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Connecting Two Worlds: Physical Models and Graph Models of Wireless Network Topologies

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Abstract—**The way the network topology is modeled in a wireless network, primarily in ad hoc and sensor networks, has a fundamental influence on protocol design and efficiency. The frequently used graph models are simpler, and more amenable to analysis and protocol development. On the other hand, physical models represent the actual radio environment much more faithfully, albeit at the price of being far less supportive to network protocol development. We consider the potential future trend of resolving this conflict, via the integration of the two approaches. We present some results and challenges in exploring the connections between the two apparently very different classes of models.**

I. Introduction: The Infamous Modeling Clash

A critical dichotomy in the modeling of multihop wireless network topologies is that the computer science (CS) and electrical engineering (EE) communities approach the issue from two fundamentally different viewpoints, with rather limited communication among each other.

- **"CS view."** For protocol/algorithm development and analysis the network topology is usually modeled by a graph. The most frequently used graph model is the Unit Disk Graph (UDG) [6], although there are also various extensions (see Section II). Unfortunately, these graph models retain little from the real behavior of the radio environment. They are, however, very amenable to analysis, protocol design and algorithm development.
- **"EE view."** The physical modeling of the actual radio environment offers a more faithful representation of radio communication. For example, the signal to interference and noise ratio (SINR) based models (see Section III) can capture much better than graph models the conditions for having successful radio communication between network nodes. The physical models, however, are far less supportive to protocol development and algorithm analysis.

The difference between results obtained in the various models can be strikingly large. As an example, consider Moscibroda's analysis [13] of the worst-case capacity of wireless sensor networks. He shows that in the SINR based physical model a per node sustainable rate of $O(\log^2 n)$ can be achieved with n nodes in the worst case. On the other hand, the corresponding sustainable rate that can be achieved in the so called protocol model (which leads to a variant of the unit disk graph topology) is only $O(1/n)$ in the worst case. This is an *exponential gap* between the two models. Such an exponentially large gap clearly demonstrates the importance of getting the modeling issue right. Below we briefly review some approaches and show our initial results that can serve as a first bridge between the two sides.

II. Graph Models

The most frequently used *Unit Disk Graph (UDG)* [6] model of the network topology is a graph that is defined by the (planar) geometry of node positions. It is assumed that each node has the same transmission radius r, and two nodes are connected by a link if and only if they are within distance r (which is often normalized to $r = 1$, hence the name). In other words, the radio range of each node is just a circular disk. As a critical difference from the physical model, in a UDG it does not matter where the rest of the nodes are located and how much interference they generate.

A clear advantage of UDGs is that a number of important algorithmic problems that are NP-complete for general graphs become solvable in polynomial time for this special class [15], thus allowing much more efficient protocols.

Unfortunately, the UDG model is quite simplistic, even among pure graph models. A generalization is the *Quasi-Unit Disk Graph (Q-UDG)* [12], in which a *shrink factor* ρ is added, with $0 < \rho < 1$, for describing the radio range of a node by two concentric circular disks, the outer one with radius r, and the inner one shrunk by the factor ρ, yielding radius ρr. If two nodes are at most ρr distance apart, then they are always connected by a link. If they are more than r apart, then they are never connected. Finally, if the distance is between ρr and r, then the link may or may not exist. Geometrically this means that the radio range of a node can have arbitrary shape, but moderated by the requirement that it should be between a circumscribed circle of radius r and an inscribed circle of radius ρr.

A nice feature of Q-UDGs is that, while providing a

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more general network topology model, they still preserve the algorithmic advantages of UDGs, at the price of an additional $1/\rho^2$ factor in complexity [12]. Thus, if the shrink factor ρ is a not too small constant, then most of the UDG advantages carry over, with only a constant factor penalty in complexity.

Another natural issue is that different nodes may transmit with different power, or have different spectrum-dependent attenuation of the transmission signal [1]. This leads to the concept of *Disk Graph (DG)*, which differs from the UDG in that each node i has its own, possibly different, transmission radius r_i, and two nodes are connected by an undirected link if they are mutually in each other's range. DGs are somewhat less friendly from the algorithmic point of view than UDGs and Q-UDGs, but still better than general graphs and still allow efficient solutions or approximations for a number of algorithmic problems, as we investigated in [16].

