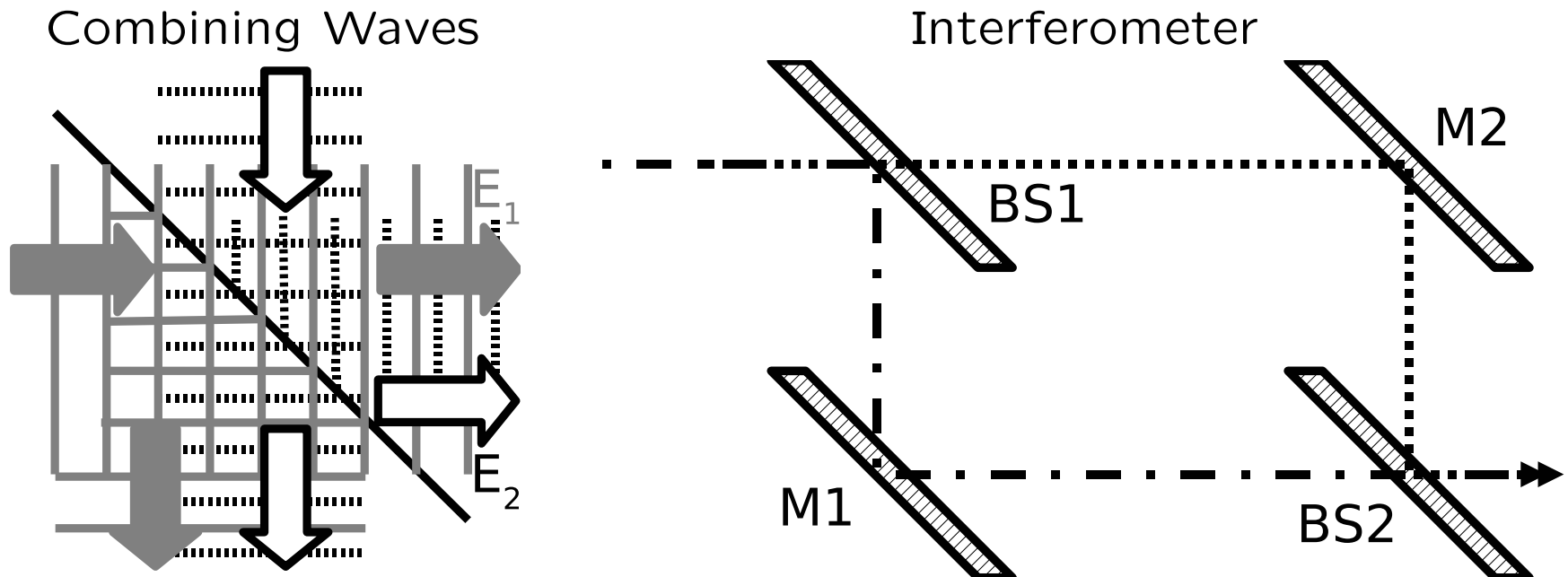# Chapter 7

Optics for Engineers

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Sept. 2012

Coherent Superposition

Combining Waves

E_1

E_2

Interferometer

M2

BS1

M1

BS2

- Output Toward the Right

$$E = E_1 + E_2$$

- Similar for Output Toward the Bottom
- Normally Common Source for Coherence

Chapter Overview

- Mach–Zehnder Interferometer

  – Alignment

  – Balanced Mixing

- Coherent Laser Radar (Lidar)

  – Sign Ambiguities

  – Quadrature Detection

- Michelson Interferometer

- Fabry–Perot Interferometer

  – Laser Longitudinal–Mode Selection

  – Laser Stabilization

- Beamsplitters

- Thin–Film Coatings (AR, Max–R, and More)

