Figure 6.9. A complete electrostatic actuator, with added elements representing the inertia of the moveable element, mechanical damping, and the source resistance of the electrical network.
Comb Drive – Another version of the electrostatic actuator.

• What is different?
  • Before, $C = \varepsilon A/g$
  • Now, $C = C_0 + C_1x$
  • Linear in $x$, the direction of motion.
  • Consequences?
• Look at one finger, treat capacitance as that of a parallel plate capacitor:
  • Then $C = \varepsilon A / g$.
  • $A = lt$.
  • $C = \varepsilon lt / g$, but remember that $g$ is a constant.
  • So, $C = \varepsilon lt / g = C_1 l$ (or $=C_1 x$)
From Earlier in the class:

\[ dW(Q, g) = Fdg + VdQ \]

\[ W(Q, g) = \frac{Q^2g}{2EA} \]

Senturia has

\[ F = \frac{dW(Q, g)}{dg} \bigg|_Q = \frac{d}{dg} \left( \frac{Q^2g}{2EA} \right) \bigg|_Q = \frac{Q^2}{2EA} \]

(as before)

\[ V = \frac{dW(Q, g)}{dQ} \bigg|_g = \frac{d}{dQ} \left( \frac{Q^2g}{2EA} \right) \bigg|_g = \frac{Qg}{EA} = \frac{Q}{2} \]

(as expected)

**Familiar:**

\[ F(e) = -\frac{dW}{dz} \]

for gravity

\[ W(z) = mgz \]

\[ F(e) = -mg \]

But why is the force in the +z direction??

**Ans.:** Because \( dW \) was written with \( F \) meaning the external force applied to put energy into the capacitor. (The electrostatic force is in the opposite direction.)
From Earlier in the class:

\[ F = \frac{Q^2}{2EA} \]
\[ z = \frac{F}{k} \quad \rightarrow \quad \text{(sentura's steps?)} \]
\[ g = g_0 - z \]
\[ g = g_0 - \frac{Q^3}{2EAk} \]

\[ (g = 0 \text{ when } g = \frac{Q^2}{2EAk} ) \]
\[ (\text{when } Q = 2EAk g) \]

\[ V = \frac{Qg}{EA} = \frac{Q \left( g_0 - \frac{Q^2}{2EAk} \right)}{EA} \]

Note: no instability - see voltage control next!
\[ F = \frac{\partial w(Q, g)}{\partial g} \bigg|_Q = \frac{\partial}{\partial g} \left( \frac{Q^2 g}{2EA} \right) \bigg|_Q = \frac{Q^2}{2EA} \]

(as before)

\[ V = \frac{\partial w(Q, g)}{\partial \varphi} \bigg|_g = \frac{\partial}{\partial \varphi} \left( \frac{Q^2 g}{2EA} \right) \bigg|_g = \frac{Qg}{EA} = \frac{Q}{C} \]

Charge control

\[ w(Q, \ell) = \frac{Q^2 g}{2EA} = \frac{Q^2 g}{2\ell e} \]

\[ F = \frac{\partial w(Q, g)}{\partial \ell} \bigg|_Q = \frac{\partial}{\partial \ell} \left( \frac{Q^2 g}{2\ell e} \right) = -\frac{Q^2 g}{2\ell e^2} \]
From Earlier in the class:

\[ Q = \left. \frac{dW^*(V, g)}{dV} \right|_g \]

\[ F = -\left. \frac{dW^*(V, g)}{dg} \right|_V \]

Now, \( W^*(V, g) = \int_0^V Q dV = \int_0^V \frac{EA}{g} V dV \)

\( W^*(V, g) = \frac{EA V^2}{2g} \quad (= \frac{1}{2} CV^2) \)

\[ F = -\left. \frac{dW^*(V, g)}{dg} \right|_V = \frac{EA V^2}{2g^2} \]
From Earlier in the class:

\[ F = \frac{EA V_{in}^2}{2g^2} \]

\[ g = g_0 - z \]

\[ z = \frac{F}{k} \]

\[ g = g_0 - \frac{EA V_{in}^2}{2k g^2} \]

Electrical force

Spring force

(Cubic equation)

