, since the protocol does assume that failure of the forwarding strategy is possible and can recover from it by entering a flooding phase. Note that, as location information is inaccurate, the most one can do, given that the forwarding strategy does not fail for any other reason, is to send the packet to the inaccurate location which approximates the destination's position. This is ambiguously defined, but we can make it more specific by stating that the forwarding strategy proceeds until the node closest, according to Euclidean distance, to the above position is reached. If the destination is not within a transmission radius from that node, we assume that a failure has occurred.

An improvement that can be applied to all protocols that we describe is *greedy forwarding*. In our general scheme we get closer to the destination in the accuracy space only when we flood. It is possible however that, as we forward a packet, we pass through a node that contains "better" information about the destination than the currently available. Intuitively, this information should be used and the current level of accuracy should be set as the one provided by that node. In this case, accuracy improvements do not happen only by flooding steps but also during forwarding. Though this may improve the

[3] which is thus an anchor point

performance of the protocol, it makes analysis much harder, so we will not take it into consideration in our later analysis.

GREP [13] and FSR [24] use by default greedy forwarding, whereas the greedy version of EASE is called GREASE [19]. FRESH [12] and DREAM [1] involve no forwarding whatsoever, since packets progress only through flooding.

Update mechanism. The update mechanism defines when and how updates take place. A proactive approach would be for each node to send periodically "hello" messages containing the necessary routing information either to the entire network or a subset of it, e.g. by using a time-to-live (TTL) field on the packet header. In this case, total overhead should take into consideration such hello packets as well.

A less expensive approach would be to piggyback information on data or other packets -like flooding packets- used by the protocol. Piggybacking can be done either aggressively or moderately. In aggressive piggybacking, each packet carries routing information about all the nodes where it has passed from. Each node that relays a packet "sniffs" routing information from the header of the packet and then adds its own information on the packet before forwarding it. This approach however does not scale well, as the size of the packets increases with the length of the path it traverses. The non aggressive approach would be to piggyback only the packet originator's information on the header, thus keeping the packet length constant but also making updates less frequent.

Piggybacking is particularly interesting in approximate information protocols for yet another reason. If updates happen using this mechanism, increased traffic should intuitively lead to more updates, and hence make information stored in flooding more accurate. On the other hand more accurate information should indicate that the protocol should resort to flooding less frequently, which thus results in decreased traffic load. Although at the present stage of our research it seems hard to model such a behaviour, intuition suggests that there exists a balancing mechanism in approximate information protocols using piggybacking as far as network load is concerned.

A less aggressive approach to updating is using *last encounters*, as happens by definition in LER. In this scheme, suggested by Grossglauser et al. [19], updates about a destination happen only when it is within transmission distance, i.e. the source and the destination become neighbours. There are at least two reasons why such an approach is interesting. First of all, the behaviour of the protocol, in terms of how costly it is to route a packet from a source to a destination, is independent of traffic load: updates are

completely independent from how many -and what kind- of packets are disseminated. This simplifies analysis to a great extent. The second reason is that such a protocol also exhibits a balancing behaviour, in terms of the speed with which nodes move: Since the information maintained is location information, if nodes move very fast this information should rapidly become degraded (less accurate). Fast moving nodes should thus be hard to locate. On the other hand, intuitively, a fast moving node should encounter more nodes than a slowly moving one, and therefore high mobility also leads to the generation of many carrier nodes, thus facilitating future route discoveries. This was noticed by Grossglauser et al. who indicated that in LER "mobility helps" routing. This is an important observation, because in most routing protocols mobility is addressed as an obstacle to the routing mechanism. However, since the speed of nodes affects the behaviour of the protocol in the above two contradicting ways, it is not clear which one of the two prevails and how the protocol reacts under high mobility.

On the other hand, protocols utilizing last encounter routing suffer from a certain lack of realism: it is highly unlikely that in a real-life ad hoc network all nodes will meet each other physically. Only under specific types of movement can one assume that all nodes (or at least most of them) become eventually adjacent to each other. This means that LER will result to flooding quite often in a real-life scenario. Note that this does not apply to protocols that use piggybacking; it is quite possible that a node broadcasts packets to all nodes in a network. In fact, since approximate information protocols do resort to flooding, which is a broadcasting mechanism, such a scenario is plausible.

GREP [13] uses scalable piggybacking as an update mechanism, whereas EASE [19] and FRESH [12] use a last encounter scheme. DREAM [1] and FSR [24] broadcast update packets.

Metric of accuracy. As already noted, the protocol must define some metric of accuracy for the information maintained at nodes. Assuming information is accurate at some moment in time, a reasonable assumption would be that a measure of accuracy is the *elapsed time* since that moment. This implies that all nodes maintain time entries corresponding to routing data, which indicate the time when these data were accurate. Thus

$$M_{ij}(t) = \text{ elapsed time since } RT_{ij}(t) \text{ was accurate}$$

In all packet-based update schemes, time entries can be replaced by *sequence numbers*

of packet headers. Whenever a packet that contains routing information is generated by a node, instead of a timestamp that indicates when it was transmitted it may receive a sequence number, which is then stored in the routing tables of nodes that receive it along with corresponding the routing entries. The idea behind this is that the metric of accuracy is unnecessary: only the partial ordering of entries, implemented by the one-to-one correspondence between possible entries and natural numbers (the sequence numbers), is required. Note however that the metric of accuracy, apart from providing a total ordering of estimates, also quantifies how much more accurate one estimate is compared to another. Other than that, the two approaches are equivalent in terms of how the protocol behaves.

EASE [19], FRESH [12] and DREAM [1] use elapsed time as a metric of accuracy, whereas GREP [13] and FSR [24] use sequence numbers.

Flooding. An important aspect of the protocol is how flooding is implemented during route discovery. One reason is that if update packets are not used, flooding during route discovery is in fact the only substantial overhead incurred by the protocol. On the other hand, if piggybacking is used as an update mechanism, flooding contributes, sometimes a lot, in how updates take place.

Assuming the network is modeled by a unit graph, in which two nodes are adjacent iff their distance is less than a constant, flooding consists of forwarding arriving packets to all neighbours. A time-to-live (TTL) field in packet headers can be used to limit flooding. By repeatedly flooding using an increasing TTL, one can implement flooding in concentric circles; however this increases the cost of flooding, in terms of the number of packets exchanged, from linear with respect to the number of nodes flooded to quadratic. An alternative approach would be to forward a packet only to one neighbor, perhaps under some probability distribution over neighbors, thus doing a *random walk query*. Such a query is obviously less effective but also less expensive than usual flooding.

Another aspect of flooding is the actual *halting condition*, i.e. when should a node not forward a flooding packet and instead reply. As defined by our general schema, queries look for nodes with better information about the destination than the current one. These nodes, which are called carrier nodes as we have already stated, are exactly those nodes that satisfy the halting condition. An extreme halting condition, employed by reactive protocols, is to flood looking for the destination itself. One plausible halting condition would be to flood until a node that contains any information which is better than the

current one is found. By our definition of approximate information protocols, this is the halting condition that defines the largest set of carrier nodes, since all other halting conditions are a subset of this. If a metric of accuracy exists, then we can impose more stringent halting conditions like flooding until a node which has at most γ $(0 < \gamma < 1)$ times better metric than the one of currently available information. Such a condition may possibly increase the expected flooding per step, since it decreases the number of carrier nodes, but may also lead to fewer steps needed in order to reach the destination. Another approach would be to flood a constant area, and use the best reply (according to the measure of accuracy) of the ones provided.

Both GREP [13] and FRESH [12] flood under the maximal flooding condition. EASE [19] on the other hand floods for node that contain information twice as good as the current. This corresponds to a $\gamma = \frac{1}{2}$ according to the terminology used above. FSR [24] does not use flooding at all, whereas DREAM [1] implements what the authors refer to as "directed" flooding (see section 2.3.1 in the next chapter for more details).

1.4 Purpose and results

The purpose of this thesis is to analyze the performance of the LER protocol under a mathematical model. Our work thus consists of the presentation of a model under which the protocol shall be studied and of the analysis which we perform based on this model.

The model we propose consists of a network and a cost model. According to our network model, nodes form a Poisson field and move according to independent Brownian motions. Our cost model consists of a function $Q(T)$, which quantifies the behaviour of our protocol in terms of the accuracy T of a source node. This function will be the solution of a Volterra integral equation. This equation will be defined in terms of two functions, $G(T)$ and $p(t, T)$, which are, with respect to the rest of our model, the expected one-step flooding area and the probability density of the improved accuracy, respectively. We will be interested in computing an upper bound on the performance measure $Q(T)$ for large values of T. Our purpose is thus to compute $G(T)$ and $p(t, T)$ based on our network model and then find an asymptotic upper bound for function $Q(T)$ by solving the corresponding Volterra equation.

Unfortunately, as we will see, the description of the above two functions in terms of the network model is far from trivial and could not be achieved within the time limitation under which this thesis was written. The main result of this thesis will therefore be an

asymptotic upper bound only for $G(T)$. A derivation of $p(t,T)$ and a bound of the cost function under our model will be left as future research.

On the other hand, we will describe a simpler model within the present thesis, under which both $p(t,T)$ and $G(T)$ can be computed, and we will prove in terms of it an asymptotic upper bound for function $Q(T)$. This will give us intuition on how to approach both $G(T)$ and $p(t,T)$ in our original model, but will also be useful in demonstrating how results acquired by solving the aforementioned Volterra equation can be interpreted.

1.5 Overview

The rest of the thesis is organized as follows. In chapter 2 we present the related work that has been done in this area. In chapter 3 we briefly review various mathematical notions that will assist our analysis. The model of the network we will use is defined in detail in chapter 4. In chapter 5 we prove a set of fundamental properties that can be derived from our model and which we will use in our later analysis. In chapter 6 we provide a simpler model than the one proposed in chapter 4 and prove under it an upper bound for the function that models the cost. The methods and the results of this simplified analysis are used in chapter 7 to prove an asymptotic upper bound for $G(T)$ and to speculate on the behaviour of $p(t,T)$ under our original model. Finally, we conclude in chapter 8.

Chapter 2

Related Work

As discussed in the introduction, approximate information protocols can be viewed as a combination of proactive, table-driven and reactive, demand-driven approaches. We will therefore briefly review the work done in these two directions. Furthermore, we will review the results on position-based routing since LER is also closely related to such protocols: information stored in LER routing tables is in fact estimations of node locations, and the forwarding strategies employed can be any location-aided forwarding strategies existing in literature. Finally, we will present several protocols that have approximate information routing features.

2.1 Topology-based routing protocols

Topology-based protocols have been widely studied and a vast amount of examples exist in literature. We will present here the most prominent proactive and reactive protocols and we will try to illuminate through them the fundamental concepts that have gained attention in this particular area of research. For more information on topology-based routing protocols we refer the reader to the survey by Royer et al. [42]. Also, a performance comparison can be found in [5].

2.1.1 Proactive or table-driven protocols

A classic proactive routing protocol is the Destination-Sequenced Distance-Vector routing protocol (DSDV) devised by Perkins et al. [37]. DSDV is one of the most prominent examples where high mobility leads to overwhelming levels of overhead. Nodes maintain

complete routing tables with next-hop information, and update this information according to the well known Bellman ford algorithm [26]. Each node periodically transmits updates including the contents of its entire routing table (called *full dump* packets) or just those entries that have been altered through time (*incremental* packets), the former being less frequent than the latter. Using sequence numbers guarantees that the most up-to-date information is considered. If two paths to a destination have the same sequence number, the one that is shorter is preferred; to implement this, nodes maintain (and transmit in update packets) the length of their paths along with next hop information.

In the Wireless Routing protocol (WRP), by Murthy et al. [34], nodes transmit updates about the state of links only to their neighbours. These transmissions happen periodically but also upon link failure. Since packets are not broadcast, the update mechanism is much more elaborate than in DSDV. Nodes have to maintain information on the possible paths to a destination from all of their neighbours. This includes the length of all such paths and the first, second and second-to-last hop. If a link among two nodes is lost, all the adjacent nodes are notified. If this link affects a path that these nodes were using - an event they can detect through second-hop information, they look for an alternate path among their neighbors and also notify the node that had the failed link about the existence of these paths. In order to accomplish this, each node in WRP maintains very detailed information on the update messages sent and the replies pending for each update message. Although WRP handles the problem known as "counting-to-infinity" and achieves loop freedom very efficiently by doing aggressive updates and consistency checks, as in DSDV, high mobility can cause significant packet overhead.

Global State Routing (GSR), by Chen et al [7], is similar to DSDV in the fact that it updates routing table entries according to how recent they are and to WRP in that updates are not broadcast but only sent to neighbours. On the other hand, in GSR each node maintains information on the entire graph that defines the network, not just next hops, hence the name "Global State".

Cluster-head Gateway Switch Routing (CGSR), proposed by C.-C. Chiang et al. [8], imposes a loose hierarchy in the ad hoc network. Groups of nodes are joined into a cluster, amd a cluster-head is elected among them in a distributed manner. Extra care is taken so that cluster-head reelections do not happen too frequently, since they consume resources of the system. The underlying routing scheme is the same as in DSDV, however CGSR exploits the existing hierarchy to minimize overhead. More specifically, nodes in different clusters exchange packets through their corresponding cluster-heads, and thus routing

among nodes is reduced to routing among cluster-heads, which are by definition less than the total number of nodes in the network. The authors also argue that organizing nodes in clusters has other benefits as well, since it enables reuse of common resources (like transmission bandwidth and encoding) into different, non adjacent clusters. However CGSR still presents the scalability problems of DSDV: in a large network, the overhead becomes increasingly large either within the same cluster (when few, large clusters are used) or in the overlying cluster-head network (when many clusters containing few nodes are used).

Iwata et al. [24] propose a protocol similar to CGSR called Hierarchical State Routing (HSR). In this protocol, nodes are organized in clusters and elect cluster-heads as in CGSR. However, cluster-heads are also organized in super-clusters, which also elect higher level cluster-heads, and so on. This creates a multi-level hierarchy, where cluster-heads of the previous level are members of clusters in the next level. However, this does not address the problem of scalability noted earlier. The reason is that high mobility hinders the consistency of the hierarchy; in other words, large overhead is needed to maintain the proper organization of nodes in clusters and super-clusters, otherwise many routing failures will happen and packets will be dropped. The authors of HSR note this trade-off between overhead and throughput.

2.1.2 Reactive or demand-driven protocols

One of the most widely studied demand-driven routing protocols is the Ad-hoc On-demand Distance Vector routing protocol (AODV), by Perkins et al. [38]. Its core concept is the core concept of reactive routing protocols: Nodes do not maintain routing table entries unless they actually participate in a currently active route. When a source node needs to establish a route towards a destination, it forwards a routing request (RREQ) packet to all its neighbours looking for that destination. Its neighbours repeat the process until the destination or a node with a "fresh" route towards the destination is reached, thus flooding the network with RREQ packets. While the RREQ packets traverse the network, nodes that receive them create temporary routing table entries with next-hop information associated with the source node. In these entries they store the neighbor from which the packet arrived. Hence, they establish a reverse path towards the source. When a RREQ packet reaches the destination for the first time, a RREP is sent to the source using the reverse path that the RREP followed. The destination will

reply only to one of the possible RREQ packets it may receive, and the RREP will use a similar mechanism to the one described above to establish a path from the source to the destination (namely, the path that the original RREQ followed). A similar procedure takes place if, instead of the destination, a node with a "fresh" path to it is reached. The above process we just described is the route discovery phase mentioned in section 1.2.

All routing information recorded in nodes in the above fashion (i.e. through the dissemination of RREQs and RREPs) expire after a certain time period if they remain unused. Node entries are thus temporary. This helps eliminate "stale" paths. The "fresh" paths we mentioned above are paths defined by entries that have not yet expired.

AODV handles link failures by falling back to route discovery: when a link breaks, the node where the failure occurred propagates back to the source a link failure notification. The source then initiates a new flooding in order to establish a new path to the destination.

The Dynamic Source-vector Routing protocol (DSR), devised by Johnson et al. [25], is also a prominent reactive protocol. Its main differences from AODV are the following. First of all, each node maintains much more information. Instead of one path per destination, many different paths are maintained. In that sense, DSR has a *routing cache* instead of a routing table. Furthermore, entire paths are stored, as opposed to AODV where only next-hop information is stored. During the route discovery phase, nodes that forward RREQ packets append their identification numbers to the RREQ packet. Hence, every node that receives a RREQ can update its cache on all nodes from which the packet passed, whereas in AODV only next-hop entries towards the source are updated. Another difference is that the destination replies to *all* routing request and not just the first. Finally, upon link failure, notification packets are propagated backwards towards the source, however, instead of reinitiating route discovery by default, the source can use one of its other cached routes. Note also that routes do not expire after some time and thus may become stale since failures at inactive routes are not monitored.

As noted by Royer et al. [10], DSR is much more aggressive than AODV, since it generates and disseminates much more information. Its route discovery phase incurs a greater overhead than AODV, since all RREQs are replied, and the packets involved have headers that increase with the size of the network. On the other hand, DSR can handle link failures more gracefully, since it does not resort necessarily to the route discovery phase again. For a thorough comparison of the two protocols we refer the reader to [42] and [5].

Another protocol considered as a benchmark in demand-driven protocols is the Temporally Ordered Routing Algorithm (TORA), by Park et al. [36]. It is the result of the merging of a proactive and reactive algorithm [15, 9]. TORA has a novel approach on storing and maintaining nodes. Once routes are established, each node considers its adjacent edges either as incoming or outgoing, with respect to a destination. All edges in the network are thus directed. The direction is such that all edges and the nodes form a directed acyclic graph (DAG) rooted at the destination. This imposes a partial ordering of nodes with respect to the distance in hops from the destination, and an associated metric among nodes (a "height", with the destination being the lowest point). Upon a link failure, if a node used that link to reach destination, it can reset its height with respect to its neighbors, thus reversing the direction of edges. This reversal can be propagated until a new path is established. We refer the reader to [36] for the exact details of this technique, which the authors call *link reversal*.

TORA has many interesting properties not existing in the aforementioned on-demand routing protocols. By default, multiple paths are maintained for each destination. Furthermore, upon link failure the necessary updates are localized and only happen to nodes that are immediately affected. TORA also does not focus on the cost of established routes; finding as many routes as possible is considered more important than establishing "good" routes.

Contrary to TORA, both AODV and DSR prefer during the route discovery phase routes that have the minimum number of hops. Alternate protocols have been suggested that focus on different metrics of the value of routes in ad hoc networks. For example, the Associativity Based Routing (ABR) protocol, by Toh [48], and the Signal Stability-based Adaptive routing protocol (SSA), by Cube et al. [11], can be seen reactive protocols where routes generated during routing discovery are weighted according to a different metric than next-hops. Both protocols are directed towards maintaining routes that are more probable to be long-lived.

In ABR, each node periodically transmits beacon messages to all its neighbours. When a node receives such a message, it increases its associativity measure on the source of the beacon. Neighbours with high associativity are probably moving slowly, whereas nodes with low associativity are probably moving fast and the link between them is more likely to be short-lived. Associativity measures are set to zero when a link ceases to exist. Upon route discovery, the destination replies to the routing request that has traversed the path with the highest overall associativity. Hence, among paths with equal number of

hops the one that is more "robust", in the sense that its links are more stable, is selected.

SSA adopts a similar approach. However, the quality of a link is indicated by the intensity of the reception signal. Links are thus characterized as "strong", if the signal of the adjacent node is strong, and "weak" otherwise. Routing requests are only forwarded over "strong" links, thus eliminating the generation of paths to the destination which might be weak and therefore short-lived.

2.1.3 Hybrid protocols

Hybrid protocols, that combine ideas of both proactive and reactive methods, have also been proposed. The most common approach is to employ proactive routing locally, e.g. within a small distance from the source, where maintaining accurate routing tables is tractable, and reactive routing globally. An example is the Zone Routing Protocol [22] by Haas et al. which separates the network into zones and employs proactive routing within a zone and reactive routing between nodes in different zones. Note that approximate information protocols are also hybrid protocols, although in a different sense than the one described above.

2.2 Position-based routing protocols

Designing an ad hoc, distributed location service for position-based routing is a challenging task, as stated in section 1.2. As we have already discussed, the nature of approximate information protocols makes a location service unnecessary: all nodes maintain position information about all possible destinations, which may of course be inaccurate. We therefore do not discuss the location services proposed in literature in detail. Apart from the references listed below, a survey on several proposed location services can be found in [33].

In brief, one way the issue of location service has been addressed in the past is by storing the position of a node in many locations and creating multiple, redundant copies, which are not however all kept up-to-date. This idea leads to the organization of the location service in *quorums* [21, 46], a concept widely used in databases and distributed information systems. Another approach is the use of a global hash function to identify where location information should be stored [17, 44]. A similar approach is also used in the Grid project [32].

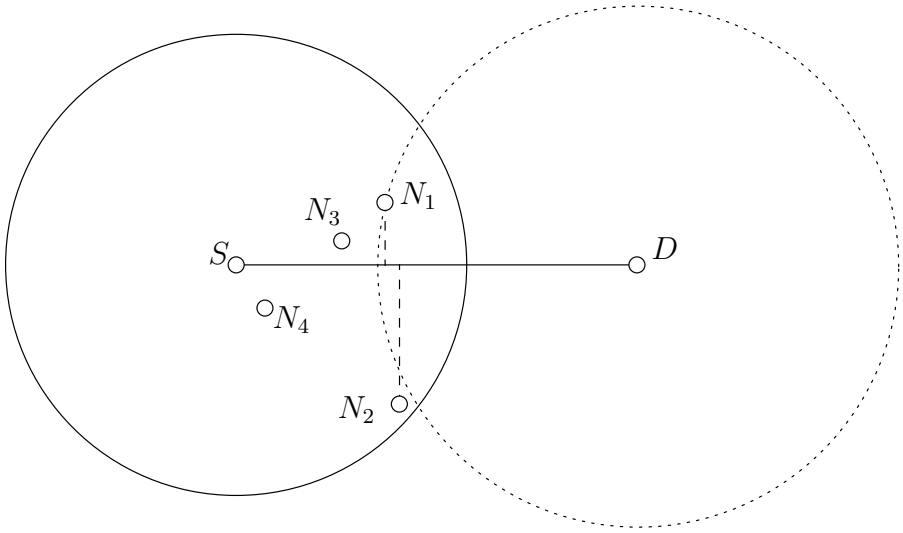

Figure 2.1: Greedy Forwarding. The circle around S designates its transmission range. N_1 is the closest neighbor to destination D, whereas N_2 is the most forward within radius, N_3 is chosen by compass routing and N_4 is the nearest with forward progress.

The simplest forwarding method using location information is using a greedy strategy. In such a method, a node forwards a packet to one of its neighbours based only on its current position and the position of the destination and no other information about the topology of the network. Several greedy forwarding strategies have been proposed. In the following, we will denote with S the node that makes the routing decision, with N the neighbour of S to which the packet is forwarded and with D the destination. In all of the following protocols, only neighbours N of S that are closer to the destination D than S are considered. Finn [14] first suggested the simple greedy scheme, in which S forwards the packet towards the neighbour that is closest to the destination, i.e. N is such that the distance between N and D is minimized (node N_1 in figure 2.1). Takagi et al. [47] suggested the most forward within radius scheme. In this scheme, N is such that the projection of the segment ND on the line that passes from S and D is minimized (node N_2 in figure 2.1). The compass routing method, by Kranakis et al. [31], also employs the line between the source and the destination: the neighbour N that minimizes the angle $\angle DSN$ is chosen (node N_3 in figure 2.1). All of the above schemes aim at minimizing the number of hops necessary to reach the destination. A different approach is the nearest with forward progress scheme, by Hou et al. [23], in which the nearest neighbour that is closer to the destination than the current node is chosen (node N_4 in figure 2.1). Such a scheme is useful in a setting where nodes are able to adapt their transmission range

in order to avoid packet collisions. Also, choosing any neighbour that is closer to the destination arbitrarily [35] is also useful when one wants to minimize the computational load and thus also the delay of forwarding.

However, greedy forwarding strategies fail if no neighbour is closer to the destination than the current node, although a path to the destination exists. Several methods have been proposed in order to address this issue. One example is the face-2 algorithm, by Bose et al. [4], which is used by the Greedy Perimeter Stateless Routing protocol (GPSR), by Karp et al. [28]. This algorithm guarantees packet delivery, given that a path to the destination exists and that the nodes remain static (or, any changes to their positions are insignificant) during the time necessary to route a packet. We refer the reader to the original papers for further details.

Forwarding strategies that utilize (loose) hierarchical features of a protocol have been proposed. For example, the Terminodes project [2] routing happens proactively if the destination is within a certain hop distance, otherwise greedy position based routing is used. A similar approach is used in the Grid project [32].

For more information on position-based protocols, we refer the reader to the surveys on the subject by Mauve et al. [33] and by Stojmenovic [45].