Measuring the Field Amplitude Is Hard

- Easy for Ocean Wave Height
- Easy for Acoustic Pressure
- Even Easy for Radio Waves
- No Direct Measurement for Light
  - Terahertz Frequencies
  - Sub–Micrometer Wavelengths
- Use Interferometry
  - Mix With Known Reference Wave
  - Measure Irradiance
  - Variations in Space or Time
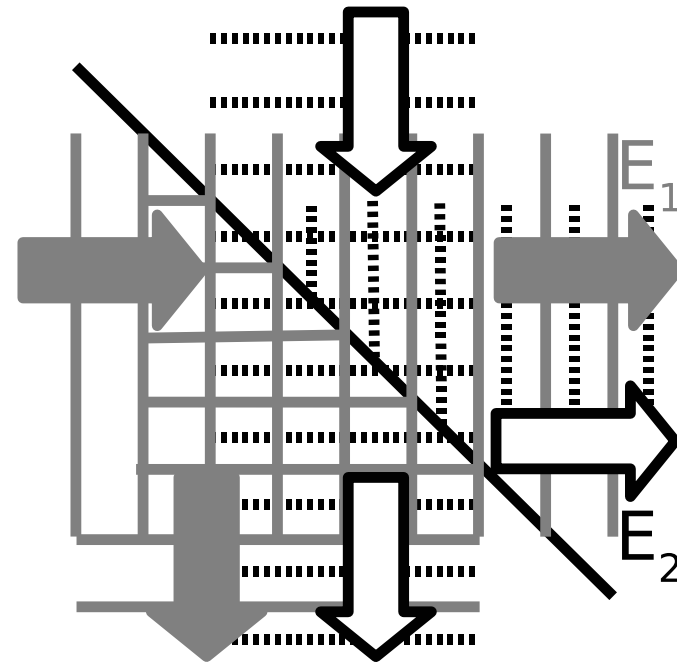
$$I = \frac{|E|^2}{Z} = \frac{EE^*}{Z}$$

$$I = \frac{\left(E_1^* + E_2^*\right)\left(E_1 + E_2\right)}{Z}$$

Interferometry Equations

- Irradiance

$$I = \frac{\left(E_1^* + E_2^*\right)\left(E_1 + E_2\right)}{Z}$$

- Expand

$$I = \frac{E_1^* E_1 + E_2^* E_2 + E_1^* E_2 + E_1 E_2^*}{Z}$$

 – First Two Terms are "DC" Terms
 – Third and Fourth are "Mixing" Terms
 – Complex Conjugate Pair (Real Sum)

$$I_{mix} = \frac{E_1 E_2^*}{Z} \qquad \text{and} \qquad I_{mix}^* = \frac{E_1^* E_2}{Z}$$

Mixing Terms

- Complex Conjugates Add to Real Value

$$I_{mix} = \frac{E_1 E_2^*}{Z} \qquad \text{and} \qquad I_{mix}^* = \frac{E_1^* E_2}{Z}$$
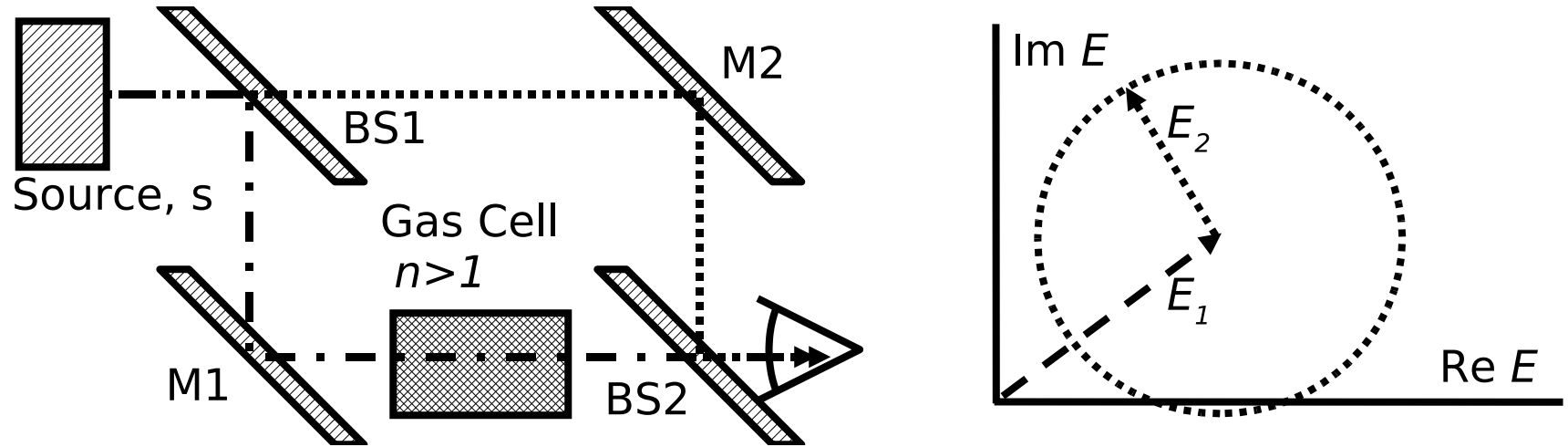
- Magnitude