Set equal
\[ Q = -\left. \frac{dw^*(V, g)}{dV} \right|_g \]

\[ F = -\left. \frac{dw^*(V, g)}{dg} \right|_V \]

but for voltage control:

\[ w^*(V, e) = \int_0^V Q \, dV = \int_0^V \frac{e t e V^2}{2g} \, V \, dV \]

\[ w^*(V, e) = \frac{e t e V^2}{2g} \left( \frac{1}{2} \right) (V^2) \]

\[ F = -\left. \frac{dw^*(V, e)}{de} \right|_V = -\frac{e t e V^2}{2g} \quad (\text{Constant in } e!) \]
\[ F = -\frac{\epsilon e V^2}{2g} \]

\[ e = \frac{E}{k} \]

\[ e = -\frac{\epsilon e V^2}{2g K} \]

(well-behaved, no instability)
Revisit

\[ F = \frac{dW(Q, g)}{dg} \bigg|_Q = \frac{d}{dg} \left( \frac{Q^2g}{2EA} \right) \bigg|_Q = \frac{Q^2}{2EA} \]

(as before)

\[ V = \frac{dW(Q, g)}{dQ} \bigg|_g = \frac{d}{dQ} \left( \frac{Q^2g}{2EA} \right) \bigg|_g = \frac{Qg}{EA} = \frac{Q}{C} \]

Charge control

\[ W(Q, \ell) = \frac{Q^2g}{2EA} = \frac{Q^2g}{2\ell e} \]

\[ F = \frac{dW(Q, g)}{d\ell} \bigg|_Q = \frac{d}{d\ell} \left( \frac{Q^2g}{2\ell e} \right) = -\frac{Q^2g}{2\ell e^2} \quad ??? \]

Non-ideal effects?

Folded Flexure Spring
Figure 6.9. A complete electrostatic actuator, with added elements representing the inertia of the moveable element, mechanical damping, and the source resistance of the electrical network.
Dynamics: A realistic system:

\[ Q = I = \frac{1}{R} (V_{in} - V_c) = \frac{1}{R} \left( V_{in} - \frac{Q}{C} \right) = \frac{1}{R} \left( V_{in} - \frac{Q_g}{EA} \right) \]

\[ F = \frac{Q^2}{2EA} \]

(can't just use C - must include motion explicitly)
\[
\dot{Q} = I = \frac{1}{R} (V_{\text{in}} - V_c) = \frac{1}{R} (V_{\text{in}} - \frac{Q}{C}) = \frac{1}{R} (V_{\text{in}} - \frac{Q g}{EA})
\]

\[
F = \frac{Q^2}{2EA}
\]

\[
\frac{Q^2}{2EA} + b \dot{g} + m \ddot{g} + k(g - g_0) = 0
\]

In state variable form, identify 3 state variables

\[
x_1 = Q
\]
\[
x_2 = g
\]
\[
x_3 = \dot{g}
\]

\[
\dot{x}_1 = \frac{1}{R} \left( V_{\text{in}} - \frac{x_1 x_2}{EA} \right)
\]
\[
\dot{x}_2 = x_3
\]
\[
\dot{x}_3 = -\frac{1}{m} \left( \frac{x_1^2}{2EA} + k(x_2 - g_0) + b x_3 \right)
\]

Direct integration (Simulink, MATLAB ... H.W.)

Linearization about operating point
Linearization about operating point.

\[ f(x) \]

\[ s f(x) = \left. \frac{df(x)}{dx} \right|_{x_0} s x \quad \text{for small } s \]

So if we study small changes in \( x \) (small-signal analysis, EE terms) the equation is linear. In EE we have a bias point (DC) and often a small AC signal.
Linearization about an operating point.