2.3 Approximate information protocols

2.3.1 DREAM

The Distance Routing Effect Algorithm for Mobility (DREAM), proposed by Basagni et al. [1], is a location based protocol that bears some interesting approximate information properties. The authors of the original paper [1] describe it as a hybrid between proactive and reactive protocols that uses position information. As acknowledged by the authors of EASE [19], the basic concepts of DREAM are very close to the core ideas of EASE. Technically, it is not an approximate information protocol as defined by the general scheme in section 1.3.2, since its route discovery phase consists only of a single flooding. In other words, it does not alternate between the information discovery phase and the forwarding phase as approximate information protocols do. However, it does combine proactive and reactive features and can otherwise be described with the terminology we used for approximate information protocols in general.

Under this terminology, in DREAM each node i maintains a routing table RT_i with

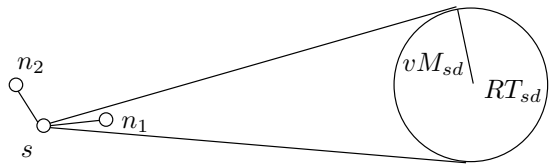

Figure 2.2: Directed flooding. Node s knows the position of node d at M_{sd} time ago and its maximum speed v. Using this information it can compute the above conic region. It the forwards a packet only to its neighbour n_1 and not to n_2, which lies outside the region.

location information about all other nodes, just as LER. The metric of accuracy M_{ij} is elapsed time since entry RT_{ij} was accurate, hence $RT_{ij}(t) = P_j(t - M_{ij}(t))$, where $P_j(t)$ is the position of node j at time t. Updates happen with the broadcasting of update packets[1] containing the current location of the node that generated them, which are transmitted periodically. However, broadcasting is constrained, i.e. it does not cover the entire network. A time-to-live (TTL) mechanism is employed to implement this. "Short-lived" update packets, i.e. packets that are sent to nodes within a certain small hop distance from the source, are transmitted frequently, whereas "long-lived" packets that reach nodes at a larger distance are sent less often.

As stated above, DREAM differs from the generic scheme of approximate information protocols in the way routing discovery takes place. Instead of alternating forwarding and flooding phases, DREAM performs a single "directed" flooding. Given that the maximum speed of the destination is known, each node can use its metric of accuracy and its estimation on the position of the destination to compute a disk in which the destination must be. It then forwards a RREQ only to its neighbours that are positioned in the conic area defined by it and this disk, as seen in figure 2.2.

Although DREAM is not an approximate information protocol, it has interesting features very much related to such protocols. An important observation made by the authors of [1] is that nodes that are further away from the destination need to know its position with less accuracy. This is due to the fact that the further two nodes are, the slower is the respective angular speed of the relative rotation of one around the other. This is referred to by the authors as the "distance effect". As far as approximate information protocols are concerned, the "distance effect" demonstrates a nice property between the distance between two nodes and the metric of accuracy: although it is plausible to assume

[1] these are called control packets in [1].

that accuracy decreases with distance, so does the level of the accuracy necessary to establish a route. Furthermore, DREAM is the first protocol in which a location service is not necessary, for the exact same reasons it is not needed in approximate information protocols.

2.3.2 FSR

The Fisheye State Routing protocol (FSR), by Iwata et al. [24] is a modification of the Global State Routing protocol (GSR), by Chen et al. [7], that adds approximate information features to it. However, like DREAM, it is not formally an approximate information protocol as defined in section 1.3.2, since it does not have any reactive properties at all: it considers routes valid even if they are inaccurate, allowing thus a margin for packet failure. In approximate information terminology, it consists only of a single forwarding phase -which is exactly the opposite of what happens in DREAM.

In FSR, as in GSR, each node maintains global topology information: the entire graph representing the network is stored in each node. As in GSR, each link is associated to a time at which it was accurate. In approximate information notation, each node i maintains a routing table RT_i, where RT_{ij} is an entire path from i to j. The metric of accuracy M_{ij} is a list of individual accuracies (elapsed times)[2], one for each link in the path RT_{ij}. Note that, since no flooding takes place, the metric of accuracy is only used to distinguish old entries during updates.

As in GSR, updates in FSR happen through the dissemination of update packets. However, contrary to GSR, each node sends information to other nodes with frequency that varies with respect to their distances in next hops from it. As seen in figure 2.3, a node sends (and receives) packets to (from) nodes with one-hop distance more often than to (from) nodes with two-hop distance, e.t.c. Thus the graph maintained in each node is increasingly inaccurate with respect to the hop distance from the node.

As noted above, FSR has only a forwarding phase, which is basically the routing mechanism of GSR: packets are forwarded according to the shortest paths computed on the global graph maintained on each node. The intuition provided by the authors on why this protocol should work is very much similar to the "distance effect" claim made in DREAM. The accuracy of topology information necessary for successful routing

[2]Technically, instead of elapsed times FSR, as GSR and DSDV use incremental packet sequence number, which are however, as discussed in section 1.3.3, equivalent to elapsed times

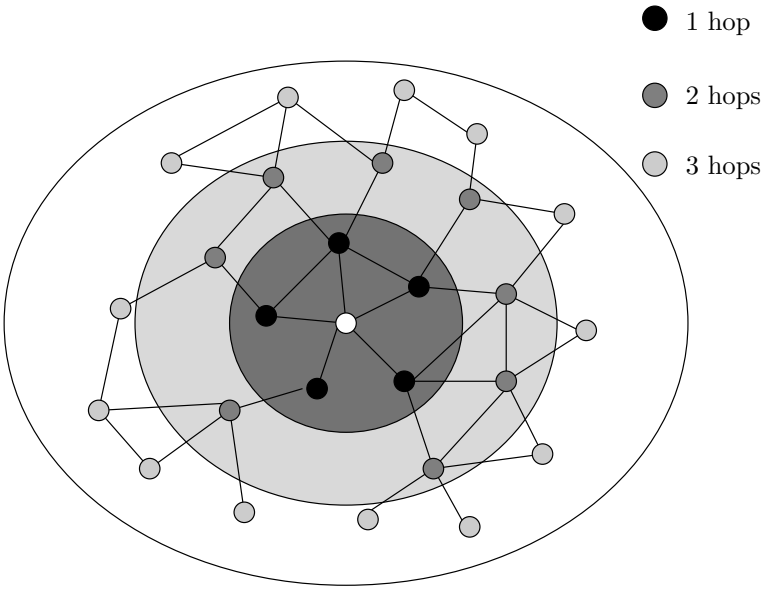

Figure 2.3: Decreasing accuracy in FSR. Nodes that are closest to the node in the center receive update packets more frequently.

decreases with the distance from the destination. The authors relate this phenomenon to the biological process that characterizes an eye of a fish (hence the name): it provides increased detail on its focal point and less accuracy away from it. This comparison was originally conceived by Kleinrock et al. [29], who used it however in a different context.

2.3.3 EASE

The Exponential Age SEarch protocol (EASE), by Grossglauser et al. [19], is essentially the LER protocol we described in section 1.3.1. Its only difference from LER, as described so far, is that the flooding halting condition is not the maximal: If the current accuracy is T, the protocol floods the network in order to locate a destination with accuracy better than $T/2$. This halting condition guarantees that the accuracy decreases exponentially with the number of steps, hence the name EASE.

In approximate information protocol notation, in EASE each node i maintains a routing table RT_i containing the coordinates of every other node j at some time in the past. The metric of accuracy M_{ij} is the elapsed time since the destination j was at that position. In other words, $RT_{ij}(t) = P_j(t - M_{ij}(t))$, where $P_j(t)$ is the position of j at time t. Updates happen according to the last encounter routing scheme (LER), defined by the authors in the same paper, according to which node i updates its routing entry

RT_{ij} every time it is adjacent to node j. Therefore, $M_{ij} = M_{ji}$ is also the elapsed time since the last encounter of i and j.

As we have already discussed, the above update mechanism is interesting because it incurs no overhead, while the claim of the authors that "mobility helps", as described in section 1.3.3, indicates that the performance of the protocol is not necessarily worse in highly mobile s networks. As we described, the faster a node moves, the more carrier nodes it creates, thus facilitating future route discoveries directed towards it. On the other hand, the faster a node moves the faster information on it become degraded (in the sense of being inaccurate) and thus the harder it is to locate. Both these two factors contribute to the overall behaviour of the protocol.

The authors provide a more technical argument to justify why EASE should behave well. Their claim is that the cost of every flooding step is comparable, as far as the overhead incurred is concerned, to the cost of every forwarding step. In that context, the number of packets generated through flooding is comparable to the number of packets needed to forward a message from one anchor point to the next one.

The argument can be summarized as follows. The authors assume that nodes move according to identically distributed, independent random walks on a grid. They first compute the probability that a carrier node will be within an area from the anchor point (referred to as "the hitting probability" of one carrier node) in terms of the current accuracy . They then compute the expected number of carrier nodes given that the destination performs a random walk in terms of the current accuracy. Using the probability computed above, the expected number of carrier nodes and the assumption that carrier nodes positions are independent, they are able to compute the probability that any carrier node will visit the an area around the anchor point (the "hitting probability" of any carrier node). From this quantity they deduce that the expected area flooded around an anchor point is proportional to $\log T$, where T is the current accuracy. They then show that this make the area comparable to the length of the path between the current anchor point and the next one.

We refer the reader to [19] for more details on the actual argument. As noted by the authors, this argument is not rigorous. There are several points in which it fails. First of all, while computing the hitting probability of one node and the expected total number of carrier nodes, they assume that the destination performs a random walk. On the other hand, in computing the hitting probability of *any* carrier node, they assume that carrier nodes move independently. However, this is true only if the destinations

trajectory is given (i.e., if the destination moves deterministically), as noted by the authors, in which case the results they calculated for the previous two quantities should be much different. This problem is better illustrated in chapters 5 and 6 of this thesis. Second, the dependence of the expected flooding area around an anchor point from the current accuracy is derived with an intuitive argument. Finally, an implicit assumption made by the authors, when they compute the expected area flooded in all steps, is that flooding at each step is not influenced by flooding at previous steps, which is also not true, as described in section 4.3.1 of this thesis.

However, the paper is quite significant, since it presents a first analysis of EASE and contains very important intuition on why EASE should work. The use of random walks as the underlying mobility process of the protocol is an approach we follow as well, since random walks present many interesting properties. However, we choose to work with Brownian motion instead of a random walk on a grid. This approach is not much different from the one used by the authors, since they too also work on the extremal properties of random walks (they make use of the central limit theorem). Another aspect of their approach that we also adopt is that the performance of the protocol is studied asymptotically, that is, for large values of the parameters involved.

2.3.4 FRESH

The FResher Encounter SearcH (FRESH) routing protocol , proposed by Ferrière et al. [12], is a simplification of the EASE protocol. Everything is defined as in LER except that nodes do not store any position information in the RT routing tables; they only keep track of the metric of accuracy M_{ij}, which is, as in EASE, the elapsed time since the last encounter of of i and j. Routing thus is performed as follows. When a node s wishes to send a packet to d it initiates a flooding, looking for a node with an accuracy better that the one s has (namely M_{sd}). When such a node is located, the same process is initiated at it.

In other words, in FRESH no forwarding takes place, and the anchor points coincide with the selected carrier nodes. Under approximate information notation, FRESH can be seen as a protocol in which the information RT_{ij} that assists routing towards the destination at a node i is *the position of the node itself*, i.e. $RT_{ij}(t) = P_i(t)$ for all j, where $P_i(t)$ is the position of node i at time t.

The authors of [12] motivate FRESH by noting that, if nodes perform random walks,

the expected distance between two nodes decreases as the elapsed time since their last encounter decreases. In that sense, as the protocol progresses and the metric of accuracy gets closer to zero, so will the expected distance of the current anchor point from the destination. This behaviour suggests that under this model of mobility the protocol converges to the destination.

This is an immediate implication of the fact that the distance between the destination and a node that encountered it some time ago for the last time is (almost) a random walk. This is explored in detail in section 5.4 in chapter 5 of this thesis. In short, the expected distance between two nodes is (almost) proportional to the square root of the elapsed time since their last encounter.

FRESH is very much related to LER. For example, the process that characterizes the generation of carrier nodes is identical in both protocols. Combined with the above observation, we believe that an analytical description of LER should lead to one for FRESH as well.

2.3.5 GREP

The Generalized Route Establishment Protocol (GREP), by Ferrière et al. [13], is an approximate information protocol that uses piggybacking as its update mechanism. Nodes maintain next-hop routing tables, i.e. RT_{ij} is the next hop that i should use in order to forward a packet to j. When a node receives a packet it immediately updates its information on the neighbor that forwarded it and the node that initially generated it (the source). The metric of accuracy is again time, although it is implemented through packet sequence numbers. Forwarding a packet is rather straight forward: packets are passed to the indicated next hop until a link failure occurs. In that case, a flooding is initiated until the destination or a node with better information (a later sequence number) than the one at the link break is reached.

An interesting property of GREP is that, by default, it performs greedy forwarding and that such behaviour converges and is loop free, as the authors prove. In other words, a next-hop node is guaranteed to have an accuracy at least as good as the current, i.e. the sequence number it will have will be at least as good. The reason is, intuitively, that since a node is designated as a next-hop node for a particular destination, a packet originated from that destination must have also passed from that node. Hence, the next-hop node has an accuracy greater or equal to the current one, since a packet of the same

accuracy passed through it. This is important because accuracy in GREP is improved not only during flooding but during forwarding as well. That is also why the accuracy used at flooding is actually the accuracy at the current anchor point -contrary to EASE, for example.

The actual argument is a little more complicated than the one described. In order for loop freedom to be guaranteed, extra caution must be taken so that paths created during flooding also have the increasing accuracy property. We refer the reader to the proof in [13] for further details.

As noted in section 1.3.3 of the introduction, protocols such as GREP display a balancing behaviour as far as the total number of packets is concerned. In a network with heavy traffic, updates will happen quite often and hence many carrier nodes will exist per flooding step. This, on the other hand, will facilitate future routing discovery and will reduce the overhead, and thus also the overall traffic. There seems thus to be an interesting relation between the load in the network and the overhead: reducing one should increase the other and vice versa, perhaps in some way that an invariant exists.

However, such a behaviour is hard to model. In GREP, it is very hard to quantify and analyze the accuracy of a path, because of the nature of topology (path) information: a path either exists or not, and it is hard to create finer distinctions. However, a protocol identical to GREP as far as the update mechanism is concerned in which however location information is used may be amenable to analysis. Even so, as noted in section 1.4 of the introduction, describing the process of carrier node generation is considerably hard, that is why we chose to discuss about the LER protocol instead.

Chapter 3

Technical Preliminaries

In this chapter we introduce certain mathematical concepts that will be useful in our analysis. First of all, we give a brief review of the Landau notation. We will use this notation to describe the asymptotic behaviour of functions. Furthermore, in our analysis, the network will be modeled as a Poisson field, whereas the nodes will move according to a Brownian motion. We define and describe the basic properties of the above two mathematical objects.

As we will see in the rest of the thesis, an interesting feature of the protocol will be the number of carrier nodes that exist at a given time. This number constitutes an arrival process and, conditional on the trajectory of the destination, this process is a non-homogeneous Poisson process. Hence, in the following, we shall give a brief review of arrival processes in general and Poisson processes in particular.

Finally, in the last section of this chapter we present Volterra equations and methods to solve them. Volterra equations will be a useful tool in modeling the behaviour of the network.

3.1 Landau notation

The following definitions are from Hardy and Wright [49]. Let $f : \mathbb{R}^+ \to \mathbb{R}$ and $g : \mathbb{R}^+ \to \mathbb{R}^+$. Then

$$f = O(g)$$

means that

$$\lim_{t\to\infty} \frac{|f(t)|}{g(t)} \leq c$$

where c is a positive constant. In the special case where

$$\lim_{t\to\infty} \frac{f(t)}{g(t)} = 0$$

we write

$$f = o(g).$$

For example, $3t^2 + 6t = O(t^2)$ whereas $\log t = o(t)$. Intuitively, $f = O(g)$ means that g grows at least as fast as the absolute value of f. If $f = o(g)$, then g grows faster than the absolute value of f. The symbols O and o, sometimes referred to as "big-o" and "little-o", are also called Landau symbols, in honour of Edmund Landau (1877-1938) who first introduced them.

For reasons of completeness, we present two more symbols.

$$f = \Omega(g)$$

denotes that

$$\lim_{t\to\infty} \frac{|f(t)|}{g(t)} \geq c$$

where c is a positive constant, and

$$f = \Theta(g)$$

means that both $f = O(g)$ and $f = \Omega(g)$ hold.

3.2 Arrival processes

Arrival processes will be important in establishing the number of carrier nodes in our protocol. The following definitions are taken from Gallager [16].

An *arrival process* can be seen as a stochastic process that describes arrivals in an initially empty system. The *epochs* of arrivals in the system are usually denoted with S_i, $i \geq 1$, whereas the *interarrival intervals*, i.e. the time intervals between two consecutive arrivals, are denoted with X_i, $i \geq 0$. Each arrival process is related to a corresponding

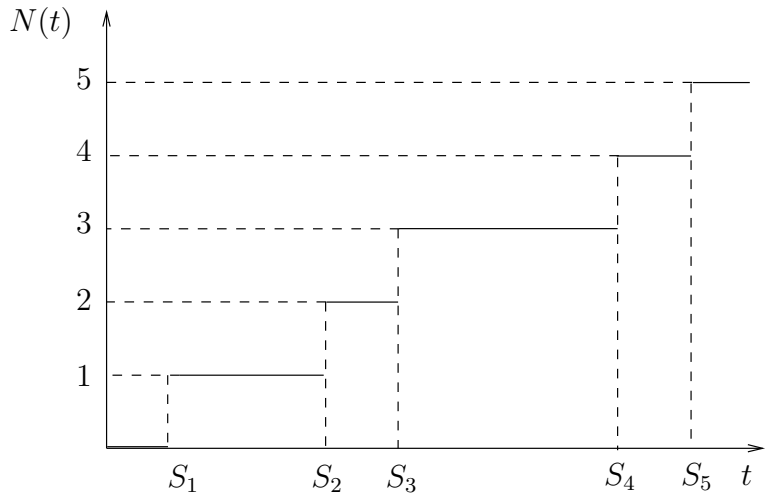

Figure 3.1: An example of an arrival process $\{S_1, S_2, \ldots\}$ and the corresponding counting process $\{N(t), t \geq 0\}$.

counting process. This process, denoted usually with $N(t)$, $t \geq 0$, shows the number of arrivals in an initially empty system that have occurred up to and including time t. Figure 3.1 contains an example of an arrival process and the corresponding counting process.

More formally, an arrival process is a set of random variables $\{S_1, S_2, \ldots\}$, the arrival epochs, such that the interarrival intervals, defined as $X_1 = S_1$ and $X_i = S_i - S_{i-1}$ for $i > 1$, are positive random variables, i.e.

$$\mathbf{P}\{X_i > 0\} = 1$$

for all $i \geq 1$. Note that $S_n = \sum_{i=1}^{n} X_i$. An arrival process can be specified by providing the joint probability distributions of either $\{S_1, S_2, \ldots, S_n\}$ or $\{X_1, X_2, \ldots, X_n\}$ for all $n \geq 1$.

A counting process $N(t), t \geq 0$ is a family of non-negative integer valued random variables, one for each real number $t \geq 0$, with the properties that $N(\tau) \geq N(t)$ for all $\tau \geq t$ (i.e., $N(\tau) - N(t)$ is a non negative random variable) and $N(0) = 0$ with probability 1. An arrival process and the corresponding counting process are linked by the equality of the following two events

$$\{S_n \leq t\} = \{N(t) \geq n\}. \tag{3.1}$$

A *renewal process* is an arrival process in which the interarrival intervals are independent identically distributed (positive) random variables.

3.3 The Poisson process

The number of carrier nodes in our protocol is related to a special kind of arrival process, the non-homogeneous Poisson process. We shall provide here the necessary definitions and will prove certain properties of Poisson processes that will be useful to our analysis.

3.3.1 Definitions

There are several equivalent definitions of a Poisson process. The ones most related to our analysis are the following two, which are taken from Gallager [16]:

Definition 3.1. *A Poisson process of rate λ is a counting process $\{N(t), t \geq 0\}$ that satisfies*

$$\mathbf{P}\{N(t) = n\} = \frac{(\lambda t)^n}{n!} e^{-\lambda t} \tag{3.2}$$

and has the following two properties:

1. *For all t, t' such that $t' \geq t \geq 0$, the random variable $\widetilde{N}(t, t') = N(t') - N(t)$ follows the same distribution as as $N(t' - t)$.*

2. *For any given k and for all $t_1, t_2, \ldots, t_k$ with $0 < t_1 < t_2 < \ldots < t_k$ the random variables $N(t_1), \widetilde{N}(t_1, t_2), \widetilde{N}(t_2, t_3), \ldots, \widetilde{N}(t_{k-1}, t_k)$ are independent.*

The properties mentioned in the above definition are known as the *stationary increment property* and the *independent increment property* respectively. The probability distribution in (3.2) is none other than the *Poisson distribution*. The following definition is equivalent to the one above:

Definition 3.2. *A Poisson process of rate λ is a counting process $\{N(t), t \geq 0\}$ for which the following hold:*

1. *$\{N(t), t \geq 0\}$ satisfies the stationary and independent increment properties.*

2. *The random variable $\widetilde{N}(t, t') = N(t') - N(t)$ $(0 \leq t \leq t')$ satisfies*

$$\begin{aligned} &\mathbf{P}\{\widetilde{N}(t, t+\delta) = 0\} = 1 - \lambda\delta + o(\delta) \\ &\mathbf{P}\{(\widetilde{N}(t, t+\delta) = 1\} = \lambda\delta + o(\delta) \\ &\mathbf{P}\{(\widetilde{N}(t, t+\delta) \geq 2\} = o(\delta) \end{aligned} \tag{3.3}$$

where $o(\delta)$ is any function f such that $\lim_{\delta \to 0} f(\delta)/\delta = 0$.

3.3.2 Non-homogeneous Poisson process

A non-homogeneous Poisson process can be thought as a generalization of a Poisson process in which the arrival rate λ is a function of time. The following definition, as well as Lemma 3.1, are from Gallager [16]:

Definition 3.3. *A non-homogeneous Poisson process with arrival rate $\lambda(t)$, where λ is right-continuous, is a counting process $\{N(t); t \geq 0\}$ for which the following hold:*

1. *$\{N(t); t \geq 0\}$ has the independent increment property.*
2. *The random variable $\tilde{N}(t, t') = N(t') - N(t)$ for all $t \geq 0$, $\delta \geq 0$ satisfies:*

$$\begin{aligned} \mathbf{P}\{\tilde{N}(t, t+\delta) = 0\} &= 1 - \delta\lambda(t) + o(\delta) \\ \mathbf{P}\{\tilde{N}(t, t+\delta) = 1\} &= \delta\lambda(t) + o(\delta) \qquad (3.4) \\ \mathbf{P}\{\tilde{N}(t, t+\delta) \geq 2\} &= o(\delta) \end{aligned}$$

$$(3.5)$$

An interesting lemma that relates a non-homogeneous Poisson process to the Poisson distribution is the following:

Lemma 3.1. *For a non-homogeneous Poisson process $\{N(t); t \geq 0\}$ with right-continuous arrival rate $\lambda(t)$ bounded away from zero, the distribution of $\tilde{N}(t, t') = N(t') - N(t)$ $(t' \geq t)$ satisfies*

$$\mathbf{P}\{\tilde{N}(t, t') = n\} = \frac{[\tilde{\mu}(t, t')]^n \exp[-\tilde{\mu}(t, t')]}{n!}$$

where $\tilde{\mu}(t, t') = \mathrm{E}\left[\tilde{N}(t, t')\right] = \int_t^{t'} \lambda(u)du$.

In this thesis, we will be also interested in a process that differs from a non homogeneous Poisson process in the fact that bulk arrivals may happen at time 0.

Definition 3.4. *A stochastic process $\{N(t), t \geq 0\}$ is a non-homogeneous Poisson process with bulk arrivals at time zero if $N(t) - N(0)$ is a non-homogeneous Poisson process independent of $N(0)$ and $N(0)$ is a Poisson random variable.*

This is not a counting process according to the definition given in section 3.2, since the probability that more than one arrivals happen at time zero is positive. Equivalently, this process can be seen as a non-homogeneous Poisson process with rate

$$\lambda'(t) = \mathrm{E}[N(0)]\,\delta(t) + \lambda(t),$$

where $\delta(t)$ is the Dirac function and λ is the rate of $N(t)-N(0)$, or as a non-homogeneous Poisson process with expectation $\mu(t) = \mathrm{E}[N(t)]$ such that $\mu(0)$ is positive.

3.3.3 The distribution of arrivals of Poisson process conditioned on their number at time T

The following lemma, from [30, p. 76], will be useful in our analysis.

Lemma 3.2. *Let $\{N(t), t \geq 0\}$ be a non-homogeneous Poisson process with $\mathrm{E}[N(t)] = \mu(t)$. Given that $N(T) = n$, for $T \geq 0$ and $n > 0$, we have that*

$$N(t) = \sum_{k:s_k \leq t} 1, \qquad 0 \leq t \leq T$$

where $s_1, \ldots, s_n$ are independent random variables distributed as

$$\mathbf{P}\{s_k \leq t\} = \frac{\mu(t)}{\mu(T)}, \qquad 0 \leq t \leq T.$$

An immediate consequence of the above lemma is the following corollary.