$$|I_{mix}| = |I_{mix}^*| = \sqrt{I_1 I_2}$$

- Total Irradiance

$$I = I_1 + I_2 + 2\sqrt{I_1 I_2} \cos\left(\phi_2 - \phi_1\right)$$

Mach–Zehnder Interferometer

$$I = I_1 + I_2 + 2\sqrt{I_1 I_2} \cos\left(\phi_2 - \phi_1\right)$$

- Add Gas Pressure to Cell; $n \uparrow$, $\Delta = \Delta OPL \uparrow$

$$\Delta = \delta\left(n\ell_c\right)$$

$$\delta\phi_1 = k\Delta = 2\pi\frac{\Delta}{\lambda} = 2\pi\frac{\ell_c}{\lambda}\delta n$$

Interpreting the Interference

- Phase Change

$$\delta\phi_1 = k\Delta = 2\pi\frac{\Delta}{\lambda} = 2\pi\frac{\ell_c}{\lambda}\delta n$$

- Beginning at $\phi_2 - \phi_1 = 0$

$$\frac{dI}{d\left(n\ell_c\right)} = 2\sqrt{I_1 I_2}\left[-\sin\left(\phi_2 - \phi_1\right)\right]\frac{2\pi}{\lambda} = 0$$

- Quadrature Point $\phi_2 - \phi_1 = \pi/2$

$$\frac{dI}{d\left(n\ell_c\right)} = \frac{2\pi}{\lambda}I \qquad \delta\left(n\ell_c\right) = \frac{\lambda}{2\pi}\frac{dI}{I}$$

- Very Sensitive Phase Measurement
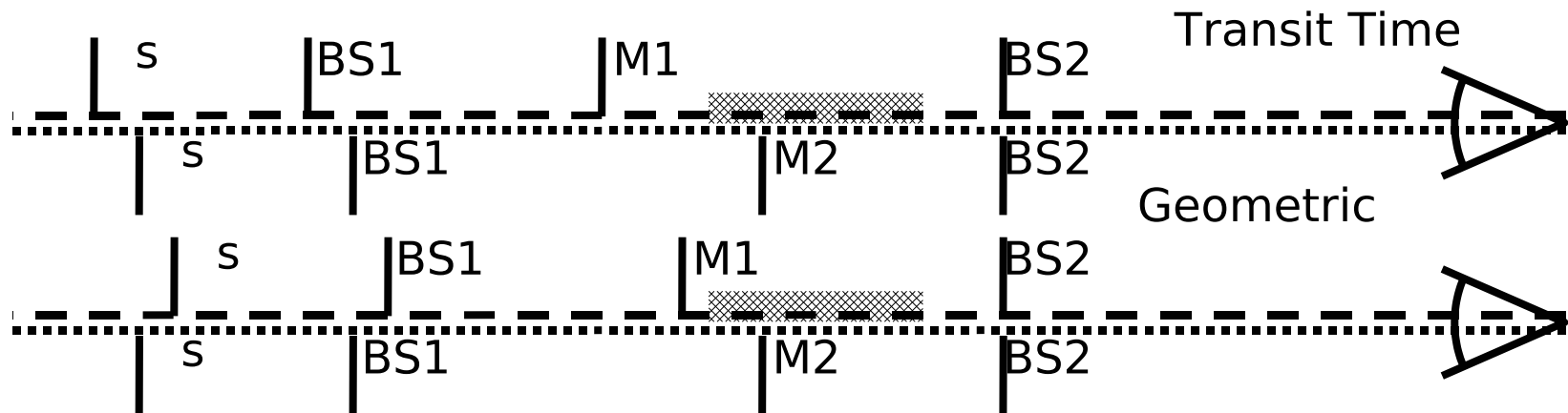 - For $\lambda = 500\text{nm}$ and $\ell_c = 1\text{cm}$, $d\phi/dn = 1.25 \times 10^5$
  - For $\delta n = 10^{-6}$, $\delta I = 1.25 \times 10^5 \times I \times \delta n = 0.125 \times I$. (12%)
- Potential Problem: Maintaining the Quadrature Point