Similarly to the generalization that leads to the Q-UDG concept, one can also introduce *Quasi-Disk Graphs (Q-DG)*, by adding a shrink factor ρ that allows to refine the radio range description as for Q-UDG.

All the above graph models can naturally be generalized to higher dimensions, replacing the disks by balls in the appropriate space.

A common nontrivial generalization of all these graphs, the *Bounded Independence Graph (BIG)* model is also worth mentioning [15]. It is defined by the requirement that the maximum number of independent nodes[1] within the k-hop neighborhood $\mathcal{N}_k(v)$ of any node v is bounded by a polynomial of k. Although this definition is based purely on the graph structure and does not have a direct geometric meaning, it can still be related to geometry through the concept of *doubling metric spaces* [15]. These are metric spaces in which any ball of radius r can be covered by a finite number of balls of radius $r/2$. This property does not hold for all metric spaces, although it holds for Euclidean spaces of any finite dimension[2]. It can be shown that if a geometric graph is defined in a doubling metric space, in analogy with UDG or DG, then it is always a Bounded Independence Graph [15].

All these graph classes are related to some kind of geometric insight. It is not surprising, since geometry and distance play a key role in forming the radio network topology. On the other hand, radio propagation (with possible obstacles and other irregularities) can induce much more complicated distances that may not satisfy the mathematical distance axioms, primarily the triangle inequality. Nevertheless, even in this more complicated situation, it is still possible to meaningfully analyze geometric-like graphs and prove nontrivial results about important properties, such as connectivity, as we have proved in [7], [9].

[1]A set of nodes in a graph is called independent if there is no edge between any two of them.

[2]Radio propagation properties may lead to a "radio-distance" that is quite different from Euclidean.

III. Topology Analysis through the Physical SINR Model

The *Signal to Interference and Noise Ratio (SINR)* model (see [2], and further references therein) captures the physical conditions of receiving the radio signal with a satisfactory quality.

A. The SINR Concept

Let $v_1, \ldots, v_n$ be radio nodes and, with a slight abuse of notation, let each v_i also represent the position of the node (in the plane, for simplicity). Let $d(x, y)$ be the Euclidean distance and assume that node v_i transmits with power P_i. Then, the *reception zone* of node v_i consists of all those points x in the plane, where $SINR_i(x) \geq \beta$ holds with some parameter β, that is, the received power of v_i at x is at least β times larger than a noise power N, plus the interference from all other nodes at x, assuming that attenuation is proportional to a power of distance. In formula,

$$SINR_i(x) = \frac{P_i d(v_i, x)^{-\alpha}}{N + \sum_{j \neq i} P_j d(v_j, x)^{-\alpha}} \geq \beta. \qquad (1)$$

In the simplest case, the parameters β, N and α are assumed known constants; α is called *path loss exponent* and it usually falls in the range $2 \leq \alpha \leq 6$.

Now, one may ask the natural question: What kind of network topology arises from this model? We can define it by connecting any two nodes v_i, v_k whenever both $SINR_i(v_k) \geq \beta$ and $SINR_k(v_i) \geq \beta$ hold, that is, they are mutually in each other's reception zone, so they can communicate. This generates a graph (network topology) that describes which pairs of nodes are capable of communicating.

The arising (undirected) graph, however, does not seem to belong to any special class with nice properties, since the existence of any given edge depends, in a complicated, nonlinear way, on *all* the node positions and transmission powers, not only on the end nodes of the particular link. For example, if any single node v_i changes position or power, then all $SINR_i(x)$ values change, so the *whole* graph may become different. The effect may not remain local, in sharp contrast with the philosophy of graph models. Thus, the SINR approach does not seem to offer any good opportunity to apply the special graph classes and properties, along with the rich treasury of results that build on them. Next, however, we show the surprising and unexpected fact that it is still possible to build bridges between the "different worlds."