Corollary 3.1. *Let $\{N(t), t \geq 0\}$ be a non-homogeneous Poisson process with $\mathrm{E}[N(t) = \mu(t)]$. Given that $N(T) = n$, let X be a random variable defined as*

$$\mathbf{P}\{X = S_i \mid N(t) = n\} = \frac{1}{n}$$

where S_i is the epoch of the i-th arrival, $1 \leq i \leq n$. Then

$$\mathbf{P}\{X \leq t \mid N(T) = n\} = \frac{\mu(t)}{\mu(T)}, \qquad 0 \leq t \leq T.$$

Note that, for a homogeneous Poisson process, X is uniformly distributed over $[0, T]$ given $N(T) = n$. Also, it is easy to prove that the above lemma and corollary hold even if we extend the definition of Poisson processes to the case where bulk arrivals can happen at time 0, i.e. when $\mu(0) \neq 0$.

3.4 Poisson field and Brownian motion

The positions of nodes in our network will be modeled as a Poisson field and each node in the network will move independently of other nodes according to a stochastic process called Brownian motion. Poisson fields moving according to independent Brownian

motions have interesting properties. Most importantly, as stated in section 3.4.3, the nodes of such a network maintain the property of forming a Poisson field through time. Furthermore, later in the thesis we will see how under such a model the distribution of the number of carrier nodes in the network can be described.

3.4.1 Poisson Field

A set $\boldsymbol{\pi} = \{P_1, P_2, \ldots\}$ where P_i are random variables in $\mathbb{R}^2$ is called a *point process in* $\mathbb{R}^2$. The following definition is from Révész [41].

Definition 3.5. *A point process $\boldsymbol{\pi}$ in $\mathbb{R}^2$ is called a Poisson field of density ρ ($\rho > 0$) if*

$$\mathbf{P}\{\boldsymbol{\pi}(\mathcal{A}) = k\} = \frac{(\rho|\mathcal{A}|)^k}{k!}\exp(-\rho|\mathcal{A}|) \qquad (k = 0, 1, 2, \ldots)$$

and $\boldsymbol{\pi}(\mathcal{A}_1), \boldsymbol{\pi}(\mathcal{A}_2), \ldots, \boldsymbol{\pi}(\mathcal{A}_n)$ are independent random variables where $\boldsymbol{\pi}(\mathcal{A})$ is the number of points of the process $\boldsymbol{\pi}$ in $\mathcal{A}$, $\mathcal{A} \subset \mathbb{R}^2$ is Lebesque measurable, $|A|$ is the Lebesque measure of $\mathcal{A}$ and $\mathcal{A}_1, \mathcal{A}_1, \ldots, \mathcal{A}_n$ are disjoint Lebesque measurable subsets of $\mathbb{R}^2$.

Notice that

$$\mathrm{E}\left[\boldsymbol{\pi}(\mathcal{A})\right] = \mathrm{Var}\left[\boldsymbol{\pi}(\mathcal{A})\right] = \rho|\mathcal{A}|.$$

This property indicates that $\boldsymbol{\pi}$ is homogeneous. Although we do not provide a rigorous definition of homogeneity, what the above sentence intuitively means is that in a Poisson field, every (measurable) subset of the plane has the same expected density, i.e. the same expected number of points per unit area.

3.4.2 Brownian motion

In our model, nodes will move according to a stochastic process called *Brownian motion* or *Wiener process*. The following definition is from Borodin and Salminen [3].

Definition 3.6. *A stochastic process $B(t)$ ($t \geq 0$) is called a standard one-dimensional Brownian motion or a Wiener process initiated at x on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ if*

1. *$B(0) = x$ a.s.,*

2. *$t \mapsto B(t)$ is continuous a.s.,*

3. *for all* $0 = t_0 < t_1 < \ldots < t_n$ *the increments*

$$B(t_n) - B(t_{n-1}), B(t_{n-1}) - B(t_{n-2}), \ldots, B(t_1) - B(t_0)$$

are independent and normally distributed with

$$\mathrm{E}\left[B(t_i) - B(t_{i-1})\right] = 0, \qquad \mathrm{Var}\left[B(t_i) - B(t_{i-1})\right] = t_i - t_{i-1} \quad (i = 1, \ldots, n)$$

Brownian motion has the following properties, as stated by Borodin and Salminen [3]. Assuming that B is a Brownian motion starting at the origin:

1. *Spatial homogeneity:* For every $x \in \mathbb{R}$ the process $x + B$ is a Brownian motion initiated at x.

2. *Symmetry:* $-B$ is a Brownian motion.

3. *Scaling:* For every $c > 0$ the process $\{\sqrt{c}B(t/c) : \ t \geq 0\}$ is a Brownian motion.

4. *Time reversibility:* For a given $t > 0$ the process $\{B(s) : 0 \leq s \leq t\}$ is identically distributed as $\{B(t - s) - B(t) : 0 \leq s \leq t\}$.

The spatial homogeneity and symmetry properties indicate that a Brownian motion remains a Brownian motion even if the origin is translated or the axis is inversed. The scaling property suggests that if we "speed up" a point that performs a Brownian motion by a factor c, the resulting process will be a Brownian motion if we also rescale the unit length of the axis on which the point moves by $\sqrt{c}$. Finally, the time reversibility property means that given the position at time t of a point that has been performing a Brownian motion in the interval $[0, t]$, its position in the past -with the time moving "backwards"- is a Brownian motion.

Definition 3.7. *A standard two-dimensional Brownian motion* $B(t)$ *is a vector* $\{B_1(t), B_2(t)\}$ *where* $B_i(t)$ *(*$i = 1, 2$*) are one-dimensional Brownian motions.*

Note that the vector $B(t') - B(t)$ where $t' \geq t$ follows a two-dimensional omnidirectional normal distribution with joint density

$$f_{X_1,X_2}(x_1, x_2) = \frac{1}{2\pi(t' - t)} \exp\left(-\frac{(x_1 - B_1(t))^2 + (x_2 - B_2(t))^2}{2(t' - t)}\right). \tag{3.6}$$

In our model, we assume that nodes move independently from each other according to two-dimensional non-standard Brownian motions with variance proportional to v. In

other words, nodes' positions are independent two-dimensional Wiener processes, similar to the one defined above, having however variances in the interval (t, t')

$$\mathrm{Var}\left[B_i(t') - B_i(t)\right] = v(t' - t) \qquad i \in \{1, 2\}. \tag{3.7}$$

Parameter v is related to how far a node can travel in a given time interval, and can be used to quantify the intensity of the mobility process in the network[1]. From now on, when we refer to a Brownian motion as "with variance v" we will mean that it is in general non-standard and its variance is proportional to v. Non-standard and standard two-dimensional Brownian motions are related according to the following lemma:

Lemma 3.3. *Let $B(t)$ be a two-dimensional Brownian motion with variance v. Then $B^*(t) = B(t/v)$ is a standard two-dimensional Brownian motion.*

The above Lemma can be easily proved from definitions of standard and non-standard Brownian motions.

For further details about Brownian motion we refer the reader to [3], whereas more information about Brownian motion in the context of random walks can be found in [40].

3.4.3 Brownian motion in a Poisson field

The following theorem from Révész [41] shows how a Poisson field interacts with Brownian motion.

Theorem 1. *Let $\boldsymbol{\pi} = \{P_1, P_2, \ldots\}$ be a Poisson field of density ρ in $\mathbb{R}^2$. Let $B_1(t)$,$B_2(t)$,... $(t \geq 0)$ be a sequence of independent $\mathbb{R}^2$ valued Brownian motions being also independent from $\boldsymbol{\pi}$ and define*

$$P_i(t) = P_i + B_i(t) \qquad (i = 1, 2, \ldots)$$

and

$$\boldsymbol{\pi}(t) = \{P_1(t), P_2(t), \ldots\}.$$

Then for any fixed $t \geq 0$ the point process $\boldsymbol{\pi}(t)$ is a Poisson field of density ρ.

This theorem is important because it indicates that $\boldsymbol{\pi}(t)$ remains a Poisson field (and thus also maintains its homogeneity) through time. In other words, a stochastic point process $\boldsymbol{\pi}(t)$ defined as above can be treated as a Poisson field of density ρ for any $t \geq 0$.

[1] Note that the standard Brownian motion can be considered as a non-standard Brownian motion with variance proportional to one.

3.5 Volterra equations

The overhead incurred by our protocol will be modeled as the solution of a linear Volterra equation of the second kind. The following introductory remarks for Volterra equations are taken from Brunner and van der Howen [6], except corollary 3.2, which we prove since it will be useful in our analysis. The bibliography on the subject of integral equations and Volterra equations in particular is quite extensive. The above text is a treatise on numerical solutions for Volterra equations. For a more theoretical discussion of Volterra equations we refer the reader to Gripenberg et al. [18].

Integral equations are functional equations where the unknown function appears under an integral sign. A linear Volterra equation of the second kind is an integral equation of the form

$$y(t) = g(t) + \int_0^t K(t,s)y(s)ds, \quad t \in I \tag{3.8}$$

where $I = [0, +\infty)$, $g : I \to \mathbb{R}$ and $K : S \to \mathbb{R}$ where

$$S = \{(t,s) : t \in I \wedge 0 \leq s \leq t\}. \tag{3.9}$$

Function $y : I \to \mathbb{R}$ is the unknown function. Function K is called the *kernel* of the equation. Vito Volterra (1860-1940) first showed that the following theorem holds.

Theorem 2 (Volterra). *Let functions g and K of the integral equation* (3.8) *be continuous on $I \subset \mathbb{R}$ and S respectively. Then this equation has a unique solution $y : I \to \mathbb{R}$ given by*

$$y(t) = g(t) + \int_0^t R(t,s)g(s)ds, \quad t \in I \tag{3.10}$$

where $R : S \to \mathbb{R}$ is called the resolvent kernel of kernel K. The resolvent kernel satisfies the following two identities (called the Fredholm identities)

$$R(t,s) = K(t,s) + \int_s^t K(t,\tau)R(\tau,s)d\tau \tag{3.11}$$

and

$$R(t,s) = K(t,s) + \int_s^t R(t,\tau)K(\tau,s)d\tau \tag{3.12}$$

for all $(t,s) \in S$.

This theorem can be extended to Volterra equations on a domain of the form $[\alpha, +\infty)$

Corollary 3.2. *The Volterra equation of the form*

$$y(t) = g(t) + \int_\alpha^t K(t,s)y(s)ds, \quad t \in I_\alpha \tag{3.13}$$

where $I_\alpha = [\alpha, +\infty)$ *and* g *is continuous in* I_α *and* K *is continuous in*

$$S_\alpha = \{(t,s) : t \in I_\alpha \wedge \alpha \leq s \leq t\} \tag{3.14}$$

has a unique solution given by

$$y(t) = g(t) + \int_\alpha^t R(t,s)g(s)ds, \quad t \in I_\alpha \tag{3.15}$$

where $R : S_\alpha \to \mathbb{R}$ *is the resolvent kernel of* K *and satisfies the Fredholm identities* (3.11) *and* (3.12) *in* S_α.

Proof. Let $\hat{y}(t) = y(t+\alpha)$, $t \in I$, where $I = [0, +\infty)$. Then

$$\begin{aligned}
\hat{y}(t) &= y(t+\alpha) \\
&= g(t+\alpha) + \int_\alpha^{t+\alpha} K(t+\alpha, s)y(s)ds && \text{by (3.13)} \\
&= g(t+\alpha) + \int_0^t K(t+\alpha, s'+\alpha)y(s'+\alpha)ds' && \text{where } s' = s - \alpha \\
&= \hat{g}(t) + \int_0^t \hat{K}(t,s')\hat{y}(s')ds'
\end{aligned}$$

where

$$\hat{g}(t) = g(t+\alpha), \quad t \in I \tag{3.16}$$

and

$$\hat{K}(t,s) = K(t+\alpha, s+\alpha) \tag{3.17}$$

where $(t,s) \in S$. Thus Theorem 2 applies to $\hat{y}(t)$. The resolvent kernel $\hat{R}(t,s)$ of $\hat{K}$ is such that, for $(t,s) \in S$

$$\begin{aligned}
\hat{R}(t,s) &= \hat{K}(t,s) + \int_s^t \hat{K}(t,\tau)\hat{R}(\tau,s)d\tau \\
&= K(t+\alpha, s+\alpha) + \int_s^t K(t+\alpha, \tau+\alpha)\hat{R}(\tau,s)d\tau && \text{by (3.17)} \\
&= K(t+\alpha, s+\alpha) + \int_{s+\alpha}^{t+\alpha} K(t+\alpha, \tau')\hat{R}(\tau'-\alpha, s)d\tau' && \text{where } \tau' = \tau + \alpha
\end{aligned}$$

By setting $(t', s') = (t + \alpha, s + \alpha)$, where $(t', s') \in S_\alpha$ we get

$$\hat{R}(t' - \alpha, s' - \alpha) = K(t', s') + \int_{s'}^{t'} K(t', \tau')\hat{R}(\tau' - \alpha, s' - \alpha)d\tau'.$$

We define $R(t', s') = \hat{R}(t' - \alpha, s' - \alpha)$, where $(t', s') \in S_\alpha$. Then we have

$$R(t', s') = K(t', s') + \int_{s'}^{t'} K(t', \tau')R(t', s')d\tau'.$$

Thus the resolvent kernel $\hat{R}$ of $\hat{K}$ is such that $R(t', s') = \hat{R}(t'-\alpha, s'-\alpha)$ where $R : S_\alpha \to \mathbb{R}$ satisfies (3.11). Similarly it can be shown that R satisfies (3.12). From Theorem 2 we have

$$\hat{y}(t) = \hat{g}(t) + \int_0^t \hat{R}(t, s)\hat{y}(s)ds \qquad\qquad t \in I$$

By replacing $\hat{y}(t)$ with $y(t + \alpha)$ and $\hat{g}(t)$ with $g(t + \alpha)$ we get

$$\begin{aligned} y(t + \alpha) &= g(t + \alpha) + \int_0^t \hat{R}(t, s)y(s + \alpha)ds \\ &= g(t + \alpha) + \int_\alpha^{t+\alpha} \hat{R}(t, s' - \alpha)y(s')ds' \qquad \text{where } s' = s + \alpha \end{aligned}$$

By setting $t' = t + a$, $t' \in I_\alpha$ we get

$$\begin{aligned} y(t') &= g(t') + \int_\alpha^{t'} \hat{R}(t' - \alpha, s' - \alpha)y(s')ds' \\ &= g(t') + \int_\alpha^{t'} R(t', s')y(s')ds' \end{aligned}$$

where $t' \in I_\alpha$ and $(t', s') \in S_\alpha$. □

Closed solutions for equations (3.11) and (3.12) exist for a variety of kernels. We will be interested in a special class of kernels called *separable* or *decomposable*.

Lemma 3.4. *If $K(t, s) = A(t)B(s)$, then its resolvent kernel is*

$$R(t, s) = A(t)B(s) \exp\left(\int_s^t A(u)B(u)du\right). \tag{3.18}$$

Proof. It can be easily seen that R satisfies (3.11) and (3.12). □

Chapter 4

The Model

In this chapter we present the model we will use in this thesis in order to analyze the LER protocol. Before doing so, we define LER rigorously, by giving precise definitions of the accuracy, the routing table entries and the routing algorithm.

Our model consists of two parts. The first is the network model, which describes how nodes are distributed on the plane and how they move. The second part is a model for the cost, i.e. a performance measure that we will use in order to describe how the protocol behaves.

4.1 The LER routing protocol

In this section we define LER, the last encounter routing protocol, more rigorously. Nodes maintain routing tables with the positions of destinations during their last encounters. This means that the metric of accuracy is

$$M_{ij}(t) = \min\{\tau : ||P_i(t-\tau) - P_j(t-\tau)|| \leq r_0,\ \tau \geq 0\}$$

where $P_i(t)$, $P_j(t)$ are the positions of nodes i and j respectively at time t and r_0 is the transmission radius. The information stored in the routing table is the position of the destination at the last time of encounter, i.e.

$$RT_{ij}(t) = P_j(t - M_{ij}(t)).$$

We assume that the speed of the moving nodes is such that the change of node positions during packet routing is negligible. The routing protocol is defined in figure 4.1. Note that the protocol is actually resorting to flooding if $M_{id} \geq \alpha$, where α is some positive value.

Route a packet p from s to d
{
    $i = s$
    while $(M_{id} \geq \alpha)$
    {
        forward p to the position RT_{id}
        let a be the node reached with this forwarding
        flood the network from a until a node n is reached
                such that $M_{nd} < M_{id}$
        $i = n$
    }
    forward p to d using RT_{id};
}

Figure 4.1: Definition of the LER routing protocol.

If $M_{id} < \alpha$, we assume that the protocol can reach the destination without resorting to flooding. This simplifies analysis, but is also a plausible assumption since within some level of accuracy, the destination can be reached without flooding in real life protocols.

The halting condition is the maximal and the set of carrier nodes is

$$C = \{n \in \mathbb{N}^+ : M_{nd} < M_{id}\}.$$

According to the definition of carrier nodes provided in section 1.3.2 the destination also should be included in the set of carrier nodes since $M_{dd} = 0 < \alpha \leq M_{id}$. In our analysis however we shall not consider it as a carrier node. In other words, when we refer to the set of carrier nodes we mean all the nodes that satisfy the flooding halting condition except the destination. This is merely for reasons of conciseness, since we will quite often need to distinguish carrier nodes from the actual destination. That said, the fact that the destination is considered during flooding guaranties that flooding will halt, since at least one node that satisfies the halting condition exists (namely, the destination).

Flooding is implemented by expanding concentric circles, i.e. the area that we flood is a disk centered at the anchor point a and radius

$$r = \min_{n \in C} ||P_a - P_n||$$

and therefore the carrier node that the protocol chooses is $n \in C \cup \{d\}$ such that $||P_a - P_n||$ is minimized, i.e. n is the carrier node (or the destination) closest to the anchor point a.

4.2 The network model

In our model, the network will consist of an infinite, countable number of nodes. If for each node i, $P_i(t) \in \mathbb{R}^2$ will denote the position of i at time t. The set

$$\boldsymbol{\pi}(t) = \{P_1(t), P_2(t), \ldots\}$$

of nodes in the network will be a Poisson field of density ρ spanning over the entire plane $\mathbb{R}^2$. Each point $P_i(t)$ will move independently according to a two-dimensional Brownian motion of variance v.

The destination will be a point d *not belonging to* $\boldsymbol{\pi}$, which will also move according to a Brownian motion with variance v and independent of each $P_i(t) \in \boldsymbol{\pi}(t)$. Each node maintains a routing table in which an entry about destination d exists.

The parameters describing the network will be the density ρ and v. The distance unit will be scaled is such a way so that the transmission radius around each node P_i is $r_0 = 1/\sqrt{\pi}$. Equivalently, the area of the disk $\{x : ||x - P_i(t)|| \leq r_0\}$ around every node i will be equal to one. Two nodes will be considered as neighbouring nodes if their distance is less than r_0, i.e. the underlying graph of the network will be a unit graph.

We will assume that nodes have been moving in the interval $(-\infty, 0]$ and at time $t = 0$ a node initiates a route discovery to node d. Note that since nodes have been moving for an infinite amount of time every node will have encountered d with probability one. Route discoveries will be considered to last for a negligible period of time, so that nodes can be considered static while they take place.

Since the purpose of a routing protocol is to send a packet from a source node s to a destination d, it is important that, under the connectivity model that we used (i.e. the unit graph), a path from s to d exists. However, we will not deal with connectivity issues in our analysis. More specifically, we will assume a path that can be found by our protocol exists between s and d. This is justified by the fact that, the probability that such a path exists increases with the density of the network. We will therefore assume that ρ is large enough so that such a path exists with high probability.

4.3 A model for the cost

The goal of this thesis is to measure the performance of the aforementioned approximate information protocols. We will quantify this performance with a cost function.

Assume that we try to route a packet from node s to a node d, where $M_{sd} = t$. We wish to measure the expected overhead induced during route discovery by our protocol. We will assume that this overhead is attributed solely to flooding. Furthermore, we will assume that the this overhead is proportional to the total area covered by the protocol during flooding. In that sense, we wish to compute the expected total area flooded while routing from node s to d. While calculating this total area, we might include regions of $\mathbb{R}^2$ more than once, since multiple flooding phases during route discovery may overlap.

4.3.1 An expression for the expected flooding area

Let $\boldsymbol{\pi}(t) = \{P_1(t), P_2(t), \ldots\}$ be the point process that forms the network and d be the destination not belonging to the above process, with P_d its position. Assume that nodes are ordered according to their accuracies on the position of the destination at time 0, i.e. if $t_i = M_{id}(0)$, $i \geq 1$, then

$$0 = t_0 < t_1 < t_2 < \ldots$$

where t_0 is the accuracy of the destination itself. We will assume that accuracies are strictly increasing (i.e. no two nodes have the same accuracy). This makes the definition of carrier nodes easier and, as we will see, it is true for our protocol: only nodes with accuracy zero may appear more than once with non-zero probability. This can be easily taken into consideration in the forthcoming analysis.

We wish to find the expected total area we need to flood in order to send a packet from a source node to d. Assume that the source node is n, with $t_n \geq \alpha$. According to the LER protocol, a flooding will commence at an anchor point looking for carrier nodes with accuracy better than t_n. The anchor point is in fact $P_d(-t_n)$, but we will assume w.l.o.g. that the anchor point is at the origin.

All nodes with indices less than n will be carrier nodes, according to the definition of the protocol. The positions of carrier nodes will be distributed around the anchor point according to some probability distribution. Let x_i denote the position of carrier node i, $1 \leq i < n$, and x_0 denote the position of the destination. Furthermore let $y_i = P_d(-t_i)$

be the estimation that carrier node i has[1]. We denote with

$$F(t_0, t_1, \ldots, t_{n-1}, x_0, x_1, \ldots, x_{n-1}, y_0, y_1, \ldots, y_{n-1} \mid t_n) \tag{4.1}$$

the joint cumulative density distribution of the accuracies, the positions and the estimations of all carrier nodes, given that the current anchor point is at the origin and the current accuracy is t_n.

Let A be the one-step flooding area, i.e. the area that we flood in the first step of flooding, t_c be the accuracy of the carrier node that we locate with this flooding and y_c be the estimation of the position of the destination that this carrier node contains. Note that y_c is in fact $P_d(-t_c)$. Furthermore, according to the definition of the LER protocol, A is the area that we need to flood in order to locate the carrier node that is closest to the origin. Hence, the random variables A, t_c and y_c can be defined in terms of the random variables t_i, x_i and y_i, $0 \leq i < n$, as follows

$$A = \pi \cdot \min |x_i|^2$$

$$t_c = t_{\arg\min |x_i|}$$

$$y_c = y_{\arg\min |x_i|}$$

Therefore, we can compute from the joint distribution of eq. (4.1) the joint cumulative distribution of A, t_c and y_c given that the current anchor point is at the origin and the current accuracy is t_n. We denote this distribution as

$$F(A, t_c, y_c \mid t_n). \tag{4.2}$$

Let Q be the total area we need to flood from the current anchor point in order to reach the destination. Then obviously

$$Q = A + Q_c.$$

Where A is the one-step flooding area as defined above and Q_c is the total area flooded from the second anchor point. We therefore have that

$$\mathrm{E}\left[Q \mid t_n\right] = \iiint A \; dF(A, t_c, y_c \mid t_n) + \iiint \mathrm{E}\left[Q_c \mid A, t_c, y_c, t_n\right] . dF(A, t_c, y_c, \mid t_n) \tag{4.3}$$

If $F(A \mid t_n)$ is the marginal distribution of A, the expected area of the first flooding step can be further simplified to

$$\iiint A \; dF(A, t_c, y_c \mid t_n) = \int A \; dF(A \mid t_n). \tag{4.4}$$

[1]Note that $y_0 = x_0$.

The quantity $\mathrm{E}[Q_c \mid A, t_c, y_c, t_n]$ is the expected total area we will flood from the second anchor point given that the second anchor point is located at y_c, the accuracy at it is t_c and the first flooding (initiated at the origin) covered an area of size A. We will try to establish how the values of these random variables affect Q_c.

Let $\mathrm{E}[Q \mid t_c]$ be the expected total area flooded from an anchor point located at the origin with accuracy equal to t_c. Because of the translation invariance property of both the Poisson field and Brownian motion, this is equal to the expected total flooding area from an anchor point located at y_c. Furthermore, $\mathrm{E}[Q \mid t_c]$ is independent of the trajectory of the destination in the interval $(-\infty, -t_c)$. This is due to the independence of Brownian motion of its value at any time within that interval, given its value at time $-t_c$.

We are thus tempted to say that $\mathrm{E}[Q \mid t_c] = \mathrm{E}[Q_c \mid A, t_c, y_c, t_n]$. However, this is not true. To see this, note that $\mathrm{E}[Q \mid t_c]$ can be written in the form of (4.3). The joint distributions involved can again be expressed in terms of a distribution just like the one in (4.1), namely

$$F(t_0, t_1, \ldots, t_m, x_0, x_1, \ldots, x_m, y_0, y_1, \ldots, y_m \mid t_c) \tag{4.5}$$

where m is the largest index such that $t_m < t_c$. If we translate the origin to y_c, $\mathrm{E}[Q_c \mid A, t_c, y_c, t_n]$ is the expected total area from an anchor point at the origin with accuracy t_c, given that a disk of area A around the point $-y_c$ contains *no carrier nodes nor the destination*. This affects $\mathrm{E}[Q \mid t_c]$ in two ways.