OPL and Geometric Path

- Straight–Line Layout

- OPL Mismatch (Time Difference: Loss of Coherence: Ch. 10)

$$\ell = \ell_c n$$

- Geometric–Path Mismatch (Curvature Mismatch: Dense Fringes)
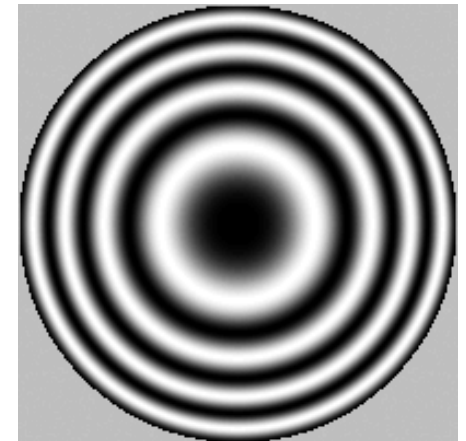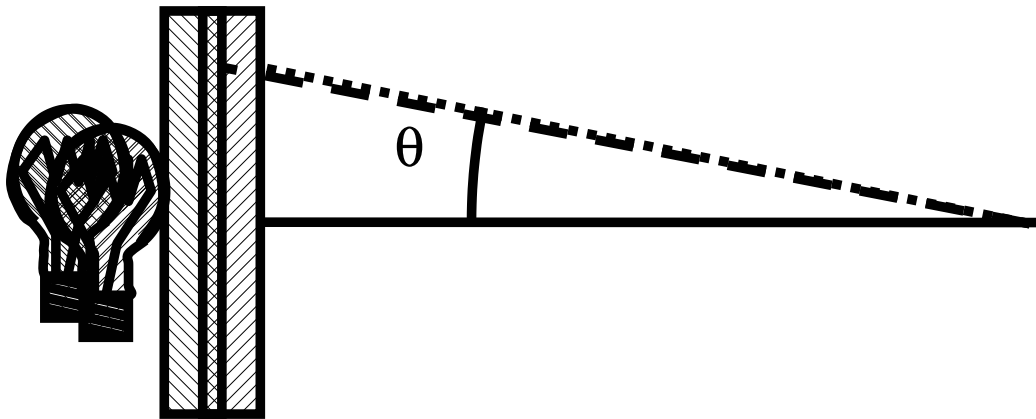
$$\ell_{geom} = \frac{\ell_c}{n}$$

Mach–Zehnder and Extended Source

- Distance from Axis at Observer to Point in Source

$$r = \sqrt{x^2 + y^2 + z^2}$$

- Change in z Adds Δ to Optical Path: $r \to \Delta + r$

$$\Delta = \frac{\ell}{\cos\theta} = \ell\sqrt{\frac{x^2 + y^2 + z^2}{z^2}}$$

Paraxial Approximation

- For $x \ll z$ and $y \ll z$ Taylor's Series for Square Root

$$\sqrt{\frac{x^2 + y^2 + z^2}{z^2}} \approx 1 + \frac{1}{2}\frac{x^2 + y^2}{z^2},$$

- Simplify

$$r \approx z + \frac{x^2 + y^2}{2z}$$

$$\phi_1 = kr = \frac{2\pi}{\lambda}r \approx \frac{2\pi z}{\lambda} + 2\pi\frac{x^2 + y^2}{2\lambda z} \qquad \frac{x^2 + y^2}{2\lambda z} \ll 1$$