In this case the operating point will be some \( g_0, \varphi_0 \), and we will look at operation over a small range of \( g \) and \( \varphi \) around \( g_0, \varphi_0 \).

generally:

\[
\begin{align*}
\dot{x}(t) &= X_0 + \delta x(t) \\
\dot{u}(t) &= U_0 + \delta u(t)
\end{align*}
\]

\[
\dot{x} = f(x, u)
\]
Senturia: \[ x(t) = X_0 + \int x(t) \]

\[ x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix} \]

\[ \dot{x} = f(x, u) \]

\[ \dot{x} = f(x, u) = \left( \frac{\partial f}{\partial x} \right) \delta x(t) + \left( \frac{\partial f}{\partial u} \right) \delta u(t) \]

\[ \text{Jacobian, } n \times n, n = \# \text{ of state variables, } n \times m, m = \# \text{ of inputs} \]

\[ \Rightarrow \text{proceed to linearize set of three state equations} \]
\[
\dot{x}(t) = \left( \frac{df}{dx} \right)_{x_0, u_0} x(t) + \left( \frac{df}{du} \right)_{x_0, u_0} u(t)
\]

\[
\text{\{ Jacobian \}} \quad n \times m \\
\text{(h = \# of state variables)} \quad (m = \# of inputs)
\]

\[
\begin{pmatrix}
\dot{x}_1 \\
\vdots \\
\dot{x}_n
\end{pmatrix} =
\begin{pmatrix}
\frac{df_1}{dx_1} & \cdots & \frac{df_1}{dx_n} \\
\vdots & \ddots & \vdots \\
\frac{df_n}{dx_1} & \cdots & \frac{df_n}{dx_n}
\end{pmatrix}
\begin{pmatrix}
x_1 \\
\vdots \\
x_n
\end{pmatrix}
\begin{pmatrix}
X_0 \\
\vdots \\
X_0
\end{pmatrix}
+ \\
\begin{pmatrix}
\frac{df_1}{du_1} & \cdots & \frac{df_1}{du_m} \\
\vdots & \ddots & \vdots \\
\frac{df_m}{du_1} & \cdots & \frac{df_m}{du_m}
\end{pmatrix}
\begin{pmatrix}
u_1 \\
\vdots \\
u_m
\end{pmatrix}
\begin{pmatrix}
X_0 \\
\vdots \\
X_0
\end{pmatrix}
\]
\[ \begin{align*}
X_1 &= \eta \\
X_2 &= j \\
X_3 &= \dot{j}
\end{align*} \]

\[ \begin{align*}
\dot{X}_1 &= \frac{1}{R} V_{in} - \frac{X_1 X_2}{EAR} \\
\dot{X}_2 &= X_3 \\
\dot{X}_3 &= -\frac{X_1^2}{2 m \varepsilon A} - \frac{V}{m} (X_2 - \eta) - \frac{b}{m} X_3
\end{align*} \]

For operating point, \( \dot{X}_1 = 0 \) \( \dot{X}_2 = 0 \) \( \dot{X}_3 = 0 \) solve equations!
This is now in the form of a linear problem.

How to proceed - find operating point

\{-
\quad \text{Find Jacobians}
\quad \text{Use to check stability of operating point}
\quad \text{Analyze system}
\}\n
Apply to Electrostatic actuator.

\begin{align*}
\dot{x}_1 &= 0 \\
\dot{x}_2 &= g \\
\dot{x}_3 &= g
\end{align*}

\[ J = \begin{pmatrix}
-\frac{X_2}{RE A} & -\frac{X_1}{RE A} & 0 \\
0 & 0 & 1 \\
-\frac{X_2}{mE A} & -\frac{k}{m} & -\frac{b}{m}
\end{pmatrix} + \begin{pmatrix}
\frac{1}{R} \\
0 \\
0
\end{pmatrix} \]
Notes:

7.3.3

\[ J = \begin{pmatrix}
    -\frac{X_2}{RE_A} & -\frac{X_1}{RE_A} & 0 \\
    0 & 0 & 1 \\
    -\frac{X_1}{RE_T} - \frac{k}{m} - \frac{b}{m} & 0 & 0 \\
\end{pmatrix} \]

For state variables

For input (voltage source)

\[ \uparrow \text{need this to obtain 7.58} \]

Eq. 7.60 \[ X_3 = -\frac{1}{200} X_1^2 - x_2 + 1 - 0.5X_3 \]

Note: error in sentencia (my printing)
\[ \delta X_1 = \delta Q = \frac{X_2}{REA} \delta x_1 - \frac{X_1}{REA} \delta x_2 + \frac{1}{R} \delta V_n \]

\[ \delta y = \delta X_2 = \delta X_3 \]

\[ \delta y = \delta X_3 = \frac{X_1}{MEA} \delta x_1 - \frac{k}{m} \delta x_2 - \frac{b}{m} \delta x_3 \]