First of all, since this area contains no carrier nodes, the joint cumulative distribution of 4.5 should be conditioned on the fact that no carrier nodes (or the destination) exist in this area. In other words, x_0, x_1,..., x_m cannot assume values within the aforementioned disk. Given the possible interdependencies between t_i, x_i and y_i for all i, the joint cumulative distribution can change significantly compared to the one used in $\mathrm{E}[Q \mid t_c]$.

Another way that 4.5 is affected is due to the fact that a carrier node i that contains information with accuracy t_i did not move independently of the destination in the interval $[-t_i, 0]$. As a matter of fact, it moved in such a way so that it avoided the destination (in other words, it did not meet it again). Hence, any information on the position of such a carrier node at time 0 gives some information on the position of the destination - namely, that it is not within a radius r_0 from that position. In that sense, the information that no carrier nodes exist within the aforementioned disk affects the position of the destination, i.e. x_0, and thus, through the possible interdependencies of (4.5), the values of the rest

of the variables as well.

4.3.2 A function that approximates the cost

Evaluating the expected total flooding area through equation (4.3) is quite hard. This is due to the interdependency between flooding at intermediate flooding steps and flooding at previous steps, as presented above. We will thus try to measure performance using a different quantity than the expected total flooding area.

Assume that we are performing a flooding at an anchor point with accuracy T. Let $G(T)$ be the expected one-step flooding area and $p(t, T)$, $0 \leq t < T$, be the conditional probability density of the accuracy t achieved after the first flooding step, given that the accuracy at the anchor point is T. In terms of the cumulative distribution in (4.2), $G(T)$ is defined by equation 4.4 and $p(t, T)$ is the marginal density of t_c given that $t_n = T$. We define the function $Q(T)$ that approximates the cost as a function that satisfies the system of equations:

$$Q(T) = G(T) + \int_{\alpha}^{T} Q(t)p(t, T)dt \quad T \geq \alpha \tag{4.6a}$$

$$Q(T) = 0 \quad 0 \leq T < \alpha \tag{4.6b}$$

Function $Q(T)$ would be equal to the expected total flooding area if the total flooding area from an intermediate anchor point were independent of previous floods. More precisely, according to the terminology introduced in the previous section, $Q(T)$ would be equal to the expected total flooding area if the total area Q_c flooded after the anchor point with accuracy t_c given A, y_c, t_n and t_c were independent of A, y_c, t_n and equal to the total flooding area given t_c. In such a case, flooding at any intermediate anchor point, such as $P_d(t_c)$, will cover the same area, on average, as a new flooding that was initiated at that point. To see that in such circumstances $Q(T)$ coincides with the total flooding area, note that if this is true then the integrant of equation (4.6) becomes

$$\mathrm{E}\left[Q_c \mid A, t_c, y_c, t_n\right] = \mathrm{E}\left[Q \mid t_c\right] \tag{4.7}$$

i.e. the expected area we need to flood from the second anchor point depends only on the accuracy of the second anchor point and is independent of the previous flood. Using (4.7) , for $F(t_c \mid t_n)$ the marginal distribution of t_c we get

$$\mathrm{E}\left[Q \mid t_n\right] = \mathrm{E}\left[A \mid t_n\right] + \int \mathrm{E}\left[Q \mid t_c\right] \; dF(t_c \mid t_n). \tag{4.8}$$

Since, by definition, $\mathrm{E}[Q \mid T] = 0$ for $T < \alpha$, by replacing $Q(T)$ with $\mathrm{E}[Q \mid T]$, $G(T)$ with $\mathrm{E}[A \mid T]$ and $p(t,T)dt$ with $dF(t_c \mid t_n)$ in equation (4.8) we get

$$Q(T) = G(T) + \int_0^T Q(t)p(t,T)dt \quad T \geq \alpha$$
$$Q(T) = 0 \quad 0 \leq T < \alpha$$

These equations imply (4.6).

As described in the previous section, since intermediate flooding steps depend on previous ones, function $Q(T)$ does not coincide with the expected total flooding area. However, given that the actual total flooding area is hard to describe, we believe that the cost estimated under this assumption should be an indicator of the overall behaviour of the protocol. Even if the protocol does not behave as stated, the results provided should give some intuition on how the total flooding area should look like and, in a non-rigorous manner, approximate it.

Equation (4.6) is linear Volterra equation of the second kind where the unknown function is $Q(T)$. If functions $G(T)$ and $p(t,T)$ can be computed, the problem of finding function $Q(T)$ can thus be reduced to solving (4.6). A major part of the thesis will be devoted to deriving functions $G(T)$ and $p(t,T)$ and then using them to find a solution for equation (4.6). We shall refer to them as the *expected one-step flooding area* and the *distribution of the improved accuracy* respectively.

Chapter 5

Fundamental Properties of the LER Protocol

In this chapter, we will prove a set of fundamental properties about the carrier nodes in the LER protocol that we will use in our later analysis.

We will first investigate how many carrier nodes exist per flooding step. Intuitively, this affects the behaviour of the protocol since the more nodes we have, the less area we need to flood per step in order to find any of them. Furthermore, it is interesting to know the accuracy of the information maintained in carrier nodes. Intuition here suggests that the more accurate carrier nodes are on predicting the destinations position, the better the protocol should behave.

Finally, we shall address the issue of how carrier nodes are positioned around an anchor point. These spatial distributions are important for two reasons. First, they can be used to determine the expected one-step flooding area $G(T)$, i.e. the area that we need to flood around the anchor point in order to find a carrier node. Second, they can provide information on whether some carrier nodes are more likely to be located through flooding than others with respect to their accuracy, influencing thus the distribution of improved accuracy $p(t, T)$.

5.1 Poisson Field crossing a moving disk

In the LER protocol, carrier nodes update their routing tables when they encounter the destination. Therefore, in order to find how many carrier nodes exist, one needs to investigate how many nodes cross the unit-area disk around the destination. This is the

subject of this section.

5.1.1 Poisson Field crossing a disk moving according to a deterministic function

Assume that $P_i(t)$, $i \in \{1, 2, \dots\}$, are two-dimensional independent Brownian motions of variance v, and that the point process $\boldsymbol{\pi}(t) = \{P_1(t), P_2(t), \dots\}$ is a Poisson field of density ρ. Assume that a point d, not belonging to $\boldsymbol{\pi}(t)$, moves according to a deterministic continuous function $f(t) : [0, +\infty) \to \mathbb{R}^2$ i.e.

$$P_d(t) = f(t), \quad t \geq 0.$$

Let

$$\mathcal{D}_f(t) = \{x \in \mathbb{R}^2 : ||x - f(t)|| \leq r_0\} \tag{5.1}$$

be a closed unit-area disk centered at $P_d(t) = f(t)$. By "unit-area" we mean that r_0 is such that

$$|\mathcal{D}_f(t)| = \pi r_0^2 = 1. \tag{5.2}$$

Furthermore, let

$$N_f(t) = \#\{i : \exists 0 \leq \tau \leq t \text{ such that } P_i(\tau) \in \mathcal{D}_f(\tau)\ \} \tag{5.3}$$

be the number of distinct particles of $\boldsymbol{\pi}$ that enter the disk around d in the interval $[0, t]$. Obviously

$$\mathbf{P}\{N_f(0) = k\} = \frac{(\rho|\mathcal{D}_f(0)|)^k}{k!} e^{-\rho|\mathcal{D}_f(0)|} = \frac{\rho^k}{k!} e^{-\rho}. \tag{5.4}$$

The case where $f(t) = f(0)$ for all $t \geq 0$ is the special case in which point d does not move but remains fixed. To distinguish this case we will use the notation A_0,$N_0(t)$ and $\mu_0(t)$ for a fixed disk, the number of points crossing it and its mean respectively. Révész [41] studied the behaviour of $N_0(t)$ in the case where the particles forming the Poisson field $\boldsymbol{\pi}$ moved according to standard Brownian motions. We will generalize his results to any continuous function f and to Brownian motions of variance v.

Theorem 3. *Let $\boldsymbol{\pi}(t)$ be a two-dimensional Poisson field of density ρ whose points move according to independent Brownian motions with variance v. Let $N_f(t)$ be the number*

of distinct points of $\boldsymbol{\pi}$ *that enter the unit-area disk centered at* $f(t)$ *in the interval* $[0,t]$*, where* $f(t):[0,+\infty)\to\mathbb{R}^2$*, as defined in equation* (5.3)*. Then*

$$\mathbf{P}\{N_f(t)=k\}=\frac{(\mu_f(t))^k}{k!}e^{-\mu_f(t)} \tag{5.5}$$

where $\mu_f(t)=\mathrm{E}\left[N_f(t)\right]$*,* $t\geq 0$ *is a strictly increasing function and* $\mu_f(0)=\rho$*. Furthermore* $N_f(t)$ *satisfies the independent increment property.*

Proof. The proof follows roughly the proof of Theorem 1.3 of Révész [41]. Let $z\in\mathbb{R}^2$ and

$$p_f(z,t)=\mathbf{P}\{\exists 0\leq\tau\leq t \text{ for which } P_i(\tau)\in\mathcal{D}_f(\tau)\mid P_i(0)=z\} \tag{5.6}$$

be the probability that a point starting from z visits $\mathcal{D}_f$ up to time t. Let

$$\mathcal{B}_z=\mathcal{B}(z,\Delta x,\Delta y)=\{z':z_x\leq z'_x\leq z_x+\Delta x, z_y\leq z'_y\leq z_y+\Delta y\} \tag{5.7}$$

be a rectangle on z and

$$\boldsymbol{\pi}(\mathcal{B}_z)=\#\{i:P_i(0)\in\mathcal{B}_z\} \tag{5.8}$$

be the number of points in $\mathcal{B}_z$ at time zero. Then

$$\mathbf{P}\{\boldsymbol{\pi}(\mathcal{B}_z)=k\}=\frac{(\rho|\mathcal{B}_z|)^k}{k!}e^{-\rho|\mathcal{B}_z|} \tag{5.9}$$

where

$$|\mathcal{B}_z|=\Delta x\Delta y. \tag{5.10}$$

For any z with $||z||>r_0$ let

$$N_f(z,t)=\#\{i:\exists 0\leq\tau\leq t \text{ such that } P_i(\tau)\in\mathcal{D}_f(\tau), P_i(0)\in\mathcal{B}_z\} \tag{5.11}$$

be the number of points which started from $\mathcal{B}_z$ and visited $\mathcal{D}_f$ in the interval $[0,t]$. Because of the independence of the movement of points P_i, intersections of different points with the moving disk $\mathcal{D}_f$ are also independent. Thus, as $\Delta X\to 0$ and $\Delta Y\to 0$, we have

$$\begin{aligned}
\mathbf{P}\{N_f(z,t)=k\} &\simeq \sum_{j=k}^{\infty}\frac{(\rho|\mathcal{B}_z|)^j}{j!}e^{-\rho|\mathcal{B}_z|}\binom{j}{k}\left(p_f(z,t)\right)^k\left(1-p_f(z,t)\right)^{j-k}\\
&=e^{-\rho|\mathcal{B}_z|}\frac{(\rho|\mathcal{B}_z|p_f(z,t))^k}{k!}\sum_{j=k}^{\infty}\frac{(\rho|\mathcal{B}_z|)^{j-k}}{(j-k)!}\left(1-p_f(z,t)\right)^{j-k}\\
&=e^{-\rho|\mathcal{B}_z|}\frac{(\rho|\mathcal{B}_z|p_f(z,t))^k}{k!}\exp\left[\rho|\mathcal{B}_z|\left(1-p_f(z,t)\right)\right]\\
&=\frac{\Lambda^k}{k!}e^{-\Lambda}
\end{aligned} \tag{5.12}$$

where

$$\Lambda = \rho|\mathcal{B}_z|p_f(z,t) = \rho p_f(z,t)\Delta x\Delta y. \tag{5.13}$$

Hence, as the sum of Poisson distributed independent random variables, $N_f(t)$ is a Poisson random variable with distribution

$$\mathbf{P}\{N_f(t) = k\} = \frac{\mu_f^k}{k!}e^{-\mu_f} \tag{5.14}$$

where

$$\mu_f = \mu_f(t) = \rho + \rho \iint_{\overline{\mathcal{D}_f}(0)} p_f(z,t)dxdy \tag{5.15}$$

and $\overline{\mathcal{D}_f}(0) = \mathbb{R}^2 \setminus \mathcal{D}_f(0)$.

This proves (5.5). We now need to prove that N_f has the independent increment property, i.e. for any given k and for all $t_1, t_2, \ldots, t_k$ with $0 \leq t_1 < t_2 < \ldots < t_k$ the random variables $N_f(t_1)$, $N_f(t_2) - N_f(t_1)$, $\ldots$, $N_f(t_k) - N_f(t_{k-1})$ are independent. In proving this, we follow again [41].

Let $P_i(0) \in \mathcal{B}_z$. Then the process $P_i(t)$ meets $\mathcal{D}_f$ up to time t with probability $p_f(z,t)$. Hence the points located in $\mathcal{B}_z$ at time $t = 0$ and *not* visiting $\mathcal{D}_f$ up to time t form a Poisson field, *independent* from $\{N_f(\tau) : 0 \leq \tau \leq t\}$[1]. Let $\boldsymbol{\pi}^*(\overline{\mathcal{D}_f}(0), t)$ be the points in $\overline{\mathcal{D}_f}(0)$ that do not meet $\mathcal{D}_f$ before t. Then $\boldsymbol{\pi}^*(\overline{\mathcal{D}_f}(0), t)$ is a Poisson field independent from $\{N_f(\tau) : 0 \leq \tau \leq t\}$. Now, $N_f(t_k) - N_f(t_{k-1})$ depends only on $\boldsymbol{\pi}^*(\overline{\mathcal{D}_f}(0), t_{k-1})$ and is thus independent from $\{N_f(\tau) : 0 \leq \tau \leq t_{k-1}\}$. Therefore $N_f(t_k) - N_f(t_{k-1})$ and $N_f(t_{k-1})$ are independent random variables. The general statement can be proved similarly. □

Corollary 5.1. *Process $N_f(t)$ is non-homogeneous Poisson process with bulk arrivals at time 0.*

The mean values μ_f cannot be described with basic functions. This is true even for μ_0. However, Révész cites the following result from Spitzer regarding the asymptotic behaviour of μ_0 [43]

Theorem 4. *For a Poisson field moving according to standard independent Brownian motions,*

$$\mu_0(t) = \rho\left(\frac{2\pi t}{\log t} + (d + o(1))\frac{t}{\log^2 t}\right) \tag{5.16}$$

[1] Splitting with probability p a Poisson random variable with mean λ creates two *independent* Poisson random variables with means $p\lambda$ and $(1-p)\lambda$.

where d is a constant.

This result can be extended to non-standard Brownian motions of variances v.

Corollary 5.2. *For a Poisson field moving according to independent Brownian motions with variances v,*

$$\mu_0(t) = \rho\left(\frac{2\pi vt}{\log vt} + (d + o(1))\frac{vt}{\log^2 vt}\right) \tag{5.17}$$

where d is a constant.

5.1.2 Poisson Field crossing a disk performing a Brownian motion

In section 5.1.1 we described how points forming a Poisson field and performing independent Brownian motions cross a closed unit-area disk which moves according to a deterministic function. We will now see how these results are affected if the movement of the disk is also stochastic. In particular, we will investigate the case where the center of the disk also performs a Brownian motion.

Assume that $P_i(t), i \in \{1, 2, \dots\}$, are two-dimensional independent Brownian motions of variance v, and that the point process $\boldsymbol{\pi}(t) = \{P_1(t), P_2(t), \dots\}$ is a Poisson field of density ρ. Assume that a point d, not belonging to $\boldsymbol{\pi}(t)$, moves according to a a two-dimensional Brownian motion $B(t)$ with variance v, i.e.

$$P_d(t) = B(t), t \geq 0.$$

It is easy to derive the following lemma from the definitions of a Brownian motion and a Poisson field.

Lemma 5.1. *Let $\hat{P}_i(t) = P_i(t) - P_d(t)$, $i \in \{1, 2, \dots\}$, be the vector distance of point $P_i(t)$ from point $P_d(t)$. Then*

1. *$\hat{P}_i(t)$, $i \in \{1, 2, \dots\}$, are Brownian motions with variance $2v$ and*
2. *The point process $\hat{\boldsymbol{\pi}}(t) = \{\hat{P}_1(t), \hat{P}_2(t), \dots\}$ is a Poisson field of density ρ.*

Let

$$\mathcal{D}_B(t) = \{x \in \mathbb{R}^2 : ||x - B(t)|| \leq r_0\} \tag{5.18}$$

where r_0 is such that (5.2) holds be a closed unit-area disk centered at $P_d(t) = B(t)$ and

$$N_B(t) = \#\{i : \exists 0 \leq \tau \leq t \text{ such that } P_i(\tau) \in \mathcal{D}_B(\tau) \} \tag{5.19}$$

be the number of distinct particles of $\boldsymbol{\pi}$ that enter the disk around d in the interval $[0, t]$.

By fixing the origin on $P_d(t)$, $\mathcal{D}_B$ becomes $\mathcal{D}_0$, i.e. a fixed unit-area disk centered at the origin, and by Lemma 5.1 $\hat{P}_i(t)$ are Brownian motions and that form a Poisson field. Does Lemma 5.1 imply that we can use Theorem 3 for $N_0(t)$ to describe $N_B(t)$? The answer is no, because $\hat{P}_i(t)$ *are not independent.*

However, $\hat{P}_i(t)$ are independent conditional on $P_d(t)$. As a matter of fact, conditional on $P_d(t) = f(t)$, $t \geq 0$, $N_B(t)$ is none other than $N_f(t)$. Theorem 3 implies that $N_B(t)$, as an average of Poisson distributed variables $N_f(t)$ over all possible f paths the destination may follow, *is not* Poisson distributed and $\tilde{N}_B(t) = N_B(t) - N_B(0)$ *is not* a non-homogeneous Poisson process. However, the expected number of particles crossing the moving disk can still be computed.

Proposition 5.1.

$$\mu_B(t) = \mathrm{E}\left[N_B(t)\right] = \mu_0(2t) \tag{5.20}$$

where $\mu_0(t)$ is the expected number of points that cross a fixed unit-area disk in the time interval $[0, t]$.

Proof. Let $\delta_i(t)$ be one if point i has crossed $\mathcal{D}_B$ up to time t and zero otherwise. Then

$$N_B(t) = \sum_i \delta_i(t)$$

thus

$$\begin{aligned}
\mathrm{E}\left[N_B(t)\right] &= \mathrm{E}\left[\sum_i \delta_i(t)\right] \\
&= \sum_i \mathrm{E}\left[\delta_i(t)\right] \\
&= \sum_i \mathbf{P}\{\exists 0 \leq \tau \leq t : P_i(\tau) \in \mathcal{D}_B(\tau)\} \\
&= \sum_i \mathbf{P}\{\exists 0 \leq \tau \leq t : \hat{P}_i(\tau) \in \mathcal{D}_0(\tau)\}
\end{aligned}$$

For each i, the probability $\mathbf{P}\{\exists 0 \leq \tau \leq t : P_i(\tau) \in \mathcal{D}_B(\tau)\}$ is the probability that i will cross the moving disk $\mathcal{D}_B$ up to time t. This is equal to the probability that

$\hat{P}_i = P_i - P_d$ will cross a unit-area disk fixed at the origin. By Lemma 5.1, if initial Brownian motions had variances v, $\hat{P}_i$ will be (not independent) Brownian motions with variances $2v$. However, each one of these probabilities is the same as the probability that $\hat{P}_i$ would cross a disk at the origin if $\hat{P}_i$ were independent. Therefore their sum is equal to the expected number of distinct particles crossing a unit-area disk at fixed the origin while moving *independently* with variance proportional to $2v$. □

This proposition means that although the number of distinct particles crossing a moving disk $\mathcal{D}_B$ is distributed differently than the number of particles crossing a unit-area disk $\mathcal{D}_0$ fixed at the origin while moving independently with twice the variance, their expected values are the same.

5.1.3 Comparing a moving and a static destination

As we have already stated, $N_B(t)$ is the average of random variables $N_f(t)$ over all possible f paths the destination may follow. Hence, $\mu_B(t)$ is also the average of $\mu_f(t)$ over all possible f paths the destination may follow. Since $\mu_0(t)$ is a strictly increasing function and $\mu_0(2t) > \mu_0(t)$, Proposition 5.1 implies that there exist paths f the destination may follow such that $\mu_f(t) \geq \mu_0(t)$. We claim that something stronger holds.

Proposition 5.2. *For every continuous function* $f : [0, +\infty) \to \mathbb{R}^2$ *and every* $t \geq 0$

$$\mu_f(t) \geq \mu_0(t). \tag{5.21}$$

This proposition indicates that a particle that moves will meet more particles, on average, than a particle that does not. If its movement is a Brownian motion, we already know this is true: Proposition 5.1 suggests that it meets as many particles as a stationary would meet in *double* the time. The interesting point of Proposition 5.2 is that if it moves according to *any* deterministic two-dimensional function $f(t)$, it will still meet more particles!

Proving this requires an elaborate technique, the presentation of which is beyond the scope of this thesis. However, we will show how this problem can be reduced to one that is intuitively more simple to address, and shall provide a proof of this by Quastel [39] in the appendix.

Let $f(t) \in \mathbb{R}^2$, $t \geq 0$, be a two-dimensional vector and $\mathcal{D}_f(t)$ be a closed unit-area

disk around $f(t)$ as defined in (5.1). Furthermore, let

$$\begin{aligned}\mathbf{A}_f(t) &= |\bigcup_{0\le\tau\le t} \mathcal{D}_f(\tau)| \\ &= |\{x \in \mathbb{R}^2 : x \in \mathcal{D}_f(\tau) \text{ for some } 0 \le \tau \le t\}|\end{aligned}$$

be the Lebesque measure -the area- of the region covered by the moving disk $\mathcal{D}_f$ in the interval $[0, t]$. If B is a two-dimensional Brownian motion, the above region of $\mathbb{R}^2$ is called a two-dimensional *Wiener sausage*. For an exact definition of a Wiener sausage and basic properties we refer the reader to [41], pp. 210-212. The expected area of a Wiener sausage is closely related to the mean μ_f, as the following proposition implies.

Proposition 5.3. *Let $f(t)$, $t \ge 0$ be a two-dimensional vector and B a two-dimensional Brownian motion. Let $\mu_f(t) = \mathrm{E}[N_f(t)]$. Then for all $t \ge 0$*

$$\mu_f(t) = \rho \mathrm{E}[\mathbf{A}_{B-f}(t)]$$

where ρ is the density of the Poisson field and $\mathbf{A}_{B-f}(t)$ is the area of the Wiener sausage of a Brownian motion with drift $-f$.

Proof. W.l.o.g. we assume that $f(0)$ is the zero vector, i.e. f starts at the origin. Equation (5.15) in the proof of Theorem 3 gives us

$$\begin{aligned}\mu_f(t) &= \rho + \rho \iint_{\overline{\mathcal{D}_f}(0)} p_f(z,t)dxdy \\ &= \rho \iint_{\mathbb{R}^2} p_f(z,t)dxdy\end{aligned}$$

where

$$p_f(z,t) = \mathbf{P}\{\exists 0 \le \tau \le t \text{ for which } P_i(\tau) \in \mathcal{D}_f(\tau) \mid P_i(0) = z\}$$

is the probability that a point starting from z visits $\mathcal{D}_f$ up to time t. Let $B(t)$, $t \ge 0$ be a two-dimensional Brownian motion starting at the origin. Then p_f can be written as

$$\begin{aligned}p_f(z,t) &= \mathbf{P}\{\exists 0 \le \tau \le t : ||\,(z + B(\tau)) - f(\tau)|| \le r_0\} \\ &= \mathbf{P}\{\exists 0 \le \tau \le t : ||\,(B(\tau) - f(\tau)) - (-z)|| \le r_0\} \\ &= \mathbf{P}\{\exists 0 \le \tau \le t : -z \in \mathcal{D}_{B-f}(\tau)\}\end{aligned}$$

which is the probability a unit-area disk whose center performs a Brownian motion with drift $-f$ will pass from the point $-z$ in the interval $[0,t]$. Hence μ_f becomes

$$\begin{aligned}\mu_f(t) &= \rho \iint_{\mathbb{R}^2} \mathbf{P}\{\exists 0 \le \tau \le t : -z \in \mathcal{D}_{B-f}(\tau)\} dxdy \\ &= \rho \iint_{\mathbb{R}^2} \mathbf{P}\{\exists 0 \le \tau \le t : z \in \mathcal{D}_{B-f}(\tau)\} dxdy \\ &= \iint_{\mathbb{R}^2} \mathrm{E}\left[\delta_{B-f}(z,t)\right] dxdy \\ &= \mathrm{E}\left[\iint_{\mathbb{R}^2} \delta_{B-f}(z,t) dxdy\right] \\ &= \mathrm{E}\left[\mathbf{A}_{B-f}(t)\right]\end{aligned}$$

where $\delta_{B-f}(z,t)$ is one if $\mathcal{D}_{B-f}$ crosses z within the time interval $[0,t]$ and zero otherwise. □

Hence, proving Proposition 5.2 is equivalent to proving that for every continuous function $f : [0,+\infty) \to \mathbb{R}^2$ and every $t \ge 0$

$$\mathrm{E}\left[\mathbf{A}_{B+f}(t)\right] \ge \mathrm{E}\left[\mathbf{A}_B(t)\right]$$

i.e. the expected area covered by a unit-area disk performing a Brownian motion is less than or equal to the expected area covered by a unit-area disk performing a Brownian motion with any (continuous) drift. This is a more illuminating perspective on the problem, because it relates it to only one Brownian motion rather than the infinitely many that were involved in the original problem. The proof of this in a one-dimensional setting is rather straightforward, however for two or more dimensions it becomes quite involved. We present a proof by Quastel [39] in the appendix.