Mach–Zehnder and Extended Source Revisited

- Equation for Extended Source

$$\Delta = \frac{\ell}{\cos\theta}$$

$$= \ell\sqrt{\frac{x^2 + y^2 + z^2}{z^2}}$$

- Paraxial Approximation

$$\delta\phi = \phi_2 - \phi_1$$

$$\delta\phi = k\Delta = \frac{2\pi\Delta}{\lambda} =$$

$$\frac{2\pi\ell}{\lambda}\left(1 + \frac{1}{2}\frac{x^2 + y^2}{z^2}\right)$$

- Irradiance for Each Path Separately

$$I_1 = I_0 R_1 T_2$$

$$I_2 = I_0 T_1 R_2$$

- Fields

$$E_1 = E_0 \rho_1 \tau_2 e^{jknz_1}$$

$$E_2 = E_0 \tau_1 \rho_2 e^{jknz_2}$$

- Interference Pattern

$$I = I_0 \left(R_1 T_2 + T_1 R_2 + \cdots \right.$$

$$\left. 2\sqrt{R_1 T_2 T_1 R_2} \cos\delta\phi \right)$$

Fringe Pattern for Mach–Zehnder

$$I = I_0 \left(R_1 T_2 + T_1 R_2 + 2\sqrt{R_1 T_2 T_1 R_2} \cos \delta\phi \right)$$
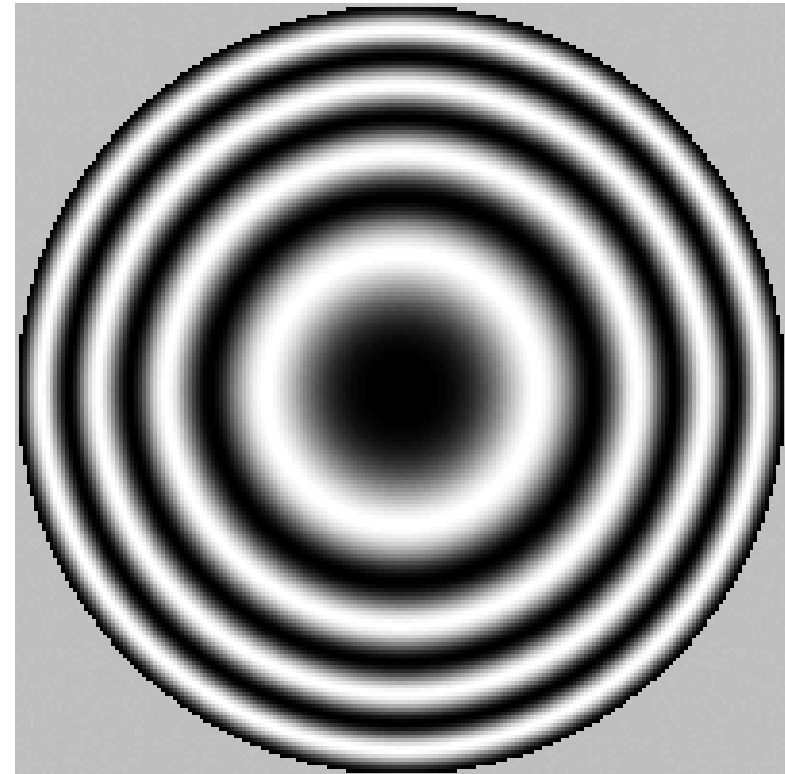
- Fringe Count

$$N = \left(\frac{\rho}{z}\right)^2 \frac{\delta\ell}{\lambda}$$

- Invert

$$\ell = \lambda \left(\frac{z}{\rho}\right)^2 N$$

- Sensitivity

$$\frac{d\ell}{\ell} = \frac{dN}{N}$$

Fringe Amplitude and Contrast

- Total Signal

$$I = I_0 \left(R_1 T_2 + T_1 R_2 + 2\sqrt{R_1 T_2 T_1 R_2} \cos \delta\phi \right)$$