B. The First Bridge Linking the CS and EE Approaches

Building on the involved analysis of [2] about the geometry that the SINR condition (1) generates, we prove a result (see Theorem 1 below), which shows the unexpected fact that the "messy" SINR based topology is in fact not too far from a simpler, standard graph model, at least for some choice of the parameters.

Theorem 1 *Assume that all nodes transmit with the same power and the path loss exponent is $\alpha = 2$. Then, for arbitrary $\beta \geq 1$ and for arbitrary node positions, the network topology that arises from the SINR model is a Quasi-Disk Graph with shrink factor*

$$\rho = \frac{\sqrt{\beta} - 1}{\sqrt{\beta} + 1}. \tag{2}$$

Proof. Avin et al. prove in [2] the following (rather nontrivial) geometric fact: under the conditions of Theorem 1 the reception zones will be convex domains with the additional property that the ratio between the radius of an inscribed and circumscribed circle is at least $(\sqrt{\beta} - 1)/(\sqrt{\beta} + 1)$. Let r_i be the radius of this circumscribed circle around the reception zone of node v_i. Set $\rho = (\sqrt{\beta} - 1)/(\sqrt{\beta} + 1)$. Then we obtain that the transmission of node v_i cannot be received beyond distance r_i, but it is guaranteed to be received within distance ρr_i. If the distance is between ρr_i and r_i, then the transmission may or may not be received. If we consider a Quasi-Disk Graph with the r_i, $i = 1, \ldots, n$, and ρ parameters, it behaves exactly the same way. For the intermediate case, when the distance is between ρr_i and r_i, we can arbitrarily choose between the options of reception and non-reception so we can always choose the same that happens in the SINR model. Thus, there must exist a Q-DG that gives precisely the network topology generated by the SINR model under the conditions of the Theorem.

♡

Theorem 1 shows the unexpected fact that despite the messy "everything depends on everything" nature of the physical SINR model, the arising graph still belongs to a special class that is part of the toolkit of multihop wireless network modeling (in the "CS view").

C. Trade-Off Between SINR and Global Properties of the Network Topology

Equation (2) implies that with $\beta \to \infty$ the Quasi-Disk Graph in Theorem 1 approaches a Disk Graph, since ρ tends to 1. Depending on the node positions, however, it may not be a UDG, only a DG. On the other hand, if the node positions are random (chosen independently from the same distribution), then, by symmetry considerations, each node will have the same *expected* radius r. Due to independence, one can also expect a strong concentration of the actual random r_i values around their common expected value r. Taking into account that with large enough β, the shrink factor will be $\rho \approx 1$, we obtain that, under these conditions, the SINR generated network topology can asymptotically be well approximated by a UDG.

This fact is encouraging from the viewpoint of analysis of network level properties, such as the connectivity of the network topology, since much is known about these issues in the UDG model. Additionally, in the random setting it is known that if n nodes are placed uniformly at random in a unit disk, then the network will be connected with probability approaching 1, as $n \to \infty$, if and only if the transmission radius r satisfies

$$\pi r^2 = \frac{\log n + c(n)}{n} \tag{3}$$

where $c(n)$ is an arbitrary function with $c(n) \to \infty$ [11]. Various generalizations are also known in this direction, see, e.g., [7], [9]. It is also important that both DGs and UDGs have efficient routing and labeling schemes, as well as spanners and separators, etc., with good properties (see [17] for definitions and further references).

These considerations point to the unexpected fact that the "messy" SINR generated network topology and the abstract UDG model are in fact not as far from each other as one might expect. This may allow the extension of a lot of useful analytical and protocol design results from UDGs to SINR based models.

IV. Common Generalization of SINR and Graph Models

The SINR model of Section III-A and, in particular, its special case used in Theorem 1, is only the simplest version of this type of physical models. In reality there is a good number of additional complications that need to be taken into account for a faithful representation of the radio environment. For example, the transmission power of each node may be different and randomly changing. The received power may be subject to fading. The noise level may be time and location dependent and randomly changing. The path loss exponent may be different from $\alpha = 2$; it may also depend on direction, time and location. The distance may not be Euclidean, to reflect additional effects, such as obstacles to radio propagation, etc.