5.1.4 Stochastic domination

A random variable X whose range is A stochastically dominates another random variable Y, whose range is also A, if

$$\mathbf{P}\{X > a\} \ge \mathbf{P}\{Y > a\} \tag{5.22}$$

for all $a \in A$. An immediate consequence of (5.22) is that $\mathrm{E}[X] \ge \mathrm{E}[Y]$. We will prove a series of lemmas about stochastic domination that will assist our analysis.

Lemma 5.2. *Let X_1, X_2 be two Poisson distributed random variables with $\mathrm{E}[X_1] = \mu_1$ and $\mathrm{E}[X_2] = \mu_2$. If $\mu_1 \geq \mu_2$, then X_1 stochastically dominates X_2.*

Proof. It suffices to prove that for X Poisson distributed with expectation μ

$$\mathbf{P}\{X > k\} = 1 - \sum_{i=0}^{k} e^{-\mu} \frac{\mu^i}{i!} \tag{5.23}$$

is an increasing function in terms of $\mu \geq 0$ for all integers $k \geq 0$. Indeed

$$\begin{aligned} \frac{d}{d\mu}\mathbf{P}\{X > k\} &= \frac{d}{d\mu}\left(-\sum_{i=0}^{k} e^{-\mu}\frac{\mu^i}{i!}\right) \\ &= e^{-\mu} + \sum_{i=1}^{k}\left(e^{-\mu}\frac{\mu^i}{i!} - e^{-\mu}\frac{\mu^{i-1}}{(i-1)!}\right) \\ &= e^{-\mu}\frac{\mu^k}{k!} \end{aligned}$$

which is non-negative for all $\mu \geq 0$ and for all $k \geq 0$. □

Lemma 5.3. *Let*

$$N_B(t) = \#\{i : \exists 0 \leq \tau \leq t \text{ such that } P_i(\tau) \in \mathcal{D}_B(\tau)\ \} \tag{5.24}$$

be the number of distinct particles of $\boldsymbol{\pi}$ that enter the disk around a node that performs a Brownian motion in the interval $[0, t]$, as defined in section 5.1.2. Furthermore, let

$$N_0(t) = \#\{i : \exists 0 \leq \tau \leq t \text{ such that } P_i(\tau) \in \mathcal{D}_0(\tau)\ \} \tag{5.25}$$

be the number of distinct particles of $\boldsymbol{\pi}$ that enter a fixed disk in the interval $[0, t]$, as defined in section 5.1.1. Then $N_B(t)$ stochastically dominates $N_0(t)$ for all $t \geq 0$.

Proof. Conditioned on a path f the destination may take, N_B is none other than N_f. Then, by Proposition 5.2 and Lemma 5.2, we get that N_f stochastically dominates N_0. Furthermore, N_B can be seen as an average of N_f over all paths the destination may take. Hence,

$$\begin{aligned} \mathbf{P}\{N_B > a\} &= \mathrm{E}\left[\mathbf{P}\{N_f > a \mid f\}\right] \\ &\geq \mathrm{E}\left[\mathbf{P}\{N_0 > a \mid f\}\right] \\ &= \mathbf{P}\{N_0 > a\} \end{aligned}$$

□

5.2 The accuracy of carrier nodes

The analysis in section 5.1 provides some insight in how many carrier nodes exist. However, it is not clear how the accuracy of the information maintained in these nodes is distributed. We shall address this issue by introducing the concepts of *first entry* and *last exit* epochs.

5.2.1 First entry epochs vs. last exit epochs

As an arrival process, $N_B(t)$ models the first entry epochs into $\mathcal{D}_B$ within the interval $[0, t]$ of points of the Poisson field. To make this more precise, we give a more elaborate definition of the arrival process.

First of all, we assume that particles P_i perform Brownian motions of variance v in the time interval $(-\infty, +\infty)$. Note that at any time $t \in \mathbb{R}$ the point process $\boldsymbol{\pi}(t) = \{P_1(t), P_2(t), \ldots\}$ forms a Poisson field. We define the first entry epoch of point i into $\mathcal{D}_B$ after time t as

$$t_{fen}(\mathcal{D}_B, i, t) = \min\{\tau : \tau \geq t \wedge P_i(\tau) \in \mathcal{D}_B(\tau)\}. \tag{5.26}$$

Lemma 5.4. *The first entry epoch $t_{fen}(\mathcal{D}_B, i, t)$ of a point i in $\mathcal{D}_B$ is a well-defined random variable.*

Proof. It is easy to see that $\{\tau : \tau \geq t \wedge P_i(\tau) \in \mathcal{D}_B(\tau)\}$ is non-empty a.s., since every point will enter the disk eventually with probability one [3]. Furthermore, since all its elements are lower-bounded by t an infimum exists. In order for $t_{fen}(\mathcal{D}_B, i, t)$ to be a random variable, we need to show that a minimum exists a.s. Let $t' \geq t$ denote the infimum of the above set. In order for the minimum to exist, t' must belong in the above set, i.e. $P_i(t') \in \mathcal{D}_B(t')$. Suppose that $P_i(t') \notin \mathcal{D}_B(t')$. Since $P_i(t)$, $B(t)$ are Brownian motions, their difference $P_i'(t) = P_i(t) - B(t)$ is continuous. Thus, for every $\epsilon > 0$ there exists a $\delta > 0$ such that $|P_i'(t' + x) - P_i'(t')| < \epsilon$ for every $0 < x < \delta$. Note that $P_i(t) \in \mathcal{D}_B(t)$ is equivalent to $P_i'(t) \in \mathcal{D}_0$, where $\mathcal{D}_0$ a closed unit-area disk fixed at the origin. Since $\mathcal{D}_0$ is a closed set, its complement $\overline{\mathcal{D}_0}$ is an open set. By our hypothesis, $P_i'(t') \in \overline{\mathcal{D}_0}$. Therefore there exists an $\epsilon' > 0$ such that $P_i'(t') + \vec{y} \in \overline{\mathcal{D}_0}$ for all $0 < |P_i'(t') - \vec{y}| < \epsilon'$. By setting $\epsilon = \epsilon'$, we get that there exists δ such that $|P_i'(t' + x)| \in \overline{\mathcal{D}_0}$. Hence, a lower bound of the set $\{\tau : \tau \geq t \wedge P_i(\tau) \in \mathcal{D}_B(\tau)\}$ grater than t', a contradiction. □

Note that the first entry epoch of a point that is already in $\mathcal{D}_B$ at t is t. We define the following process

$$N_{fen}(\mathcal{D}_B, T, t) = \#\{i : t_{fen}(\mathcal{D}_B, i, T) \leq t\} \quad t \geq T \tag{5.27}$$

This process is counting the number of first entry epochs in the interval $[T, t]$. Interestingly enough, $N_{fen}(\mathcal{D}_B, 0, t)$ is none other than N_B.

Lemma 5.5.

$$N_{fen}(\mathcal{D}_B, 0, t) = N_B(t)$$

a.s. for all $t \geq 0$.

Proof. For every point i that is counted by $N_B(t)$ there exists a $\tau \in [0, t]$ such that $P_i(\tau) \in \mathcal{D}_B(\tau)$. Thus $\{\tau : 0 \leq \tau \leq t \wedge P_i(\tau) \in \mathcal{D}_B(\tau)\}$ is non-empty and the minimum of $\{\tau : \tau \geq 0 \wedge P_i(\tau) \in \mathcal{D}_B(\tau)\}$ (which exists by Lemma 5.4) will be less or equal to t. Therefore i is also counted by $N_{fen}(\mathcal{D}_B, 0, t)$. Similarly, if a point is counted by $N_{fen}(\mathcal{D}_B, 0, t)$ the set $\{\tau : \tau \geq 0 \wedge P_i(\tau) \in \mathcal{D}_0\}$ is non-empty and at least one element in it (its minimum) is less or equal to t, so i will also be counted by N. □

Note again that

$$\mathbf{P}\{N_{fen}(\mathcal{D}_B, T, 0) = k\} = \frac{(\rho)^k}{k!} e^{-\rho} \tag{5.28}$$

i.e. bulk arrivals happen at time 0. However, at subsequent epochs only single arrivals may happen, i.e. the probability that two points enter the disk simultaneously at epoch $t > 0$ is 0.

Apart from the first entry epochs it is interesting to know the *last exit epochs* from disk $\mathcal{D}_B$. We define the last exit epoch of point i from $\mathcal{D}_B$ prior to time t as

$$t_{lex}(\mathcal{D}_B, i, t) = \max\{\tau : \tau \leq t \wedge P_i(\tau) \in \mathcal{D}_B(\tau)\}. \tag{5.29}$$

One can show just as in Lemma 5.4 that last exit epochs constitute well-defined random variables. According to the above definition the last exit epoch of a point that is still in $\mathcal{D}_B$ at time t is t. Similarly to (5.27) we define the following process

$$N_{lex}(\mathcal{D}_B, t, T) = \#\{i : t_{lex}(\mathcal{D}_0, i, T) \geq t\} \quad t \leq T. \tag{5.30}$$

This process is counting the number of last exit epochs in the interval $[t, T]$. For a fixed

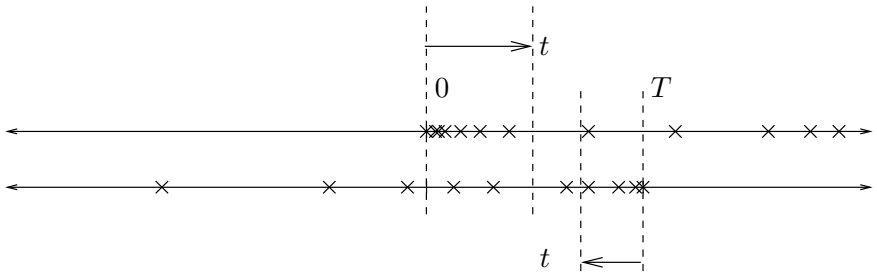

Figure 5.1: First entry and last exit processes.

T, $N_{lex}(\mathcal{D}_B, T-t, T)$ can be seen as a counting process. Note that, as in (5.28)

$$\mathbf{P}\{N_{lex}(\mathcal{D}_B, T, T) = k\} = \frac{(\rho)^k}{k!}e^{-\rho} \tag{5.31}$$

since the points whose last exit epoch is T are exactly the ones that are in disk $\mathcal{D}_B$ at that time. This is not the only similarity that N_{lex} has with N_{fen}, as it can be seen by the following lemma.

Lemma 5.6. *For any fixed T, $N_{lex}(\mathcal{D}_B, T-t, T)$ $(t \geq 0)$ is distributed as $N_{fen}(\mathcal{D}_B, 0, t)$.*

Proof. Assume that at time T all points start moving "backwards" following the trajectories that they followed in the interval $(-\infty, T]$. At time T, as at any other time, the point process $\boldsymbol{\pi}(T)$ was a Poisson field. As points move backwards they are performing independent Brownian motions by the time reversibility property (see page 42). Finally, the last exit epochs of the original process become the first entry epochs in the backward one, since the last time a point exited the disk becomes the first time it enters it now. Thus, in this backward model counting the last exit epochs of the forward model is probabilistically equivalent to counting the first entry epochs in $\mathcal{D}_B$ of points forming a Poisson field moving according to independent Brownian motions. This however is described by the process $N_{fen}(\mathcal{D}_B, 0, t)$. □

The above theorem implies that $N_{fen}(\mathcal{D}_B, 0, T)$ is distributed exactly as $N_{lex}(\mathcal{D}_B, 0, T)$. As a matter of fact something stronger holds.

Lemma 5.7.

$$N_{fen}(\mathcal{D}_B, 0, T) = N_{lex}(\mathcal{D}_B, 0, T) \quad a.s.$$

Proof. For every point i that was counted in N_{fen}, the set $\{\tau : 0 \leq \tau \leq T \wedge P_i(\tau) \in \mathcal{D}_B(\tau)\}$ is non-empty. Therefore, the maximum of $\{\tau : \tau \leq T \wedge P_i(\tau) \in \mathcal{D}_B(\tau)\}$ will be greater than 0 and i is also counted by N_{lex}. The opposite direction is proved similarly. □

5.2.2 Accuracy as a last exit epoch

In the LER protocol the accuracy of a carrier node n regarding the position of destination d is by definition

$$M_{nd}(t) = \min\{\tau : ||P_n(t-\tau) - P_d(t-\tau)|| \leq r_0,\ \tau \geq 0\}.$$

Therefore $M_{nd}(t)$ is none other than the last exit epoch of n from the unit-area disk around P_d. This means that, at time T, the number of carrier nodes with accuracy less or equal to t is none other than $N_{lex}(\mathcal{D}_B, T-t, T)$. Therefore, Lemma 5.6 can be used to describe how the accuracies of carrier nodes are distributed.

5.3 The spatial distribution of carrier nodes

The spatial distribution of carrier nodes and the destination is important in determining both the expected one-step flooding area and the density of the improved accuracy. The one-step flooding area obviously relates to this distribution since it is defined by the closest carrier node to the anchor point. The density of the improved accuracy is also influenced, for example by whether nodes that met the destination earlier rather than later are more or less likely to be located through flooding. Unfortunately, the joint spatial distribution of carrier nodes and the destination is hard to describe. To illustrate this, in this section we briefly discuss how marginal distributions relate to the first and last exit epochs and how the movements of carrier nodes and the destination with respect to time intervals defined by these epochs depend on each other.

We are ultimately interested in the positions of carrier nodes and the destination relatively to the anchor point. Let $P_d(t)$, $t \in [-T, 0]$ be the trajectory of the destination and $P_d(-T)$ be the current anchor point. A carrier node is then a node that met the destination in the interval $(-T, 0]$. Let N be the number of carrier nodes the destination meets in the interval $(-T, 0]$, and $P_i(t)$, $t \in [-T, 0]$, $1 \leq i \leq N$, denote the trajectories of carrier nodes.

Let S_i, $1 \leq i \leq N$, be the first entry epochs of carrier nodes in the unit-area disk around the destination. As we have already discussed, N is can be seen as a counting process described in section 5.2.1 (namely N_{fen}) and S_i as the arrival epochs associated with it. The distance at time 0 of a carrier node i from the anchor point can be written

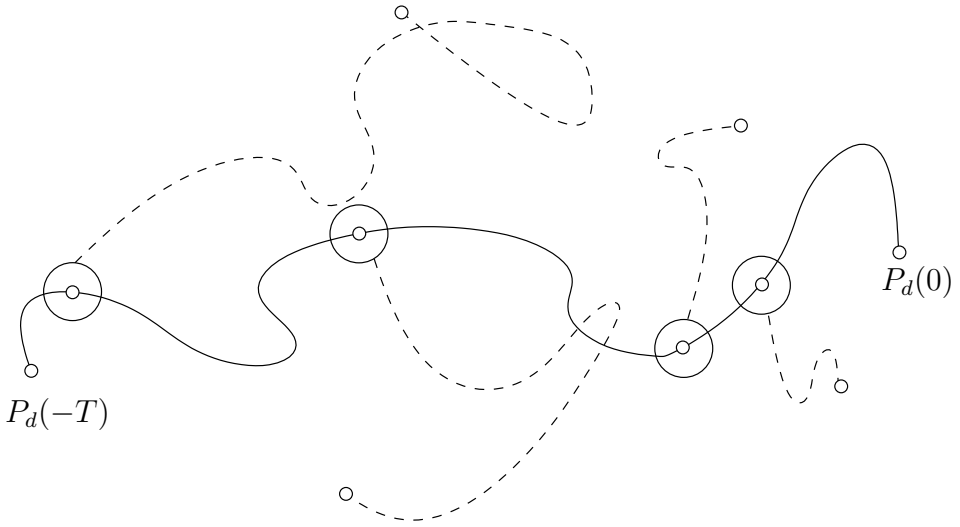

Figure 5.2: A the positions of carrier nodes relative to the anchor point.

as the sum of three random vectors as follows.

$$P_i(T) - P_d(-T) = P_d(S_i) - P_d(-T) + X_i + P_i(0) - P_i(S_i) \tag{5.32}$$

The random vector $P_d(S_i) - P_d(-T)$ is the distance of the destination from the anchor point at the time the encounter occurred. Vector X_i, which is necessarily equal to $P_i(S_i) - P_d(S_i)$, indicates the distance between the destination and the carrier node at the encounter epoch, and thus $||X_i||$ is upper-bounded by $1/\sqrt{\pi}$. Finally, $P_i(0) - P_i(S_i)$ is the distance between the carrier node at time 0 and the point at which it met the destination.

Vectors $P_d(S_i) - P_d(-T)$, $1 \leq i \leq N$, are normally distributed and independent of vectors $P_i(0) - P_i(S_i)$. Of course, they are not independent of each other. On the other hand, $P_i(0) - P_i(S_i)$, $1 \leq i \leq N$, are normally distributed and independent of each other. Furthermore, for each i, $P_i(0) - P_i(S_i)$ independent of $P_d(T) - P_d(S_i)$: carrier nodes move independently of the destination after they meet it.

The picture changes if we apply the above to last exit epochs. Let S'_i, $1 \leq i \leq N$, be the last exit epochs of carrier nodes in the unit-area disk around the destination. These form, as we discussed in section 5.2.1, a different arrival process than the one defined by first entry epochs, which follows however, albeit reversed in time, the same distribution.

Similarly to 5.32, the distance of a carrier node from the anchor point can be written

as

$$P_i(T) - P_d(-T) = P_d(S_i') - P_d(-T) + X_i + P_i(0) - P_i(S_i') \tag{5.33}$$

The trajectories of carrier nodes in the intervals $[S_i, T]$ are *not* independent of the trajectory of the destination in this interval. In fact, they are conditioned on avoiding the unit-area disk around it. Because of this property, describing the distributions of the aforementioned three vectors is harder than when referring to first entry epochs.

5.4 A function that should upper-bound the cost

The function we wish to compute is $Q(T)$, $T \geq 0$, described by the Volterra equation presented in (4.6). It is an approximation of the total expected total area the protocol floods in order to route a packet from an anchor point with accuracy T.

A question that thus arises is what value of the cost would indicate that the protocol behaves "well", at least with respect to other protocols. The minimum requirement one might have from an approximate information protocol is that the overhead is less than the one that occurs while flooding without taking any approximate information into account. This is none other than the cost of flooding from the source and looking for the destination *itself*. This is the approach taken by reactive routing protocols like AODV [38]; justifying the proactive features of the LER protocol (mainly, maintaining a routing table) requires that they outperform such an approach. Hence, it is interesting to see what is the expected flooding cost for finding the destination itself in our model.

Assume that nodes n and d move according to two-dimensional Brownian motions in the interval $[-T, 0]$ with variances v and that at time $-T$ they had an encounter. Let $A = \pi||P_n(0) - P_d(0)||^2$. Then, similarly to (5.32), the distance between n and d can be written as

$$P_n(0) - P_d(0) = (P_n(0) - P_n(-T)) - (P_d(0) - P_d(-T)) + X \tag{5.34}$$

where $X = P_n(T) - P_d(-T)$ is their distance at time $-T$. As in the previous section, $||X|| \leq 1/\sqrt{\pi}$.

Vectors $P_n(0) - P_n(-T)$ and $P_d(0) - P_d(-T)$ indicate the distances traversed by n and d respectively in the interval $[-T, 0]$. If $-T$ is not a last exit epoch, and the trajectories of n and d are independent, $P_n(t) - P_n(-T)$ and $P_d(t) - P_d(-T)$, for $t \in [-T, 0]$, are independent Brownian motions with variances v on each axis. Hence, their difference is

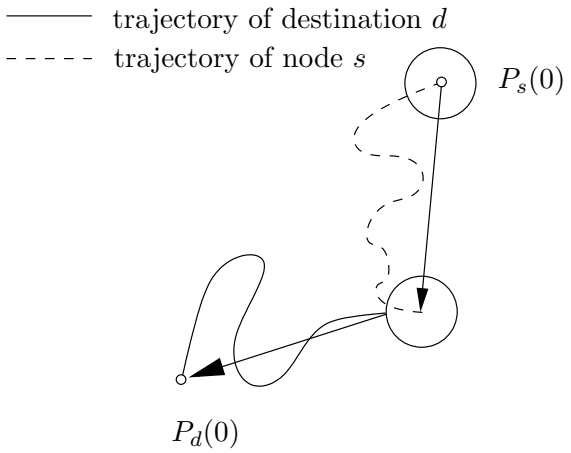

Figure 5.3: The distance between the destination d and a source n. Node n met d for the last time at T time ago.

also a Brownian motion with double the variance on each axis. Equivalently, $P_n(t)-P_d(t)$, $t \in [-T, 0]$ can be seen as a two-dimensional Brownian motion with variance $2v$ starting at X. The expectation of A can be written as

$$\begin{aligned}
\mathrm{E}[A] &= \pi \mathrm{E}\left[||(P_n(0) - P_n(-T)) - (P_d(0) - P_d(-T)) + X||^2\right] \\
&\leq \pi \mathrm{E}\left[(||(P_n(0) - P_n(-T)) - (P_d(0) - P_d(-T))|| + ||X||)^2\right] \\
&\leq \pi \mathrm{E}\left[\left(||(P_n(0) - P_n(-T)) - (P_d(0) - P_d(-T))|| + 1/\sqrt{\pi}\right)^2\right]
\end{aligned}$$

Similarly

$$\begin{aligned}
\mathrm{E}[A] &= \pi \mathrm{E}\left[||(P_n(0) - P_n(-T)) - (P_d(0) - P_d(-T)) + X||^2\right] \\
&\geq \pi \mathrm{E}\left[(||(P_n(0) - P_n(-T)) - (P_d(0) - P_d(-T))|| - ||X||)^2\right] \\
&\geq \pi \mathrm{E}\left[\left(||(P_n(0) - P_n(-T)) - (P_d(0) - P_d(-T))|| - 1/\sqrt{\pi}\right)^2\right]
\end{aligned}$$

These two bounds indicate that, for large values of T, if the trajectories of n and d are independent,

$$\mathrm{E}[A] = \Omega(\mathrm{E}\left[\pi||(P_n(0) - P_n(-T)) - (P_d(0) - P_d(-T))||^2\right]).$$

This quantity is equal to $4\pi vT$: it is an immediate consequence[2] of the fact that the distance of the two points is a Brownian motion with variance $2v$. This indicates that

[2]See Lemma 6.1 in section 6.1.1 for the computation of the variance of a two dimensional Brownian motion.

the expected area the source node needs to flood, given that it met the destination T time ago, is proportional to T.

However, in our setting, the trajectory of a source node, given that its accuracy on the destination's position is T, is not independent from the trajectory of the destination. In fact, as we stated in the previous section, the node performs a Brownian motion conditioned on not entering the unit-area disk around the destination in the interval $(-T, 0]$. Equivalently, $P_n(t) - P_d(t)$, $t \in [-T, 0]$, can be seen as a two-dimensional Brownian motion with variance $2v$, starting at the boundary of the unit-area disk around the origin and then *conditioned on not entering this disk* again in $(-T, 0]$.

This indicates that the expected area a source node has to flood in order to find the destination is different than $4\pi vT$. However, since the distance between it and the destination is a Brownian motion conditioned on not visiting the unit-area disk fixed at the origin, one can expect that this motion "wanders off" further than an unconditional Brownian motion. This means that the expected area in this setting should be greater than $4\pi vT$.

Proving that this intuitive observation is true requires mathematical notions whose presentation is beyond the scope of this thesis. We thus restrict ourselves to the aforementioned analysis, which, although non-rigorous, gives a good sense of how the expected cost grows if the source does not use any approximate information -namely, it is worse than linear in terms of the accuracy T. An immediate implication is that any sub-linear cost function for LER would indicate that the protocol outperforms a reactive protocol under our model.

Chapter 6

A Simplified Analysis

In this chapter, we will work under the following simplified version of our original model. We assume that the positions of carrier nodes follow independent, identical, omnidirectional, two-dimensional normal distributions centered around the current anchor point with variance vT on each axis, where T is the accuracy at the current anchor point. Furthermore, if $N(t)$ is the number of carrier nodes with accuracy less than or equal to t, then $\{N(t), t \geq 0\}$ is a non-homogeneous Poisson process with bulk arrivals at time zero and $\mathrm{E}[N(t)] = \mu_0(2t)$. Function $\mu_0(t)$ is the expected number of nodes encountered by a static destination within a time interval of length t and is described by equation (5.17) in Corollary 5.2.

