ECE 3321: Digital Signal Processing
Fall 2001, Final Exam
December 12, 2001

Directions
1. You have 3 hours to complete all questions on this exam
2. Solutions should be recorded and returned on the attached solution sheets
3. If you feel any question is unclear, provide in your answer sufficient justification for any assumptions you must make.
Question 1

1. (10 points) Show that the inverse of a circulant matrix is also circulant.

2. (10 points) Find a length 5 signal whose circular convolution with
   \[ h(n) = \{1.0000, -2.1180, 0.9271, -0.8820, -1.0000\} \]
   is zero (to within 3 or 4 decimal places) or explain why no such signal exists.

3. (15 points) Let \( x(n) \) be a signal of length \( N = 2^k \). Describe a “fast” algorithm for evaluating the Z transform of \( x \) at \( N \) equally spaced points on a circle of radius \( r \) starting at angle \( \theta \) and ending at angle \( 2^{\pi/2} (N-1) + \theta \), where \( v \) is an integer.

4. (15 points) In the figure below is plotted the logarithm of the absolute value of the DTFT of a square window (also known as a boxcar) and a triangular window both of length 51. Because it is the convolution of two boxcars, a triangle is “smoother” than a boxcar. Because it is smoother, the triangle should have more low frequency content than the boxcar. However, the figure shows that the mainlobe of the triangle is actually wider than that of the boxcar. This would seem to contradict the above argument. Please explain the contradiction. For reference, the DTFT of a unit amplitude, square signal running from \( n=-L \) to \( n=L \) is
   \[ \frac{\sin\left((L + \frac{1}{2})\omega\right)}{\sin\left(\frac{\omega}{2}\right)} \]
Question 2

Suppose we have a digital communications system that works in the following manner. If we want to transmit the symbol 1, a pure cosine wave of amplitude one and frequency 100Hz is turned on for 10 seconds. If we want to send the symbol zero, a unit amplitude cosine wave of frequency 1000 Hz is used instead. The goal of this problem is to build a receiver for this system; that is, a system which takes the transmitted signals and produces a 1 or 0 at the output. The receiver is going to have the structure shown below

where \( x_a(t) \) is the transmitted analog signal and \( x(n) \) is a sampled form of \( x_a(t) \)

Let’s start by ignoring the fact that the sinusoids are “on” for 10 seconds at a time. Instead let’s assume that the symbol 1 is a pure cosine wave at 100 Hz and the symbol 0 is a pure cosine at 1000Hz.

1. (5 points) Under the assumption above and given the specifications provided at the start of this problem, what is the minimum sampling rate, \( F_{min} \), required for the "Perfect Sampler" to avoid aliasing.

2. (5 points) Say we sample at \( 20F_{min} \). If the symbol 0 was sent what is the analytical expression for \( x(n) \). Repeat for the symbol 1.

We want to design \( H_0(z) \) such that its output is 0 if the input is a pure discrete sinusoid whose frequency corresponds to the symbol 1 and “large” if the input is a pure discrete sinusoid whose frequency corresponds to the symbol 0. The filter \( H_1 \) should be designed in an analogous way but to give a large value for the symbol 1. If these filters are properly built, then by looking at which output, \( s_0 \) or \( s_1 \), is “large” we can tell which symbol was sent.

3. (20 points) Again, ignoring the 10 second issue, specify rational transfer functions each with at most two poles and two zeros for these two filters to meet the above specifications. For full credit, you must justify the design choices you make being as quantitative as possible in specifying why \( s_0 \) will be larger than \( s_1 \) when a 0 is sent and vice versa for a 1.

4. (20 points) Now, let’s consider the effects of the 10 second issue. Suppose you use the filters you designed in part 3 of this problem to process the tone bursts. Will they still be useful? Is oversampling by a factor of 20 enough? Too much? In answering these questions, provide as complete an analysis as you can. Equations, while certainly allowable, are not necessary. Well chosen pictures or diagrams illustrating the basic ideas of your analysis are more than sufficient and actually preferred.
Question 3 (50 points)

On the website for the class you will find a file called final_1.mat. This file contains two matrices X and Y. X is of size 30 by 3 containing samples of the real and imaginary parts of the frequency response of some, unknown, real-valued LTI system with impulse response \( h(n) \). The first column contains the frequencies, \( \omega_i \times \pi \) \( i = 1, \ldots, 30 \), at which the samples are collected (hence the \( \omega_i \) range from 0 to 1 like the Matlab convention). The second column holds \( \text{Re}\{H(\omega_i \times \pi)\} \) \( i = 1,2,\ldots,30 \) and the third column contains \( \text{Im}\{H(\omega_i \times \pi)\} \) \( i = 1,2,\ldots,30 \). Note that the frequency samples are not equally spaced. The matrix Y is 30 by 2. The first column is a list of sample numbers, \( n_i \), while the second is the value of the impulse response at \( n_i \). That is, the second column contains \( h(n_i) \).

The goal of this problem is to design LTI systems with rational transfer functions that approximate the system whose samples are found in the file. The systems you build should be both “simple” and “accurate.” Simplicity is measured in terms of the number of “\( a_i \)” and “\( b_i \)” coefficients in your design. Accuracy is measured in terms of how well you fit the amplitude response and the impulse response of the system. If we denote by \( H_1(z) \) the transfer function of the system you design and \( h_1(n) \) the impulse response, the measures of accuracy are given by

\[
\text{Frequency accuracy} = \sum_{i=1}^{30} \left| H(\omega_i) - |H_1(\omega_i)| \right|^2
\]

\[
\text{Temporal accuracy} = \sum_{i=1}^{30} \left| h(n_i) - h_1(n_i) \right|^2
\]

At last two design methods should be proposed and analyzed. If you choose to look at linear phase FIR filtering methods, both window-based as well as remez-based methods must be evaluated. This would count as one of the two approaches. It is up to you to determine another way of solving the problem.

Please hand in the following

1. For each method, a plot of the magnitude response of your system overlaying the samples from X.
2. For each method, a plot of the impulse response of your system overlaying the samples from Y.
3. For each method, numerical values for the simplicity of your model and accuracy of the results.
4. A brief analysis as to how the two methods compare.
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