The formal introduction of such effects into the SINR model, in order to making it a more faithful description of the radio network, is not too hard in itself, since one can use the results of the extensive research that has been done in accurately modeling the physical radio environment. It is much more difficult, however, to avoid hopeless messiness from the viewpoint of networking protocols ("CS view"). In other words, it is very desirable to develop the more complex models in a way that still preserves a meaningful relationship with graph models, so that we can build further bridges in the spirit discussed in the previous sections.

In a more complex scenario it seems unlikely that Theorem 1 would directly carry over. Therefore, let us introduce a more general graph model that opens a new direction to capture the network topology, and, in a sense, serves as a common generalization of SINR and graph models. We call it *Weight Ratio Graph (WRG)* model. It is a graph characterized by the following parameters:

- A set of nodes $\{v_1, \ldots, v_n\}$.
- A positive weight $w_{i,j}$ for each pair (v_i, v_j) of nodes $(i \neq j)$.
- A parameter $\beta > 0$, which will play a similar role to the β parameter of the SINR model.

With these parameters the actual graph is defined in the following way. First, we say that node v_i can receive node v_k's transmission, if

$$\frac{w_{k,i}}{\sum_{j \neq k} w_{j,i}} \geq \beta,$$

that is, the weight $w_{k,i}$, perceived as the power that reaches v_i from v_k, is at least β times larger than the power that reaches v_i from all other nodes. Finally, there is a link between two nodes if they can both receive each other's transmission, according to the above definition.

It is not hard to see the connection between the SINR and WRG models. If we take $w_{i,j} = P_i d(v_i, v_j)^{-\alpha}$ and use the same β in both models, then WRG will produce exactly the same network topology as the SINR with $N = 0$. Therefore, at first it seems that WRG can only represent the special case when the noise is zero. A closer look reveals, however, that this is not true. In fact, surprisingly, the WRG model essentially has *universal* expressive power, as shown by the following result.

Theorem 2 *Let G be an arbitrary graph with no isolated nodes. Then the parameters of the WRG model can always be chosen such that the model will generate precisely the graph G.*

Proof. Let us choose $w_{i,j} = 1$ if there is an edge between v_i and v_j and set $w_{i,j} = \epsilon$ if there is no edge between them. Here ϵ is any constant with $0 < \epsilon < 1/2$. Let n be the number of nodes and set

$$R_{k,i} = \frac{w_{k,i}}{\sum_{j \neq k} w_{j,i}}.$$

Let us compute $R_{k,i}$ for two possible cases. If there is an edge between v_k and v_i, then $w_{k,i} = 1$ and the largest possible value of the denominator is $n - 2$, which occurs if v_k is connected to all other nodes. Hence, in this case

$$R_{k,i} = \frac{w_{k,i}}{\sum_{j \neq k} w_{j,i}} \geq \frac{1}{n-2}.$$

On the other hand, if there is no edge between v_k and v_i, then $w_{k,i} = \epsilon$ and the denominator is at least $(n-3)\epsilon + 1$, since by assumption there is no isolated node, so v_i must have a neighbor other than v_k, as v_k is not a neighbor now. Thus, in this case

$$R_{k,i} = \frac{w_{k,i}}{\sum_{j \neq k} w_{j,i}} \leq \frac{\epsilon}{(n-3)\epsilon + 1} = \frac{1}{(n-3) + \frac{1}{\epsilon}} < \frac{1}{n-2}$$

where the last inequality follows from $1/\epsilon > 1$ which is implied by the choice $\epsilon < 1/2$. Furthermore, due to symmetry, if $R_{k,i}$ satisfies either of the inequalities, then so does $R_{i,k}$. Thus, if we set

$$\beta = \frac{1}{n-2}$$

then it will separate the two cases, as desired. ♡

As we have seen, the WRG model can generate *any* graph with no isolated nodes, which, of course, also includes the SINR topology with nonzero noise, assuming there is no isolated node. As a result, we obtain the following direct, but interesting consequence.