We will denote with $G_s(T)$ and $p_s(t, T)$ the expected one-step flooding area and the density of the improved accuracy under this model respectively. Similarly to (4.6), the function $Q_s(T)$ that approximates the cost under this model will be a function that satisfies the equations

$$Q_s(T) = G_s(T) + \int_{\alpha}^{\infty} Q_s(t) p_s(t, T) dt \quad T \geq \alpha \tag{6.1a}$$

$$Q_s(T) = 0 \quad 0 \leq T < \alpha \tag{6.1b}$$

Notice that this model refers only to one flooding step of the protocol, namely the first. In other words, we do not investigate how under such a model the distributions of the above quantities (number of carrier nodes, their positions on the plane e.t.c.) will be influenced after the first step takes place. The reason is that we have restricted ourselves to approximating the flooding cost with $Q(T)$, which by definition relates only to quantities measured at the first step. In that sense, this model is a simplified model

that yields $G_s(T)$ and $p_s(t,T)$ for one step and, indirectly, a $Q_s(T)$. It is not a model that describes the mechanics of a particular network.

That said, this model differs from the first step of our original one namely in the following three aspects: First, the accuracies of carrier nodes in LER form a non-homogeneous Poisson process. Second, the positions of carrier nodes and the destination are independently distributed around the anchor point. Third, these distributions are assumed to be normal. These assumptions do not hold in our original model, as argued in the previous chapter. This fact, as we will see in chapter 7, makes $G(T)$ and $p(t,T)$ much harder to describe. On the other hand, the simplified model is also similar to the original model in some respect. The expected number of carrier nodes is in both models described by $\mu_0(2T)$. Furthermore, although the number of carrier nodes is not a Poisson random variable in the original model, it dominates a variable that is: because of this, a Poisson distribution will arise in chapter 7 in computing an upper bound for $G(T)$. Also, in Lemma 7.1 of chapter 7 we will see that this upper bound is a summation, a term of which is none other than the average one-step flooding area for n independent, normally distributed carrier nodes.

The last two properties indicate that the analysis in this chapter can be seen as a "first approach" in dealing with the LER protocol, i.e. as an introduction of the techniques employed in chapter 7. Furthermore, since LER is hard to analyze under our original model and $p(t,T)$ is not computed within this thesis, this chapter provides insight on how model attributes influence the performance of the protocol and, more specifically, how they influence $p(t,T)$. Finally, the solution of the Volterra equation obtained will indicate many intuitive properties of the protocol, as we will see in section 6.4. Knowing how to interpret such a solution and how to recognize such properties in it will assist in understanding a future solution of the Volterra equation under our original model. All of the above suggest that the analysis performed in this chapter, albeit simplified, is interesting at the present stage of our research.

We will first present several preliminary lemmas that will aid us in deriving $G_s(T)$ and $p_s(t,T)$. We will then show certain properties that can be proved about $Q_s(T)$ without actually solving the Volterra equation in (6.1). Finally we will use the Volterra equation to produce an asymptotic upper bound for $Q_s(T)$.

6.1 Preliminary lemmas

In our simplified model, the positions of carrier nodes and the destination are identically, normally and independently distributed around the current anchor point. It is interesting thus to see what is the expected one-step flooding area for a given number of nodes under such circumstances. This is presented in the following section. A rather intuitive result of this analysis is that if nodes are i.i.d. they are also equally likely to be found through flooding. Furthermore, since under our simplified model the accuracies of nodes constitute a Poisson process, we will also investigate the distribution of the accuracy achieved if chosen with equal probability from a set of accuracies generated by a Poisson process.

6.1.1 Independent Gaussian distributions in $\mathbb{R}^2$

In this section we will present a series of lemmas that describe the expected minimum area defined by points independently distributed according to Gaussian distributions on the plane. The behaviour of this quantity is of some interest since the positions of carrier nodes in our simplified model follow normal distributions around each anchor point. In that context, the minimum area, as defined in following lemmas, is none other than the flooding area around an anchor point when the positions of carrier nodes are independent.

Lemma 6.1. *Let $(X, Y) \in \mathbb{R}^2$ be a two-dimensional random variable which follows a two-dimensional omnidirectional normal distribution of variance σ^2 on each axis centered at the origin. Define $A = \pi(X^2 + Y^2)$. Then*

$$\mathrm{E}[A] = 2\pi\sigma^2. \tag{6.2}$$

Proof. For R the radial part of (X, Y), we have that $R^2 = X^2 + Y^2$. Hence

$$\begin{aligned} \mathrm{E}\left[R^2\right] &= \mathrm{E}\left[X^2 + Y^2\right] \\ &= \mathrm{E}\left[X^2\right] + \mathrm{E}\left[Y^2\right] \\ &= 2\sigma^2 \end{aligned}$$

and $\mathrm{E}[A] = 2\pi\sigma^2$. □

Lemma 6.2. *Let $(X_i, Y_i) \in \mathbb{R}^2$, $1 \leq i \leq n$, be n independent, omnidirectional, two-dimensional, normal random variables with zero mean and variance σ_i^2 on each axis. Let*

$A_i = \pi \cdot (X_i^2 + Y_i^2)$, $1 \le i \le n$, and $A = \min A_i$. Then

$$\mathrm{E}[A] = 2\pi \frac{1}{\sum_i \frac{1}{\sigma_i^2}}. \tag{6.3}$$

Proof. The density distribution of (X, Y) is

$$f_{X,Y}(x, y) = \frac{1}{2\pi\sigma^2} e^{-[x^2+y^2]/2\sigma^2}. \tag{6.4}$$

Since $dxdy = rdrd\theta$, in polar coordinates we get

$$f_{R,\Theta}(r, \theta) = \frac{r}{2\pi\sigma^2} e^{-r^2/2\sigma^2}. \tag{6.5}$$

From the joint polar density in (6.5) we get that

$$f_R(r) = \frac{r}{\sigma^2} e^{-r^2/2\sigma^2} \tag{6.6}$$

is the marginal density of the radius of a two-dimensional, omnidirectional normal random variable. The probability that the area A_i is greater than πr^2 is

$$\mathbf{P}\{A_i > \pi r^2\} = \int_r^\infty \frac{\rho}{\sigma_i^2} e^{-\frac{\rho^2}{2\sigma_i^2}} d\rho = \left[-e^{-\frac{\rho^2}{2\sigma_i^2}}\right]_r^\infty = e^{-\frac{r^2}{2\sigma_i^2}}. \tag{6.7}$$

Thus, the probability that the area A that is greater than πr^2 is

$$\begin{aligned}\mathbf{P}\{A > \pi r^2\} &= \prod_i \mathbf{P}\{A_i > \pi r^2\} = \prod_i e^{-\frac{r^2}{2\sigma_i^2}} \\ &= e^{-\frac{r^2}{2}\left(\sum_i \sigma_i^{-2}\right)}\end{aligned}$$

since A_i are independent. Therefore, the probability that the area A is greater than a value a is

$$\mathbf{P}\{A > a\} = e^{-\frac{a}{2\pi}\sum_i \sigma_i^{-2}} \tag{6.8}$$

and its expected value is

$$\mathrm{E}[A] = \int_0^\infty \mathbf{P}\{A > a\} da = 2\pi \frac{1}{\sum_i \frac{1}{\sigma_i^2}}. \tag{6.9}$$

□

Equation (6.8) suggests that the point closest to the origin is identically distributed as a point which is normally distributed around the origin with variance $\sigma^2 = \frac{1}{\sum_i \frac{1}{\sigma_i^2}}$ on each axis. In particular, if $\sigma_i = \bar{\sigma}$ for all i, this reduces to a distribution with $\sigma^2 = \frac{\bar{\sigma}^2}{n}$.

Lemma 6.3. *Let $(X_i, Y_i) \in \mathbb{R}^2$, $1 \leq i \leq n$, be n independent omnidirectional, two-dimensional, normal random variables with zero mean and variance σ_i^2 on each axis. Let $A_i = \pi \cdot (X_i^2 + Y_i^2)$, $1 \leq i \leq n$, and $I = \arg\min A_i$. Then*

$$\mathbf{P}\{I = i\} = \frac{\frac{1}{\sigma_i^2}}{\sum_j \frac{1}{\sigma_j^2}}. \tag{6.10}$$

Proof. Note that the event $\{I = i\}$, $1 \leq i \leq n$ is the event that $A_i = \min_j A_j$. We have

$$\begin{aligned}
\mathbf{P}\{I = i \mid A_i = \pi r^2\} &= \mathbf{P}\{A_j > \pi r^2,\ \forall j \neq i\} \\
&= \prod_{j \neq i} \mathbf{P}\{A_j > \pi r^2\} && \text{by the independence assumption} \\
&= \prod_{j \neq i} e^{-\frac{r^2}{2\sigma_j^2}} && \text{by eq. (6.7)} \\
&= e^{-\frac{r^2}{2}\sum_{j \neq i} \sigma_j^{-2}}
\end{aligned}$$

We therefore get

$$\begin{aligned}
\mathbf{P}\{I = i\} &= \int_0^\infty \mathbf{P}\{I_i \mid A_i = \pi r^2\} f_{R_i}(r) dr \\
&= \int_0^\infty \frac{r}{\sigma_i^2} e^{\frac{-r^2}{2\sigma_i^2}} e^{-\frac{r^2}{2}\sum_{j \neq i} \sigma_j^{-2}} dr \\
&= \int_0^\infty \frac{r}{\sigma_i^2} e^{-\frac{r^2}{2}\sum_j \sigma_j^{-2}} dr \\
&= \frac{\frac{1}{\sigma_i^2}}{\sum_j \frac{1}{\sigma_j^2}}
\end{aligned}$$

□

From the above lemma, we have that if $\sigma_i = \bar{\sigma}$ for all i, all points are equally likely to define the minimum area. Hence the following corollary holds

Corollary 6.1. *Let $(X_i, Y_i) \in \mathbb{R}^2$, $1 \leq i \leq n$, be n independent omnidirectional, two-dimensional, identically distributed normal random variables with zero mean and variance σ^2 on each axis. Let $A_i = \pi \cdot (X_i^2 + Y_i^2)$, $1 \leq i \leq n$, and $I = \arg\min A_i$. Then*

$$\mathbf{P}\{I = i\} = \frac{1}{n}. \tag{6.11}$$

It is interesting to see what the expected area is if the number of points is Poisson distributed. Of course, in order for the minimum area to make sense at least one point must exist. Note that, in our model, at least one point does exist, namely the destination.

Lemma 6.4. *Let $(X_i, Y_i) \in \mathbb{R}^2$, $1 \leq i \leq N+1$, be $N+1$ independent omnidirectional, two-dimensional, normal random variables with zero mean and variance σ^2 on each axis, where N is a Poisson distributed random variable with mean μ. Let $A_i = \pi(X_i^2 + Y_i^2)$, $1 \leq i \leq N+1$ and $A = \min A_i$. Then*

$$\mathrm{E}[A] = \frac{2\pi\sigma^2}{\mu}(1 - e^{-\mu}). \tag{6.12}$$

Proof. Given that $N = n$, by Lemma 6.2 the conditional expected minimum area is

$$\mathrm{E}[A \mid N = n] = \frac{2\pi\sigma^2}{n+1}.$$

Therefore, the expected area is

$$\begin{aligned}
\mathrm{E}[A] &= \sum_{n=0}^{\infty} \frac{2\pi\sigma^2}{n+1} \frac{e^{-\mu}(\mu)^n}{n!} \\
&= \frac{2\pi\sigma^2 e^{-\mu}}{\mu} \sum_{n=0}^{\infty} \frac{(\mu)^{n+1}}{n+1!} \\
&= \frac{2\pi\sigma^2}{\mu}(1 - e^{-\mu})
\end{aligned}$$

□

6.1.2 Equally likely carrier nodes forming a Poisson process

In this section we will see how a Poisson process can be used with equally probable carrier nodes in defining the conditional probability $p(t, T)$.

Let $\{N(t); t \geq 0\}$, be the counting process of the number of carrier nodes with accuracy in the interval $[0, t]$, and let $\mathrm{E}[N(t)] = \mu(t)$ for $t \geq 0$. Note that S_i, $i \geq 1$, the epoch of the i-th arrival of this counting process, is the elapsed time since the last encounter of a node with the destination, i.e. S_i is the entry regarding the destination at this node's routing table. W.l.o.g. we assume that the node that corresponds to the epoch S_i is node i.

We assume that $\{N(t); t \geq 0\}$ is a non-homogeneous Poissson process with bulk arrivals at time zero. This means that many nodes may know where the destination is at time 0 and their the number $N(0)$ is distributed according to a Poisson distribution with expected value $\mu(0)$. Furthermore, $\tilde{N}(t) = N(t) - N(0)$ is a non-homogeneous Poisson process independent of $N(0)$ with $\mathrm{E}\left[\tilde{N}(t)\right] = \mu(t) - \mu(0)$ (where $\mu(t) = \mathrm{E}[N(t)]$ and $\mu(0) = \mathrm{E}[N(0)] > 0$).

We assume that we commence a flooding at time 0 from the position of the destination at T time ago, i.e. at $P_d(-T)$. Furthermore, we assume that all existing carrier nodes and the destination are *equally likely to be found*. Therefore, if $N(T) = n$, the probability that the i-th carrier node will be found is $\frac{1}{n+1}$. Let X be the time estimate that we will get with this choice, i.e. the time S_i in the past that the node i had accurate information about the destination, if we located carrier node i, and zero if we located the destination. We can formally describe X as

$$\mathbf{P}\{X = S_i \mid N(T) = n\} = \frac{1}{n+1} \quad 0 \leq i \leq n \tag{6.13}$$

where S_i, $i \geq 1$, is the epoch of the i-th arrival and $S_0 \equiv 0$, corresponding to the event of finding the destination itself.

Lemma 6.5.

$$\mathbf{P}\{X = 0\} = \chi(T)$$

where

$$\chi(T) = \frac{\mu(0)}{\mu(T)} + \frac{\mu(T) - \mu(0)}{\mu(T)} \cdot \frac{1 - e^{-\mu(T)}}{\mu(T)}.$$

Proof. Given that $N(T) = n$, we have that the probability that $X = 0$ is equal to the probability that the destination was located or that a carrier node that has accuracy zero was located. The probability that the destination was located is $\frac{1}{n+1}$, since all nodes and the destination are equally likely.

The probability that a carrier node is chosen is $\frac{n}{n+1}$. We thus need to compute the probability that, given that a carrier node was chosen, it was one with accuracy zero. However, given that a carrier node is chosen, each of them is equally likely to be located with probability $\frac{1}{n}$. However, $\{N(t), t \geq 0\}$ is a non-homogeneous Poisson process with bulk arrivals at time zero, whose expected value is $\mu(0)$. Hence, by Corollary 3.1, if we chose one of them with equal probability the probability that it will be less than or equal to zero will be equal to $\mu(0)/\mu(T)$. Hence we get that

$$\begin{aligned}
\mathbf{P}\{X = 0 \mid N(T) = n\} &= \frac{1}{n+1} + \frac{n}{n+1}\frac{\mu(0)}{\mu(T)} \\
&= \frac{1}{n+1} + \left(1 - \frac{1}{n+1}\right)\frac{\mu(0)}{\mu(T)} \\
&= \frac{\mu(0)}{\mu(T)} + \left(1 - \frac{\mu(0)}{\mu(T)}\right)\frac{1}{n+1}
\end{aligned}$$

Thus $\mathbf{P}\{X = 0\}$ can be computed as the expectation over all n, since n is Poisson distributed with average $\mu(T)$. This computation is similar to the one performed in the proof of Lemma 6.4 and gives us the lemma. □

Lemma 6.6.

$$\mathbf{P}\{X \leq t \mid X \neq 0\} = \frac{\mu(t) - \mu(0)}{\mu(T) - \mu(0)} \tag{6.14}$$

for $0 < t \leq T$.

Proof. If $X \neq 0$, the carrier node we located is one of the nodes that are counted in $\tilde{N}(T)$, where $\tilde{N}(t) = N(t) - N(0)$. Furthermore, $\tilde{N}(T) \geq 1$, since at least one such carrier node exists (the one we located). We are therefore contemplating the probability

$$\mathbf{P}\{\tilde{X} \leq t \mid \tilde{N}(T) \geq 1\}$$

where $\tilde{X}$ is the epoch we get if we choose among the epochs in $(0, T]$ with equal probability, i.e.

$$\mathbf{P}\{\tilde{X} = \tilde{S}_i \mid \tilde{N}(T) = n\} = \frac{1}{n}, \qquad 1 \leq i \leq n. \tag{6.15}$$

However, $\{\tilde{N}(t), t \geq 0\}$ is a non-homogeneous Poisson process with expectation $\mu(t) - \mu(0)$. Hence, by Corollary 3.1 we get that

$$\mathbf{P}\{\tilde{X} \leq t \mid \tilde{N}(T) = n\} = \frac{\mu(t) - \mu(0)}{\mu(T) - \mu(0)}, \qquad 0 \leq t \leq T. \tag{6.16}$$

We therefore have that

$$\begin{aligned}
\mathbf{P}\{\tilde{X} \leq t \mid \tilde{N}(T) \geq 1\} &= \frac{\sum_{n=1}^{\infty} \mathbf{P}\{\tilde{X} \leq t \mid \tilde{N}(T) = n\} \mathbf{P}\{\tilde{N}(T) = n\}}{\mathbf{P}\{\tilde{N}(T) \geq 1\}} \\
&= \frac{\mu(t) - \mu(0)}{\mu(T) - \mu(0)} \frac{\sum_{n=1}^{\infty} \mathbf{P}\{\tilde{N}(T) = n\}}{\mathbf{P}\{\tilde{N}(T) \geq 1\}} \\
&= \frac{\mu(t) - \mu(0)}{\mu(T) - \mu(0)}
\end{aligned}$$

□

Lemma 6.7. *The probability density function of X given that the current estimation for the destination is T is*

$$p(t, T) = \frac{\mu'(t)}{\mu(T) - \mu(0)}(1 - \chi(T))$$

where

$$\chi(T) = \frac{\mu(0)}{\mu(T)} + \frac{\mu(T) - \mu(0)}{\mu(T)} \cdot \frac{1 - e^{-\mu(T)}}{\mu(T)}$$

for $\alpha \leq t \leq T$, $\alpha > 0$.

Proof. From Lemmas 6.5,6.6 we get

$$\mathbf{P}\{X \leq t\} = \frac{\mu(t) - \mu(0)}{\mu(T) - \mu(0)}(1 - \chi(T)) + \chi(T)$$

for $0 \leq t \leq T$, thus $p(t, T)$ is defined as the derivative $\frac{\partial}{\partial t}\mathbf{P}\{X \leq t\}$ which exists for all t bounded away from zero. □

6.2 An upper bound for $Q_s(T)$ for a given set of independent carrier nodes

We can use a simple argument to get a moderate upper bound on function $Q_s(T)$ in our simplified model for a given set of carrier nodes. This upper bound does not depend on the number of carrier nodes existing or the values of their accuracies; as such, this result does not involve the Poisson distribution of the accuracies in the simplified model at all. Hence, the only point where this proof will differ from our original model is in that carrier nodes positions will be assumed to be independently, normally distributed around the anchor point.

We briefly recapitulate the main assumptions of our simplified model that we use in order to prove the results that follow about $Q_s(t)$. At each step, carrier nodes and the destination are distributed on the plane according to independent, omnidirectional, two-dimensional Gaussian distributions centered around the current anchor point with variance vT on each axis, where T is the accuracy at the anchor point. Furthermore, by definition, function $Q_s(T)$ satisfies the set of equations (6.1).

Assume that nodes have given accuracies t_i, $i \in \{1, 2, \ldots\}$ and $t_0 = 0$ is the accuracy of the destination itself. According to our simplified model (but our original model as well), many nodes may have zero accuracies, but all other nodes have distinct accuracies with probability one. Hence, we can order nodes so that $t_i > t_j$ if $i > j > s$, for some value $s \geq 0$, and $t_i = 0$ for all $i \leq s$. By Lemma 6.2, the expected one-step flooding area from a point where the accuracy is t_n thus

$$G_s(t_n) = \frac{2\pi\sigma^2(t_n)}{n}$$

where $\sigma^2(t) = vt$ is an increasing function. The only property of this variance that we shall use in our current analysis is that it is increasing, with Proposition 6.1 being the

only exception. We will thus refer to it as $\sigma^2(t)$ to emphasize that the following results do not hold only for vt.

By Lemma 6.3, all carrier nodes and the destination are equally likely to be located through flooding, thus the density of the improved accuracy from a point with accuracy t_n, where $n > s$, is

$$p_s(t = t_i, t_n) = \frac{1}{n}, \quad 0 \leq i \leq n-1.$$

Hence, equation (6.1) becomes

$$Q_s(t_n) = \frac{2\pi\sigma^2(t_n)}{n} + \sum_{i=0}^{n-1} \frac{Q_s(t_i)}{n}, \quad t_n \geq \alpha \tag{6.17a}$$

$$Q_s(t_n) = 0, \quad \text{o.w.} \tag{6.17b}$$

Lemma 6.8. *Function Q_s, as defined in* (6.17)*, is an increasing function, i.e. for every $t_i > t_j$ we have $Q_s(t_i) \geq Q_s(t_j)$.*

Proof. Consider the case where $i = j + 1$. If $t_j < \alpha$ the lemma is trivially true. Assume that $t_j \geq \alpha$. Then, by (6.17a), $Q_s(t_{j+1}) \geq Q_s(t_j)$ becomes

$$\frac{2\pi\sigma^2(t_{j+1})}{j+1} + \sum_{i=0}^{j} \frac{Q_s(t_i)}{j+1} \geq Q_s(t_j)$$

which is equivalent to

$$\frac{2\pi\sigma^2(t_{j+1})}{j+1} + \sum_{i=0}^{j-1} \frac{Q_s(t_i)}{j+1} + \frac{Q_s(t_j)}{j+1} \geq Q_s(t_j)$$

or

$$\frac{2\pi\sigma^2(t_{j+1})}{j+1} + \sum_{i=0}^{j-1} \frac{Q_s(t_i)}{j+1} \geq Q_s(t_j)\frac{j}{j+1}$$

which by (6.17a) is

$$\frac{2\pi\sigma^2(t_{j+1})}{j+1} + \sum_{i=0}^{j-1} \frac{Q_s(t_i)}{j+1} \geq \frac{j}{j+1}\left(\frac{2\pi\sigma^2(t_j)}{j} + \sum_{i=0}^{j-1} \frac{Q_s(t_i)}{j}\right)$$

which holds iff $\sigma^2(t_{j+1}) \geq \sigma^2(t_j)$. This is true since $\sigma^2(t)$ is an increasing function. The general case is proved by induction. □

We will now show that, by adding a new carrier node, function Q_s can only be improved.