- \( X_2 \) - operating point gap
- \( X_1 \) - operating point chuga
Using Senturia's values

Area \( A \) = 100
Permittivity \( \varepsilon \) = 1
Initial gap \( g_0 \) = 1
Mass \( m \) = 1
Damping constant \( b \) = 0.5
Spring constant \( k \) = 1
Resistance \( R \) = 0.001

\[
\begin{align*}
\dot{X}_1 &= 1000 V_{\text{in}} - 10 X_1 X_2 \\
\dot{X}_2 &= X_3 \\
\dot{X}_3 &= -\frac{1}{200} X_1^2 - X_2 + 1
\end{align*}
\]

\( X_2 = 0 = X_3 = g \) (velocity = 0)

\[
\begin{align*}
0 &= 1000 V_{\text{in}} - 10 X_1 X_2 \\
0 &= -\frac{1}{200} X_1^2 - X_2 + 1
\end{align*}
\]

\[
\begin{cases}
X_1^2 - 200X_1 + 20000V_0 = 0 \rightarrow \text{solve for } X_1,
\hline
X_2 = 1 - \frac{1}{200} X_1^2
\end{cases}
\]
Can model as a linear transducer. (To be inserted in a variety of systems.)

Transforms voltage to source.

Find operating point. \( I = 0 \Rightarrow V = V_{in} \)

\( U = 0 \)
From before, \( g = g_0 - \frac{F}{4} \)

\[ F = \frac{\varepsilon A V^2}{2 g^2} \]

\[ g = g_0 - \frac{\varepsilon A V^2}{24 g^2} \]

\[ V = \frac{Q_0}{EA} \]

\[ F = \frac{Q^2}{2EA} \]

\[ F_{out} = \frac{Q^2}{2EA} - k (g_0 - g) \]
operating point gap \neq g_0 which is the zero voltage gap.

\[
\begin{pmatrix}
  \delta V \\
  \delta I
\end{pmatrix} = \begin{pmatrix}
  \frac{g_0}{E_A} & \frac{Q_0}{E_A} \\
  \frac{Q_0}{E_A} & K
\end{pmatrix} \begin{pmatrix}
  \delta Q \\
  \delta g
\end{pmatrix}
\]

\[\delta f(x) = \frac{df(x)}{dx} \bigg|_{x_0} \delta x\]
\[\delta f(x,y) = \frac{df(x,y)}{dx} \bigg|_{x_0,y_0} \delta x + \frac{df(x,y)}{dy} \bigg|_{x_0,y_0} \delta y\]

need \( SI, SQ \)

\[SQ = \int SI \, dt \]
\[\downarrow \text{s-plane of the Laplace transform} \]
\[SQ = \frac{SI}{s}\]

\[\delta g = \int SD \, dt\]
\[\delta g = \frac{SD}{s}\]
\[
(\delta V) = \left(\frac{\hat{g}_0}{\text{SEA}} \quad \frac{Q_0}{\text{SEA}} \quad \frac{k}{5}\right) \cdot (\delta I)
\]

Identify elements

\[Z_{EB} = \frac{\hat{g}_0}{\text{SEA}} \quad \text{capacitor} \quad Z_c = \frac{1}{5C}\]

\[Z_{MO} = \frac{k}{5} \quad \text{"capacitance" of spring}\]

\[T_{EM} = T_{ME} = \frac{Q_0}{\text{SEA}} \quad \phi = \frac{T_{EM}}{Z_{EB}} = \frac{Q_0}{\hat{g}_0} \quad \text{Newtons/meter (Coulombs/volt)}\]
\[ Z_{ms} = Z_{m0} (1 - K_e^2) \]

\[ K_e^2 = \frac{T_{EM}^2}{Z_{EE} Z_{m0}} = \frac{Q_{o0}^2}{EA k_0} \]

\[ Z_{ms} = \frac{k}{s} \left( 1 - \frac{Q_{o0}^2}{EA k_0} \right) \quad \text{spring, spring constant decreasing} \]

\[ \text{(spring softening)} \]

---

Direct Integration - see SIMULINK example.
- must use for large variations.
- MATLAB
- MathCad
- Matumath