Theorem 3 *For an SINR model (as defined in Section III-A) it is always possible to eliminate the effect of noise by transforming it into another SINR model with $N = 0$ that generates the same network topology, given that it contains no isolated node. The transformation can be achieved by adjusting the transmission powers from P_i to some P'_i, and possibly the value of β, with no other change in the model. That is, some positive P'_i, β' values can always be chosen such that*

$$\frac{P_i d(v_i, v_k)^{-\alpha}}{N + \sum_{j \neq i} P_j d(v_j, v_k)^{-\alpha}} \geq \beta$$

holds if and only if

$$\frac{P'_i d(v_i, v_k)^{-\alpha}}{\sum_{j \neq i} P'_j d(v_j, v_k)^{-\alpha}} \geq \beta'.$$

Moreover, this equivalence holds for all i, k simultaneously.

It is worth noting that while Theorem 3 follows from Theorem 2 as an immediate consequence, it appears to be harder to prove Theorem 3 directly, building only on the SINR model. The achieved shortcut shows the power of Theorem 2.

V. Discussion

As we have argued, the graph models ("CS view") and the physical models ("EE view") of the wireless network topology both have their advantages and drawbacks. Their integration is a challenging, but also promising project. The appropriate modeling of the network topology in large scale wireless networks is widely accepted as an essential component in various tasks, including protocol development, performance evaluation, and network design. Therefore, we believe that the research in this direction will be an important trend in the future development of distributed computing systems, that include large wireless networks.

The attractiveness of the introduced Weight Ratio Graph (WRG) model is that it contains all other models that we have discussed, including SINR generated network topologies and all the various graph models. Moreover, the WRG allows a continuous transition among them, by continuously adjusting the weights in the model.

In conclusion, let us mention some interesting research problems in connection with the WRG model.

- Given an SINR model (possibly with a number of potential generalizations, as mentioned at the beginning of Section IV), determine a functional relationship between the SINR model parameters and the weights of a WRG that generates the same network topology. Note that Theorem 2 guarantees the existence of the appropriate weights, but it is not obvious how to express them as (preferably simple, direct) functions of the SINR model parameters, such that an edge weight depends only on the parameters of nearby nodes.
- How localized are the effects of changes in the SINR parameters? For example, if a physical topology is modeled by a WRG, can we prove some bound on the changes of the weights that belong to "remote" links, if the transmission power of a single or few nodes changes?
- If the SINR model is time variant (which is often the case), then determine the WRG weights such that the arising graph provides a best approximation, in an average sense. In other words, some measure of similarity is averaged over time, and this average value is optimized.
- In case we know some statistical description of the temporal behavior of an SINR model, how does it translate into a statistical description of the WRG weights, as they change in time?
- If the weights in a WRG model are chosen randomly, according to some probabilistic model, then what can we say about the arising random graphs? How different this random graph model will be from other known random graph models used in wireless network topology modeling, such as geometric random graphs?
- Since WRGs have universal expressive power, in the sense of Theorem 2, does this mean, in some sense, that a random WRG model can capture any other random graph model? Note that there are many types of random graph models studied in the literature, see, e.g., the books [3], [4], [5], [10], [14] and hundreds of further references therein.
- How do the various parameters of such a random WRG graph behave? Examples of parameters/features of interest: degree distribution, maximum clique, diameter, average hop distance between random nodes, chromatic number, and many other graph theoretic parameters.
- Determine the WRG weights such that the arising graph is a best approximation of the SINR generated graph under the constraint that the WRG should fall in a given special graph class, such as UDG, Q-UDG, etc. Finding such a *projection* of the actual radio network topology onto a simpler class would make it possible to systematically approximate a complex SINR topology with a graph that belongs to a class with nice, well explored properties, which can be very useful in protocol development for "messy," generalized SINR models, which represent the real radio network topology. This task of finding a best approximation of a complex model in a class of simpler ones appears to be interesting not only for its potential applications, but also on its own right.

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