Lemma 6.9. *Let p be a new carrier node with accuracy t_p such that $t_{n-1} < t_p < t_n$ for some $n \geq 1$. Let $Q'_s(t)$ be the function that approximates the cost if we consider this carrier node when flooding and $Q_s(t)$ the one if we do not consider it. Then, $Q'_s(t_i) \leq Q_s(t_i)$ for all $i \in \mathbb{N}$.*

Proof. If $t_p < \alpha$ this is trivially true. Assume thus that $t_p \geq \alpha$. Notice that

$$Q'_s(t_i) = Q_s(t_i), \quad \text{for all } 0 \leq i \leq n-1 \tag{6.18}$$

since, by the definition of the functions Q_s, Q'_s, they depend only on carrier nodes with accuracy better than t_i and are thus not influenced by the existence of carrier node p. At the anchor point with accuracy t_n, $Q_s(t_n)$ is described by (6.17a). On the other hand, $Q'_s(t_n)$ is

$$\begin{aligned} Q'_s(t_n) &= \frac{2\pi\sigma^2(t_n)}{n+1} + \sum_{i=0}^{n-1} \frac{Q'_s(t_i)}{n+1} + \frac{Q'_s(t_p)}{n+1} \\ &\overset{(6.18)}{=} \frac{2\pi\sigma^2(t_n)}{n+1} + \sum_{i=0}^{n-1} \frac{Q_s(t_i)}{n+1} + \frac{Q'_s(t_p)}{n+1} \end{aligned} \tag{6.19}$$

where

$$\begin{aligned} Q'_s(t_p) &= \frac{2\pi\sigma^2(t_p)}{n} + \sum_{i=0}^{n-1} \frac{Q'_s(t_i)}{n} \\ &\overset{(6.18)}{=} \frac{2\pi\sigma^2(t_p)}{n} + \sum_{i=0}^{n-1} \frac{Q_s(t_i)}{n} \end{aligned} \tag{6.20}$$

Therefore, $Q'_s(t_n) \leq Q_s(t_n)$ becomes

$$\frac{2\pi\sigma^2(t_n)}{n+1} + \sum_{i=0}^{n-1} \frac{Q_s(t_i)}{n+1} + \frac{Q'_s(t_p)}{n+1} \leq \frac{2\pi\sigma^2(t_n)}{n} + \sum_{i=0}^{n-1} \frac{Q_s(t_i)}{n}$$

which is equivalent to

$$\left(2\pi\sigma^2(t_n) + \sum_{i=0}^{n-1} Q_s(t_i)\right)\left(\frac{1}{n} - \frac{1}{n+1}\right) \geq \frac{Q'_s(t_p)}{n+1}$$

or

$$\left(\frac{2\pi\sigma^2(t_n)}{n} + \sum_{i=0}^{n-1} \frac{Q_s(t_i)}{n}\right)\frac{1}{n+1} \geq \frac{Q'_s(t_p)}{n+1} \tag{6.21}$$

and by eq. (6.20) can be written as

$$\frac{2\pi\sigma^2(t_n)}{n} + \sum_{i=0}^{n-1} \frac{Q_s(t_i)}{n} \geq \frac{2\pi\sigma^2(t_p)}{n} + \sum_{i=0}^{n-1} \frac{Q_s(t_i)}{n}$$

which is true iff $\sigma^2(t_n) \geq \sigma^2(t_p)$. This holds again by the monotonicity of $\sigma^2(t)$. Note that inequality (6.21) implies that

$$Q_s(t_n) \geq Q'_s(t_p) \tag{6.22}$$

We prove the remaining cases by induction. We know that $Q'_s(t_n) \leq Q_s(t_n)$. Assume that $Q'_s(t_i) \leq Q_s(t_i)$ for all $n \leq i \leq k-1$ for some $k > n$. We will prove that $Q'_s(t_k) \leq Q_s(t_k)$. This is equivalent to

$$\frac{2\pi\sigma^2(t_k)}{k+1} + \sum_{i=n}^{k-1} \frac{Q'_s(t_i)}{k+1} + \frac{Q'_s(t_p)}{k+1} + \sum_{i=0}^{n-1} \frac{Q'_s(t_i)}{k+1} \leq \frac{2\pi\sigma^2(t_k)}{k} + \sum_{i=0}^{k-1} \frac{Q_s(t_i)}{k}.$$

Since $Q'_s(t_i) \leq Q_s(t_i)$ for all $n \leq i \leq k-1$ by the induction hypothesis and $Q'_s(t_i) = Q_s(t_i)$ for all $0 \leq i \leq n-1$, it suffices that

$$\begin{aligned}
&\frac{2\pi\sigma^2(t_k)}{k+1} + \sum_{i=n}^{k-1} \frac{Q_s(t_i)}{k+1} + \frac{Q'_s(t_p)}{k+1} + \sum_{i=0}^{n-1} \frac{Q_s(t_i)}{k+1} \leq \frac{2\pi\sigma^2(t_k)}{k} + \sum_{i=0}^{k-1} \frac{Q_s(t_i)}{k} \\
&\Leftrightarrow \left(2\pi\sigma^2(t_k) + \sum_{i=0}^{k-1} Q_s(t_i)\right) \frac{1}{k(k+1)} \geq \frac{Q'_s(t_p)}{k+1} \\
&\Leftrightarrow Q_s(t_k) \geq Q'_s(t_p)
\end{aligned}$$

which is true since $Q_s(t_k) \geq Q_s(t_n)$ by the monotonicity of Q_s and $Q_s(t_n) \geq Q'_s(t_p)$ by (6.22). □

Lemma 6.9, as we have already stated, shows that by considering an existing carrier node, the LER protocol can only improve its performance with respect to function Q_s under our simplified model. This has an important immediate implication.

Proposition 6.1. *The function $Q_s(T)$ is upper-bounded by $2\pi vT$.*

Proof. The quantity mentioned by the proposition is the expected area that the LER protocol has to flood in our simplified model in order to find the destination itself in one step. By Lemma 6.9, including any carrier node, which is what the LER protocol does, can only decrease function Q_s, hence the upper bound. □

This proposition is important because $2\pi(2vT)$ is a lower bound for the expected area we would flood in order to find the destination itself in any protocol that does not use approximate information, e.g. AODV [38], as described in section 5.4. There is another significant consequence of Lemma 6.9.

Proposition 6.2. *The maximal halting condition for flooding is also the optimal for the LER protocol, with respect to function Q_s.*

Proof. Any halting condition other than the maximal will discard some carrier nodes. By Lemma 6.9, function Q_s under the simplified model will be greater than the one if those carrier nodes where taken into consideration. □

This proposition suggests that a modification of the LER protocol in which the halting condition requires carrier nodes having an accuracy better than γT, where $0 \leq \gamma \leq 1$ and T the accuracy at the current anchor point, will have optimal performance for $\gamma = 1$ under our simplified model.

6.3 An asymptotic upper bound for $Q_s(T)$ as the solution of a Volterra equation

The results proved in the previous section did not use any information on the distribution of the accuracies of carrier nodes in our simplified model. In this section we will show that by using such information we can improve the upper bound of Proposition 6.1 for function $Q_s(T)$. This upper bound is asymptotic since it contains a term (the $o(1)$ term) whose behaviour we are able to describe only asymptotically.

Proposition 6.3. *Function $Q_s(T)$ that approximates the cost of the LER protocol under the simplified model is upper-bounded by*

$$\frac{1}{\rho}\left(\frac{1}{4}\log^2 2vT + (c + o(1))\log T\right) \tag{6.23}$$

where c is independent of T and ρ.

Proof. In our simplified model, $N(t)$, the number of carrier nodes with accuracy less than t, for some $t \geq 0$, is such that

$$\mathbf{P}\{N(t) = k\} = \frac{(\mu(t))^k}{k!}e^{-\mu(t)} \tag{6.24}$$

where $\mathrm{E}[N(t)] = \mu(t) = \mu_0(2t)$ and $\mu_0(t)$ is described by Corollary 5.2. Furthermore $N(t)$ satisfies the independent increment property. In addition to the above, the positions of carrier nodes are independently, normally distributed around the anchor point with variance vT on each axis, where T is the current accuracy at the anchor point. Hence, Lemma 6.4 implies that the expected one-step flooding area for the LER protocol in the simplified model will be

$$G_s(T) = \frac{2\pi vT}{\mu(T)}(1 - e^{-\mu(T)}). \tag{6.25}$$

Furthermore, Lemma 6.3 implies that carrier nodes are equally likely to be found. This can be combined with Lemma 6.7 to give that the conditional probability for the improved accuracy in the simplified model is

$$p_s(t, T) = \frac{\mu'(t)}{\mu(T) - \mu(0)}(1 - \chi(T)) \tag{6.26}$$

where χ is such that $0 \leq \chi(T) \leq 1$. Function $Q_s(T)$ satisfies the linear Volterra equation of the first kind in the system

$$Q_s(T) = G_s(T) + \int_\alpha^T Q_s(t)p_s(t, T)dt \quad T \geq \alpha \tag{6.27a}$$

$$Q_s(T) = 0 \quad 0 \leq T < \alpha \tag{6.27b}$$

where $G_s(T)$ and $p_s(t, T)$ are described by (6.25) and (6.26) respectively. The kernel $p_s(t, T)$ of this Volterra equation is separable, so by corollary 3.2 and Lemma 3.4 we have that

$$Q_s(T) = G_s(T) + \int_\alpha^T p_s(t, T) \exp\left(\int_t^T p_s(u, u)du\right) G_s(t)dt.$$

We have that

$$\begin{aligned}
\int_t^T p_s(u, u)du &= \int_t^T \frac{\mu'(u)}{\mu(u) - \mu(0)}(1 - \chi(u))du \\
&\leq \int_t^T \frac{\mu'(u)}{\mu(u) - \mu(0)}du \quad \text{since } \chi(u) \leq 1 \\
&= \int_t^T \frac{(\mu(u) - \mu(0))'}{\mu(u) - \mu(0)}du \\
&= \log \frac{\mu(T) - \mu(0)}{\mu(t) - \mu(0)}
\end{aligned}$$

Therefore

$$\begin{aligned}
Q_s(T) &\le G_s(T) + \int_\alpha^T \frac{\mu'(t)}{\mu(t)-\mu(0)} G_s(t)dt \\
&\le \frac{2\pi vT}{\mu(T)} + \int_\alpha^T \frac{\mu'(t)}{\mu(t)-\mu(0)} \frac{2\pi vt}{\mu(t)}dt \\
&\le \frac{2\pi vT}{\mu(T)-\mu(0)} + \int_\alpha^T \frac{\mu'(t)}{(\mu(t)-\mu(0))^2} 2\pi vt dt \qquad \text{since } T \ge \alpha > 0 \\
&= \frac{2\pi vT}{\mu(T)-\mu(0)} - \left[\frac{2\pi vt}{\mu(t)-\mu(0)}\right]_\alpha^T + \int_\alpha^T \frac{2\pi v}{\mu(t)-\mu(0)}dt \\
&= \int_\alpha^T \frac{2\pi v}{\mu(t)-\mu(0)}dt + \frac{2\pi v\alpha}{\mu(\alpha)-\mu(0)}
\end{aligned}$$

From Corollary 5.2 we have that

$$\mu(t) = \mu_0(2t) = \rho\left(\frac{2\pi 2vt}{\log 2vt} + (d+o(1))\frac{2vt}{\log^2 2vt}\right).$$

Where d is a constant independent of t, v and ρ. Since $\mu(0) = \rho$, the above also holds for $\mu(t) - \mu(0)$ ($-\rho$ gets absorbed in $o(1)$). We therefore have,

$$\begin{aligned}
Q_s(T) &\le \frac{1}{\rho}\int_\alpha^T \frac{2\pi v}{\frac{2\pi 2vt}{\log 2vt} + (d+o(1))\frac{2vt}{\log^2 2vt}}dt + \frac{2\pi v\alpha}{\rho\left(\frac{2\pi 2v\alpha}{\log 2v\alpha} + (d+o(1))\frac{2v\alpha}{\log^2 2v\alpha}\right)} \\
&= \frac{1}{\rho}\left(\int_\alpha^T \frac{\log 2vt}{2t}\frac{\log 2vt}{\log 2vt + (d'+o(1))} + c\right) \quad \text{where } d' = d/2\pi \\
&= \frac{1}{\rho}\left(\int_\alpha^T \frac{\log 2vt}{2t}\left(1 - \frac{d'+o(1)}{\log 2vt + (d'+o(1))}\right)dt + c\right) \\
&= \frac{1}{\rho}\left(\int_\alpha^T \frac{\log 2vt}{2t}dt - \int_\alpha^T \frac{\log 2vt}{2t}\frac{d'+o(1)}{\log 2vt + (d'+o(1))}dt + c\right) \\
&= \frac{1}{\rho}\left(\frac{1}{4}\log^2 2vT - \frac{1}{4}\log^2 2v\alpha - \int_\alpha^T \frac{\log 2vt}{2t}\frac{(d'+o(1))}{\log 2vt + (d'+o(1))}dt + c\right) \\
&= \frac{1}{\rho}\left(\frac{1}{4}\log^2 vT - \int_\alpha^T \frac{\log 2vt}{2t}\frac{(d'+o(1))}{\log 2vt + (d'+o(1))}dt + c'\right)
\end{aligned}$$

where c' independent of T, ρ. Note that

$$\begin{aligned}
\lim_{T\to\infty} \frac{\int_\alpha^T \frac{\log 2vt}{2t}\frac{d'+o(1)}{\log 2vt+(d'+o(1))}dt}{\log T} - d'/2 &= \lim_{T\to\infty} \frac{\frac{\log 2vT}{2T}\frac{d'+o(1)}{\log 2vT+(d'+o(1))}}{\frac{1}{T}} - d'/2 \\
&= 0
\end{aligned}$$

Hence

$$Q_s(T) \leq \frac{1}{\rho}\left(\frac{1}{4}\log^2 2vT - (d'/2 + o(1))\log T + c'\right)$$
$$= \frac{1}{\rho}\left(\frac{1}{4}\log^2 2vT + (c'' + o(1))\log T\right) \quad (6.28)$$

where c'' is independent of T and ρ. Notice that this implies also that $Q_s(T) = O(\log^2 T)$. □

6.4 Discussion on the results under the simplified model

As we have already mentioned, the results that we acquired under our simplified model can give us little insight on the behaviour of the protocol under the original model. Keeping this in mind, we shall attempt an interpretation of Proposition 6.3. Bound (6.23) is of the form

$$\frac{1}{\rho}\left(c_1 \log^2 T + (c_2(v) + o(1))\log T\right) \quad (6.29)$$

where c_1 is 1/4 and c_2 is a function of v only (it is independent of both ρ and T).

The first conclusion one can reach from (6.29) is that function $Q_s(T)$ is in both cases $O(\log^2(T))$. This upper bound, as sublinear, indicates that the protocol scales well for large values of T (see section 5.4).

Furthermore, if above bound is tight, it suggests an interesting balancing behaviour in terms of ρ and v. The actual measure of performance that we are interested in is the overhead, i.e. the expected number of packets exchanged during flooding. Assuming that this is proportional to the number of nodes that participated in flooding[1], given that the network forms a Poisson field, the number of nodes that were flooded should be ρ times the area flooded. This is not exactly true, since this holds only if this area is independent of the Poisson field. For example, it obviously does not hold if the area is the area of the disk around the origin with a radius defined by the point of the field closest to the origin. However, if we accept this as a heuristic approximation of the number of nodes and $Q_s(T)$ as an approximation of the area flooded, the number of nodes that participate

[1] A node should be considered many times in this calculation if it participates in multiple flooding phases.

in flooding is $\rho Q_s(T)$ on average. Equation (6.23) suggests thus that the upper bound of $\rho Q_s(T)$ is independent of ρ. This means that, if it is tight, the behaviour of the protocol under the simplified model is not affected at all by the density of the network[2]. A non-rigorous argument for this behaviour would be that although in a dense network flooding the same area is more costly than in a sparse network, a dense network will include more carrier nodes, thus facilitating the flooding.

On the other hand, (6.29) also suggests a nice behaviour in terms of v. The asymptotic behaviour of the bound for two different values of v is the same under the simplified model, since v does not affect the dominant term of (6.29). Since v is used to model the mobility in the network, (6.29) implies that, if the bound is tight, the protocol also scales well as far as mobility is concerned. The intuition behind this would be that, although fast speed makes a destination harder to locate, it also contributes to the creation of more carrier nodes. In that sense, (6.29) strengthens the claim made by the authors of EASE [19] that "mobility helps".

An important observation on the method followed is that the probability that the destination or a carrier node adjacent to it was found at any flooding step was completely ignored. This information was encapsulated in the probability $\chi(T)$, which was taken in fact to be zero in the derivation of Proposition 6.3. This means that, even under the assumptions made, the behaviour of the protocol in the simplified model is in fact better than the one derived; note that what we computed was function $Q_s(T)$ if the destination was not reached in any intermediate flooding step, but flooding stopped because an accuracy better than α was reached.

[2]As long as the network is dense enough so that it is connected, as we have assumed in section 4.2

Chapter 7

Towards an Analysis of the LER Protocol

In chapter 6 we demonstrated how the LER protocol can be analyzed under a model which is simplified compared to the one we originally proposed in chapter 4. In this simplified model, the positions of carrier nodes around an anchor point were assumed to be independent and normal and their number was assumed to be Poisson distributed. The results that we found had very intuitive interpretations, which, if true in our original model, would be very useful in understanding how the protocol works.

As noted before, our original model and the simplified one differ on the properties mentioned above: First, in our original model, the accuracies of carrier nodes in LER do not form a non-homogeneous Poisson process with bulk arrivals at time zero. Second, the positions of carrier nodes and the destination are not independently distributed around the anchor point, nor are they normal.

As we will see, the above facts make the computation of the expected one-step flooding area $G(T)$ and the distribution of the improved accuracy $p(t, T)$ much more complicated. It turns out however, that the techniques we used in the previous section can be extended to provide an asymptotic upper bound for $G(T)$. The upper bound we shall obtain differs from the expected one-step flooding area $G_s(T)$ we derived under the simplified model; it is still however sublinear, and thus can still yield an interesting upper bound for $Q(T)$.

On the other hand, we were not able to describe $p(t, T)$ within the time limitations of this thesis. Although we leave the exact distribution as future research, we discuss how the lack of independence influences it compared to the distribution $p_s(t, T)$ of our simplified model.

7.1 An asymptotic upper bound for $G(T)$

In this section we will provide an asymptotic upper bound for the expected one-step flooding area $G(T)$ in the LER protocol. There are two reasons why computing $G(T)$ exactly is difficult. First, the joint distribution of the positions of carrier nodes around an anchor point are not easy to describe, as discussed in section 5.3. Second, although the expected number of carrier nodes can be described by Proposition 5.1, the exact distribution of the number of carrier nodes is not known.

We will address these problems by using the following two techniques. We will overcome our inability to express the exact distribution of the number of carrier nodes by exploiting the fact that this number stochastically dominates a Poisson random variable, which was proved in section 5.1.4. This is an immediate consequence of Proposition 5.2, which states that more nodes are met, on average, by a moving destination than by one that is fixed.

The absence of independence among the positions of carrier nodes is resolved as follows. The expected one-step flooding area is less than the expected one-step flooding area if only a subset of the existing carrier nodes is considered while flooding. We will construct a subset for which the contribution of the independent and dependent part of the movements of carrier nodes to the expected one-step flooding area can be upper-bounded.

Though we so far have investigated the behaviour of our protocol in the interval $[-T, 0]$, to simplify notation, in this section we will look at the interval $[0, T]$. Furthermore, we will denote the trajectory of the destination in this time interval with $P_0(t)$, $t \in [0, T]$, instead of $P_d(t)$, and with

$$\mathcal{D}(t) = \{x \in \mathbb{R}^2 : ||x - P_0(t)|| \leq 1/\sqrt{\pi}\} \tag{7.1}$$

the unit-area disk around it. W.l.o.g. we will assume that $P_0(0)$ is the origin. The Poisson field that comprises the rest of the nodes in the network is $\boldsymbol{\pi}$.

At time T, a flood is initiated at $P_0(0)$. The set of carrier nodes at this time is none other than

$$C_T = \{P \in \boldsymbol{\pi} : \exists t \in (0, T] \text{ s.t. } P(t) \in \mathcal{D}(t)\}. \tag{7.2}$$

Hence, the expected one-step flooding area $G(T)$ in our protocol is

$$G(T) = \mathrm{E}\left[\min_{P \in C_T \cup \{P_0\}} \pi ||P(T)||^2\right] \tag{7.3}$$

i.e. it is the expected area at time T we need to flood from the origin in order to find the closest node in $C_T \cup \{P_0\}$.

The expected one-step flooding area $G(T)$ is upper-bounded by the area we need to flood if we look only for a subset of the aforementioned nodes. Hence, for $\tau \in (0, T]$, if we define

$$C_\tau = \{P \in \boldsymbol{\pi} : \exists t \in (0, \tau] \text{ s.t. } P(t) \in \mathcal{D}(t)\} \tag{7.4}$$

then $C_\tau \subseteq C_T$ and $G(T)$ is upper-bounded by

$$G_\tau(T) = \mathrm{E}\left[\min_{P \in C_\tau \cup \{P_0\}} \pi ||P(T)||^2\right]. \tag{7.5}$$

Set C_τ contains all the carrier nodes that met the destination in the interval $(0, T]$. Let $N = |C|$ be the size of this set. Then N is none other than

$$N = N_{fen}(\mathcal{D}, 0, \tau) \tag{7.6}$$

where N_{fen} is the counting process introduced in section 5.2.1, i.e. it is the number of nodes that entered the unit-area disk around the destination for the first time after 0 prior to τ.

We denote with $P_i(t)$, for $t \in [0, T]$ and $1 \leq i \leq N$ the trajectories of these points in C_τ. Furthermore, let σ_i, $1 \leq i \leq N$, be

$$\sigma_i = \min\{t : t \geq 0 \wedge P_i(t) \in \mathcal{D}(t)\} \tag{7.7}$$

i.e. σ_i is the first entry epoch of the i-th point, in the terminology of section 5.2.1. We also define $\sigma_0 = \tau$. We then have that

$$\sigma_i \leq \tau \tag{7.8}$$

for all $0 \leq i \leq N$. Under this notation, (7.5) can be conveniently rewritten as

$$G_\tau(T) = \mathrm{E}\left[\min_{0 \leq i \leq N} \pi ||P_i(T)||^2\right]. \tag{7.9}$$

The positions of the carrier nodes and the destination at time T can be written as

$$P_i(T) = P_0(\sigma_i) + X_i + P_i(T) - P_i(\sigma_i), \quad 0 \leq i \leq N \tag{7.10}$$

where X_i is obviously $P_i(\sigma_i) - P_0(\sigma_i)$. This is illustrated in figure 7.1. By the definition of σ_i in (7.7) we have that

$$||X_i|| \leq 1/\sqrt{\pi} \tag{7.11}$$

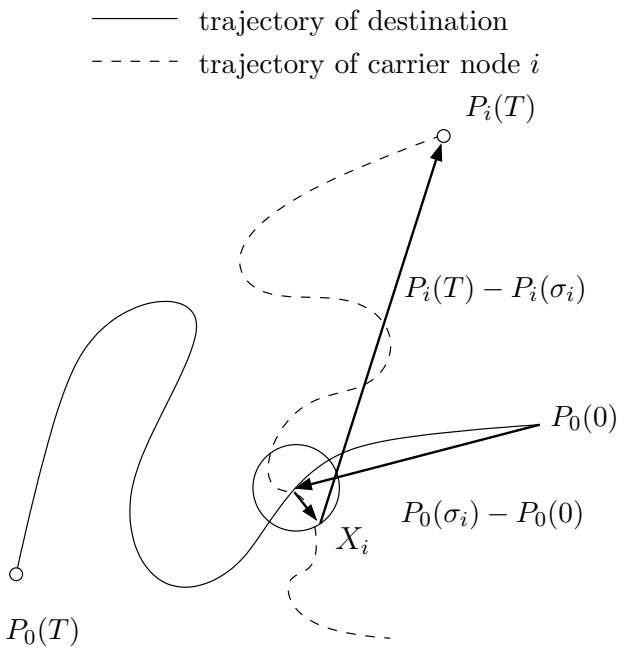

Figure 7.1: The encounter of node i with the destination at epoch σ_i.

for $0 \leq i \leq N$. Especially for the destination, we have that X_0 is trivially a zero vector. Note that $P_i(\sigma_i + t)$, $t \geq 0$, are independent Brownian motions.

We will now prove an upper bound for $G_\tau(T)$.

Lemma 7.1. *For $\tau \in (0, T]$, let $G_\tau(T)$ be the expected one-step flooding area if we consider only the carrier nodes that met the destination in the interval $(0, \tau]$, as defined above. Then,*

$$G_\tau(T) \leq \mathrm{E}\left[\frac{2\pi v T}{N+1}\right] + 2\sqrt{\mathrm{E}\left[\frac{2\pi v T}{N+1}\right]}(2\sqrt{2\pi v\tau} + 1) + (2\sqrt{2\pi v\tau} + 1)^2 \tag{7.12}$$

where the expectations above are taken over N.

Proof. By (7.9) and (7.10) we get that

$$\begin{aligned}
G_\tau(T) &= \mathrm{E}\left[\min_{0\le i\le N} \pi||P_i(T)||^2\right] \\
&= \pi\mathrm{E}\left[\min_{0\le i\le N} ||P_0(\sigma_i) + X_i + P_i(T) - P_i(\sigma_i)||^2\right] \\
&\le \pi\mathrm{E}\left[\min_{0\le i\le N} \left(||P_0(\sigma_i)|| + ||X_i|| + ||P_i(T) - P_i(\sigma_i)||\right)^2\right] \\
&= \pi\mathrm{E}\left[\left(\min_{0\le i\le N} \left(||P_0(\sigma_i)|| + ||X_i|| + ||P_i(T) - P_i(\sigma_i)||\right)\right)^2\right] \\
&\le \pi\mathrm{E}\left[\left(\max_{0\le i\le N} ||P_0(\sigma_i)|| + \max_{0\le i\le N} ||X_i|| + \min_{0\le i\le N} ||P_i(T) - P_i(\sigma_i)||\right)^2\right] \\
&= \pi\mathrm{E}\Bigg[\max_{0\le i\le N} ||P_0(\sigma_i)||^2 + \max_{0\le i\le N} ||X_i||^2 + \min_{0\le i\le N} ||P_i(T) - P_i(\sigma_i)||^2 + \\
&\qquad 2\max_{0\le i\le N} ||P_0(\sigma_i)|| \max_{0\le i\le N} ||X_i|| + 2\max_{0\le i\le N} ||P_0(\sigma_i)|| \min_{0\le i\le N} ||P_i(T) - P_i(\sigma_i)|| + \\
&\qquad 2\max_{0\le i\le N} ||X_i|| \min_{0\le i\le N} ||P_i(T) - P_i(\sigma_i)||\Bigg]
\end{aligned}$$

The expectation of the sum can be written as the sum of the expectations. The first term of the sum can be upper-bounded as follows

$$\begin{aligned}
\mathrm{E}\left[\max_{0\le i\le N} ||P_0(\sigma_i)||^2\right] &\le \mathrm{E}\left[\max_{0\le t\le \tau} ||P_0(t)||^2\right] \\
&\le 2\mathrm{E}\left[\max_{0\le t\le \tau} ||B(t)||^2\right] \quad \text{where } B \text{ is a 1-dimensional Brownian motion} \\
&\le 8\mathrm{E}\left[||B(\tau)||^2\right] \quad \text{by the reflection principle} \\
&= 8v\tau
\end{aligned}$$

By (7.11), $\max_{0\le i\le N} ||X_i|| \le 1/\sqrt{\pi}$ and $\max_{0\le i\le N} ||X_i||^2 \le 1/\pi$. The third term

$$\mathrm{E}\left[\min_{0\le i\le N} ||P_i(T) - P_i(\sigma_i)||^2\right]$$

is the expected minimum of $N+1$ Brownian motions, independent from each other and from N, where the i-th, $0 \le i \le N$, spans over the interval of $[\sigma_i, T]$. It can thus be written as

$$\begin{aligned}
\mathrm{E}\left[\min_{0\le i\le N} ||P_i(T) - P_i(\sigma_i)||^2\right] &= \mathrm{E}\left[\mathrm{E}\left[\min_{0\le i\le N} ||P_i(T) - P_i(\sigma_i)||^2 \mid N, \sigma_0, \ldots, \sigma_N\right]\right] \\
&= \mathrm{E}\left[h(v(T-\sigma_0), v(T-\sigma_1), \ldots, v(T-\sigma_N))\right]
\end{aligned}$$

where $h(x_0, x_1, \ldots, x_N)$ is described by (6.3) in Lemma 6.2 (without the constant π): it is the variance of the minimum of $N+1$ independent, omnidirectional, two-dimensional Gaussian random variables with zero means and variances x_i in each axis. The expectation above is over N and σ_1,...,σ_N (σ_0 is not random). Note that

$$h(v(T-\sigma_0), v(T-\sigma_1), \ldots, v(T-\sigma_N)) \leq h(vT, vT, \ldots, vT) = 2vT/(N+1)$$

Hence we get

$$\mathrm{E}\left[\min_{0\leq i\leq N} ||P_i(T) - P_i(\sigma_i)||^2\right] \leq \mathrm{E}\left[\frac{2vT}{N+1}\right]$$

Note that the expectation in the above upper bound is over all possible values of N, since $2vT/(N+1)$ does not depend on any σ_i, $1 \leq i \leq N$.

Since all variables involved are positive, we can use the fact that $\mathrm{E}[Y] \leq \sqrt{\mathrm{E}[Y^2]}$ for positive random variables to compute the rest of the terms. We immediately have that

$$\mathrm{E}\left[\max_{0\leq i\leq N} ||P_0(\sigma_i)||\right] \leq 2\sqrt{2v\tau}$$

and

$$\mathrm{E}\left[\min_{0\leq i\leq N} ||P_i(T) - P_i(\sigma_i)||\right] \leq \sqrt{\mathrm{E}\left[\frac{2vT}{N+1}\right]}.$$

Since $\max X_i$ are upper-bounded by a constant, the terms that contain it can be computed without concerns over the expectation of the product being equal to the product of expectations. The term

$$\mathrm{E}\left[2 \max_{0\leq i\leq N} ||P_0(\sigma_i)|| \min_{0\leq i\leq N} ||P_i(T) - P_i(\sigma_i)||\right]$$

is the expectation of the product of independent random variables: Brownian motions P_i, $1 \leq i \leq N$ are independent of the trajectory of the destination, and the trajectory P_0 of the destination after $\sigma_0 = \tau$ is independent of the trajectory of the destination prior to τ. Hence we get the upper bound in (7.12). □

The expectation appearing in Lemma 7.1 is the expected area flooded if $P_i(T)$ were independent, normal random variables. From the previous chapter, we know this can be combined with a Poisson distribution of N. However, the distribution of N is not easy to describe, as we saw in section 5.1.2. We can however use the concept of stochastic domination to derive a bound in terms of μ_0.

Lemma 7.2. *The random variable $1/(N+1)$ is dominated by $1/(N_0+1)$, where N_0 is the number of nodes in $\boldsymbol{\pi}$ that meet a static destination in the interval $[0,\tau]$.*

Proof. By (7.6), $N = N(\tau)$ is none other than $N_{fen}(\mathcal{D}, 0, \tau)$, which is none other than $N_B(\tau)$ as defined in section 5.1.2. Hence $N(\tau)$ dominates $N_0 = N_0(\tau)$, by lemma 5.3. It is easy to see from the definition of stochastic domination that if a real random variable X_1 stochastically dominates a real random variable X_2 and f is a strictly decreasing function, then $f(X_2)$ stochastically dominates $f(X_2)$. Since $1/(X+1)$ is strictly decreasing, the lemma follows. □

This gives us the following lemma.

Lemma 7.3. *For $\tau \in (0,T]$, let $G_\tau(T)$ be the expected one-step flooding area if we consider only the carrier nodes that met the destination in the interval $(0,\tau]$, as defined above. Then,*

$$G_\tau(T) \leq \frac{2\pi vT}{\mu_0(\tau)} + 2\sqrt{\frac{2\pi vT}{\mu_0(\tau)}}(2\sqrt{2\pi v\tau}+1) + (2\sqrt{2\pi v\tau}+1)^2 \tag{7.13}$$

where $\mu_0(t)$ is described by Corollary 5.2.

Proof. By Lemma 7.2, we have that

$$\begin{aligned}
\mathrm{E}\left[\frac{2vT}{N+1}\right] &\leq \mathrm{E}\left[\frac{2vT}{N_0+1}\right] \\
&= \sum_{n=0}^{\infty} \frac{2\pi vT}{n+1}\frac{e^{-\mu_0}(\mu_0)^n}{n!} \\
&= \frac{2\pi vT}{\mu_0}(1-e^{-\mu_0}) \quad \text{as in Lemma 6.4} \\
&\leq \frac{2\pi vT}{\mu_0} \quad \text{since } 1-e^{-\mu_0} \leq 1
\end{aligned}$$

where $\mu_0 = \mu_0(\tau)$ is described by (5.17) in Corollary 5.2. The above result, combined with Lemma 7.1 gives us the lemma. □

The time epoch τ can be chosen to be a function of T. For a given $\tau(T)$, such that $0 \leq \tau(T) \leq T$, the bound in Lemma 7.3 can be written in terms of T only. For example, if τ grows linearly in terms of T, Lemma 7.3 yields a linear upper bound for $G_\tau(T)$. Note that this bound is also a bound for $G(T)$. By chosing τ properly, we obtain the bound in the following theorem.

Theorem 5. *The expected one-step flooding area $G(T)$ is such that*

$$G(T) \leq 8\sqrt{\frac{\pi v T}{\rho}\log vT}(1+o(1)) \tag{7.14}$$

Proof. The expected one-step flooding area $G(T)$ is upper bounded by $G_\tau(T)$. On the other hand, inequality (7.13) can be written as

$$G_\tau(T) \leq \left(\sqrt{\frac{2\pi v T}{\mu_0(\tau)}} + 2\sqrt{2\pi v\tau} + 1\right)^2.$$

Let

$$\tau = \tau(T) = \frac{1}{4}\sqrt{\frac{T}{\rho\pi v}\log vT}. \tag{7.15}$$

By Corollary 5.2 we have that

$$\mu_0(t) = \rho\left(\frac{2\pi vt}{\log vt} + (d+o(1))\frac{vt}{\log^2 vt}\right). \tag{7.16}$$

Thus

$$\begin{aligned}
\frac{2\pi vT}{\mu_0(\tau(T))} &= \frac{2\pi vT}{\frac{2\pi\rho v\tau(T)}{\log(v\tau(T))} + (d+o(1))\frac{\rho v\tau(T)}{\log^2(v\tau(T))}} \\
&= \frac{2\pi vT}{\frac{2\pi\rho v\tau(T)}{\log(v\tau(T))}}\frac{1}{1+(d+o(1))\frac{1/2\pi}{\log v\tau(T)}} \\
&= \frac{T\log(v\tau(T))}{\rho\tau(T)}\left(1+\frac{1}{1+(d+o(1))\frac{1/2\pi}{\log v\tau(T)}}-1\right) \\
&= \frac{T\log(v\tau(T))}{\rho\tau(T)}(1+o(1)) \qquad \text{since } \lim_{T\to\infty}\tau(T) = +\infty
\end{aligned}$$

By substituting the value of $\tau(T)$ we get

$$\begin{aligned}
\frac{2\pi vT}{\mu_0(\tau(T))} &= \frac{T\log\left(v\frac{1}{4}\sqrt{\frac{T}{\rho\pi v}\log vT}\right)}{\rho\frac{1}{4}\sqrt{\frac{T}{\rho\pi v}\log vT}}(1+o(1)) \\
&= \frac{T(\log(v\sqrt{T})+O(\log\log vT))}{\rho\frac{1}{4}\sqrt{\frac{T}{\rho\pi v}\log vT}}(1+o(1)) \\
&= \frac{\frac{1}{2}T\log vT}{\frac{1}{4}\sqrt{\frac{\rho T}{\pi v}\log vT}}\left(1+\frac{O(\log\log vT)}{\log(v\sqrt{T})}\right)(1+o(1)) \\
&= 2\sqrt{\frac{\pi vT}{\rho}\log vT}(1+o(1))(1+o(1)) \\
&= 2\sqrt{\frac{\pi vT}{\rho}\log vT}(1+o(1))
\end{aligned}$$

Similarly, we have that

$$\pi v\tau(T) = \frac{1}{4}\sqrt{\frac{\pi vT}{\rho}\log vT}.$$

Hence, we have that

$$\begin{aligned}
G_\tau(T) &\leq \left(\sqrt{2}\sqrt[4]{\frac{\pi vT}{\rho}\log vT}\sqrt{1+o(1)}+2\sqrt{2\frac{1}{4}\sqrt{\frac{\pi vT}{\rho}\log vT}+1}\right)^2 \\
&= \left(\sqrt{2}\sqrt[4]{\frac{\pi vT}{\rho}\log vT}(\sqrt{1+o(1)}+1)+1\right)^2 \\
&= \left(2\sqrt{2}\sqrt[4]{\frac{\pi vT}{\rho}\log vT}(1+o(1))+1\right)^2 \\
&= \left(2\sqrt{t}\sqrt[4]{\frac{\pi vT}{\rho}\log vT}(1+o(1))\right)^2 \\
&= 8\sqrt{\frac{\pi vT}{\rho}\log vT}(1+o(1))^2 \\
&= 8\sqrt{\frac{\pi vT}{\rho}\log vT}(1+o(1))
\end{aligned}$$

□

A question that naturally arises is whether a τ different than the one chosen in the above theorem -namely the one in (7.15)- would yield a better bound for $G(T)$. It is easy to see that, for a fixed T, the optimal τ is the one that minimizes the quantity on the right hand side of (7.13) in terms of τ. This minimization gives the optimal τ as a function of T. However, minimizing the expression in (7.13) in terms of τ directly is quite hard. The reason is that μ_0 is not explicitly described; we only know its asymptotic behaviour. Computing thus the optimal value for τ remains an open problem.

However, the choice of (7.15) is not arbitrary. Since we know how $\mu_0(t)$ behaves for large values of t, we can make an educated guess on a $\tau(T)$ that will yield a "good" asymptotic upper bound. More specifically, we have that

$$G_\tau(T) \leq \left(\sqrt{\frac{2\pi v T}{\mu_0(\tau)}} + 2\sqrt{2\pi v \tau} + 1 \right)^2 .$$

By Corollary 5.2, for large values of τ, we can approximate $\mu_0(\tau)$ with $2\rho\pi v\tau / \log v\tau$. Since the term squared is positive, to minimize the upper bound for large values of τ we need to minimize a quantity which is approximately

$$\begin{aligned} V(\tau) &= \sqrt{\frac{2\pi v T}{\frac{2\rho\pi v\tau}{\log v\tau}}} + 2\sqrt{2\pi v\tau} \\ &= \sqrt{\frac{T \log v\tau}{\rho\tau}} + 2\sqrt{2\pi v\tau} \end{aligned}$$

We have that

$$\begin{aligned} \frac{dV(\tau)}{d\tau} &= \frac{\sqrt{T}}{2\sqrt{\rho}} \sqrt{\frac{\tau}{\log v\tau}} \frac{(1 - \log v\tau)}{\tau^2} + \frac{\sqrt{2\pi v}}{\sqrt{\tau}} \\ &\approx -\frac{\sqrt{T}}{2\sqrt{\rho}} \tau^{-3/2} \sqrt{\log v\tau} + \sqrt{2\pi v} \tau^{-1/2} \end{aligned}$$

since $1 - \log v\tau$ is approximately equal to $-\log v\tau$ for large values of τ. By setting this equal to 0 we get that the optimal value of τ is approximately

$$\tau \approx \frac{1}{2} \sqrt{\frac{T}{2\rho\pi v} \log v\tau}.$$

Thus the optimal τ should be asymptoticaly proportional to $\sqrt{T}$ multiplied by various constants and a sublinear function. Hence, we can substitute $\log v\tau$ with $\log v\sqrt{T}$ in the above equation and get

$$\tau \approx \frac{1}{4} \sqrt{\frac{T}{\rho\pi v} \log vT}$$

which is the value we used in Theorem 5.

7.2 The conditional distribution $p(t,T)$

Although the expected one-step flooding area $G(T)$ was upper-bounded in the context of our model, the conditional distribution of the improved accuracy $p(t,T)$ remains elusive at the current stage of our research. Estimating this distribution is important because, even if $G(T)$ remains to some extent the same, the shape of $p(t,T)$ can influence greatly the results that were obtained in chapter 6. In this section, we will present two factors that affect the shape of $p(t,T)$. More specifically, we will see that the dependence of the positions of carrier nodes has a significant impact on the conditional distribution. This impact is not reversed by the fact that more nodes, on average, contain good estimates.

First, the shape of the distribution $p(t,T)$ is influenced by the likelihood of carrier nodes being located when flooding. As we have seen, if the positions of carrier nodes are normally, independently distributed around the anchor point, they are equally likely to be found. In our protocol however, this does not hold. To investigate the influence of the lack of independence of $\pi(t,T)$, we conducted the following experiment. We assumed that the destination performs a Brownian motion, starting at the origin, in the interval $[0,T]$. While it moves, n carrier nodes are generated at equally spaced time intervals within $[0,T]$. Starting at the current position of the destination, each node "branches off" from the trajectory of the destination and performs for the remaining time a Brownian motion independent of the trajectories of other carrier nodes and the destination.

More formally, at time $i \cdot \frac{T}{n}$, $1 \leq i \leq n$, the i-th node is placed at the destination's current position. It then moves according to a Brownian motion for the interval $[i \cdot \frac{T}{n}, T]$, independently from the destination and the rest of the carrier nodes. We simulated this experiment and we recorded the frequency with which each node appeared closer to the origin than all others and thus would be chosen upon a flooding. This frequency, in terms of the time that the chosen node branched off from the trajectory of the destination, is shown in figure 7.2 for $n = 10$ and $n = 100$ respectively. As we can see, nodes that met the destination earlier are more likely to be located.

This observation can be intuitively understood by the fact that for two nodes i and j with $i < j$, given that i is not within a disk of radius r around the destination, the destination's and thus also j's likelihood of also not being in the disk is affected. Similar information on the position of j does not influence i the same way. It is more intuitive

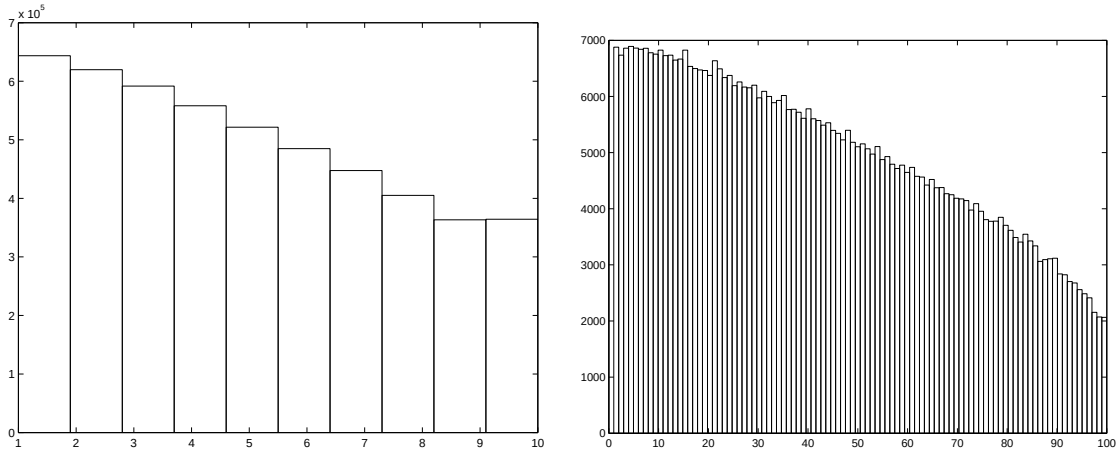

Figure 7.2: Carrier nodes branching at equally spaced intervals. The frequencies of $n = 10$ (5×10^6 repetitions) and $n = 100$ (5×10^5 repetitions) carrier nodes are shown.

in fact that i's probability will be less influenced than j's, respectively.

This is a negative result, in the sense that it shows that the distribution we assumed in our analysis in chapter 6 is too optimistic. The above experiment indicates that the distribution in the original model is skewed toward worse values (closer to T rather than 0) compared to the simplified one.

Note that the above experiment does not model the case where encounter epochs are the last exits epochs, which is what we need in order to describe the probability $p(t, T)$. If they were last exit epochs, the carrier nodes' movements after the encounter should not be considered to be independent of the movement of the destination. In fact, as we have already stated, they should be moving in such a way that they avoid the destination. Also, we ignored the fact that carrier nodes positions do not coincide with the one of the destination upon encounters; they merely are within the unit-area disk around it.

We know however that accuracies are not equally spaced in our model. As a matter of fact, the expected number of carrier nodes with accuracy t is described by the last exit epochs process and by Proposition 5.1 we know that more nodes with accuracy closer to 0 exist than nodes with accuracy closer to T. This counteracts the effect noticed in our simulation, in the sense that more nodes will branch off "later" rather than "sooner". It is interesting thus to know whether this will tilt the density function and make accuracies closer to 0 more probable than the ones closer to T.

Unfortunately, this is not the case. We repeated the experiment with the distances between branchings scaled according to a function that has a limit behaviour of $4\pi vt/\log(2vt)$,

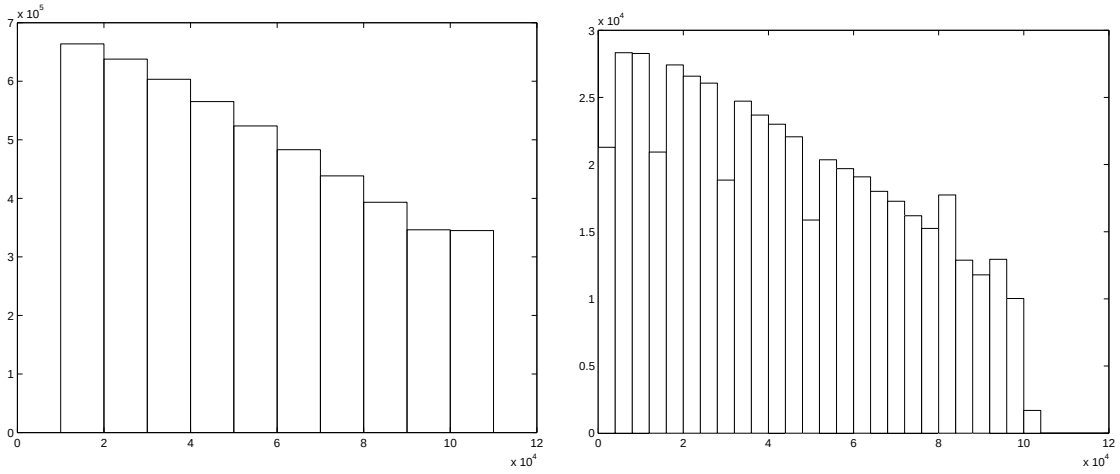

Figure 7.3: Carrier nodes branching at weighted intervals. The frequencies of $n = 10$ (5×10^6 repetitions) and $n = 100$ (5×10^5 repetitions) carrier nodes are shown. In both cases v and ρ were taken to be 1 and T was equal to 10^5.

as the expected value of the number of carrier nodes in our protocol. Unfortunately, the resulting frequency distribution showed that nodes that branched off earlier were more likely to be located. The reason is that $t/\log(t)$ is, for large values of t, almost linear. Because of this behaviour, although the intervals between nodes that branched off early are greater than the intervals between nodes that branched off later, they are still comparable and cannot diminish the effect of interdependence on the probability distribution.

In conclusion, these experiments indicate that the interdependence between carrier nodes must be taken into consideration in any attempt to find an upper bound for the behaviour of the LER protocol through a Volterra equation.

Chapter 8

Conclusions

The goal of this thesis was to provide a rigorous analysis of the LER protocol. To that purpose, we presented a model of the network under which many attributes of LER can be described, as shown in chapter 5. We also presented a cost function that quantifies the behaviour of the protocol. Approximating this cost function was reduced to obtaining the expected one-step flooding area $G(T)$ and the probability density of the improved accuracy $p(t, T)$. In chapter 7, we provided an asymptotic upper bound for the first of these two functions, which, along with $p(t, T)$, can be used to derive an upper bound for the function $Q(T)$ that approximates the cost. We were however unable to provide a concrete description of $p(t, T)$ under our model.

Our work remains thus incomplete; it indicates however that LER is amenable to analysis. As described in chapter 6, our model can yield results that are subject to very intuitive interpretations. This encourages us to continue in our goal to analyze the LER protocol in its entirety.

We intend to address the issue of the density $p(t, T)$ in future research. The techniques we have demonstrated in this thesis may prove to be useful in our attempts. Furthermore, although the upper bound of $G(T)$ we computed is promising, as sublinear, whether it is tight or not is still an open question. It is interesting thus to see if a bound lower than this can be obtained.

As stated in the introduction, LER stands out among other routing schemes that can be characterized as approximate information protocols because of the independence of the routing mechanism from network traffic. It would be interesting to extend our model in order to describe the influence of the traffic load as well, allowing us thus to analyze more elaborate protocols. As we have discussed, approximate information protocols show

an interesting balancing behaviour with respect to network traffic. We believe that this property could be verified analytically and hope that our model could be used towards that direction.

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Appendix A

Proof of Proposition 5.2

The following proof by Quastel [39] shows that the expected area covered by a Wiener sausage with a drift is larger than the expected area covered by a Wiener sausage without a drift.

For $f : [0,1] \to \mathbb{R}^2$ let

$$\Phi(f) = \mathrm{E}\left[|\{x : \exists s \in [0,1] \text{ s.t. } |B(s) - f(s) - x| \leq r_0\}|\right].$$

We wish to show that

$$\Phi(f) \geq \Phi(0)$$

Let

$$\Phi_N(f) = \mathrm{E}\left[\int_{\mathbb{R}^2} 1 - e^{-N\int_0^1 \mathbb{1}(|B(s)-f(s)-x| \leq r_0)ds}\right] dx$$

where $\mathbb{1}(x) = 1$ if x is true and $\mathbb{1}(x) = 0$ o.w. Then

$$\lim_{N\to\infty} \Phi_N(f) = \Phi(f).$$

Hence it suffices to show that

$$\Phi_N(f) \geq \Phi(f).$$

Let $V = N\mathbb{1}(|x| \leq r_0)$ and

$$r_N(t,x) = \mathrm{E}\left[e^{-\int_0^t V(B(s)-f(s)-x)ds}\right].$$

Let $u_N = 1 - r_N$. Note that

$$\Phi_N(f) = \int_{\mathbb{R}^2} u_N(1,y)dy.$$

By the Feynman-Kac formula $r_N(t,x)$ satisfies $r_N(0,x) = 1$ and

$$\partial_t r_N = \frac{1}{2}\Delta r_N - [\tau_{f(t)V}]r_N.$$

Then $u_N(t,x)$ satisfies $u_N(0,x) = 0$ and

$$\partial_t u_N = \frac{1}{2}\Delta u_N - [\tau_{f(t)}V]u_N + [\tau_f(t)V].$$

We can write the solution as

$$u_N(t,y) = \int_0^t \int_{\mathbb{R}^2} [\tau_{f(s)}V](x)p(s,x,t,y)dxds$$

where

$$p(s,x,t,y) = \mathrm{E}_x[e^{-\int_s^t [\tau_{f(u)}V](B(t+s-u))du}\delta_y].$$

So

$$\Phi_N(f) = \int_0^1 \int [\tau_{f(s)}V](x)\mathrm{E}_x[e^{-\int_s^1 [\tau_{f(u)}V](B(t+s-u))du}]dxds.$$

Let μ be the uniform distribution on the unit-area disk centered at the origin. We have

$$\begin{aligned}\Phi_N(f) &= N\int_0^1 \mathrm{E}_\mu\left[e^{-N\int_s^1 \mathbb{1}(|B(u)-[f(u)-f(s)]|\le r_0)du}\right]ds \\ &\ge N\int_0^1 \mathrm{E}_\mu\left[e^{-N\int_s^1 \mathbb{1}(|B(u)|\le r_0)du}ds\right] = \Phi_N(0)\end{aligned}$$

which concludes the proof.