- Fringe Amplitude

$$I_m = I_{max} - I_{min}$$

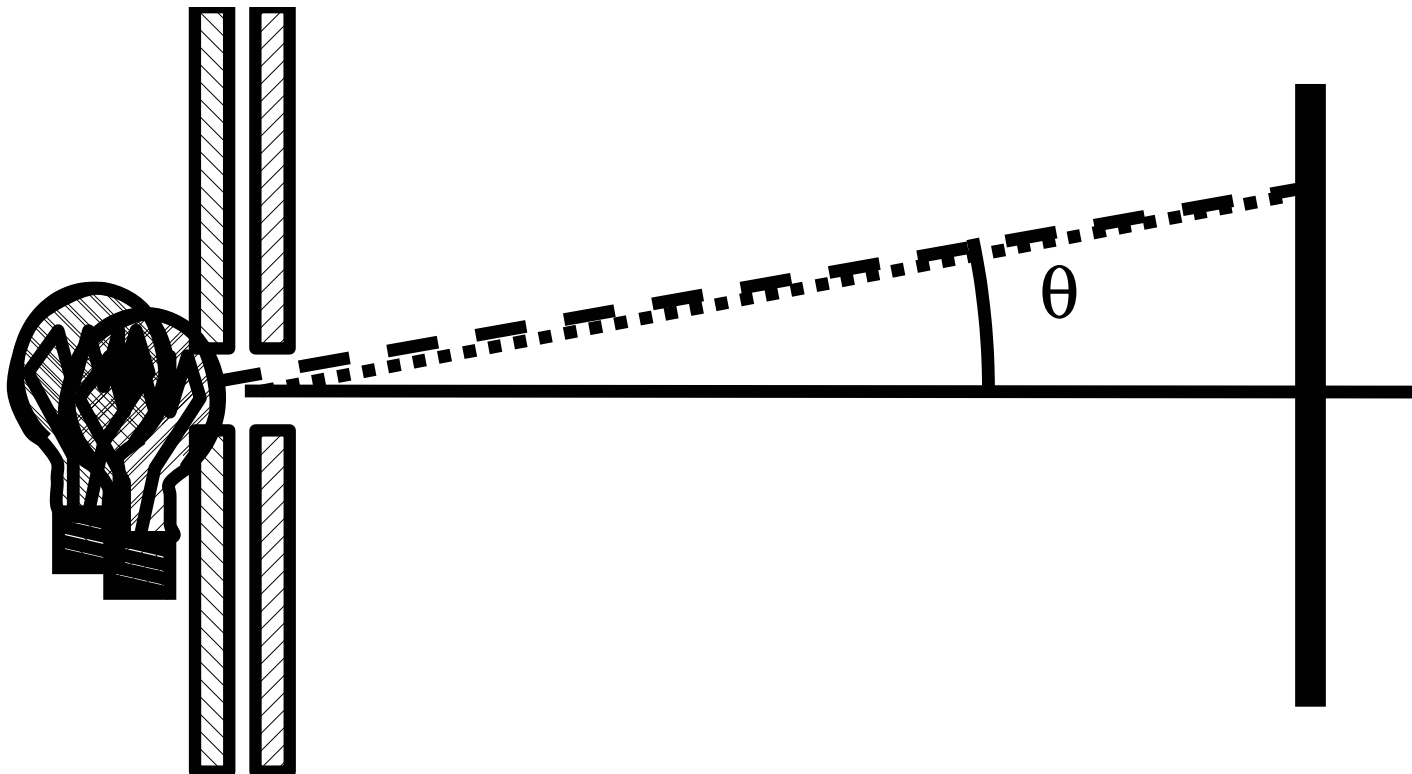$$\sqrt{R_1 T_1 R_2 T_2} I_0 = 0.25 I_0 \qquad \text{for} R_1 = R_2 = 0.5$$

- Fringe Contrast Defined

$$V = \frac{I_{max} - I_{min}}{I_{max} + I_{min}} \qquad (0 \le V \le 1) \qquad V = 2\frac{\sqrt{R_1 T_2 T_1 R_2}}{R_1 T_2 + T_1 R_2}$$

- For $R_1 = R_2 = R$ and $(T_1 = T_2 = T)$

$$I = I_0 2RT \left(1 + \cos \delta\phi \right)$$

Mach–Zehnder with Point Source

- Same Path Length Equations as Extended Source
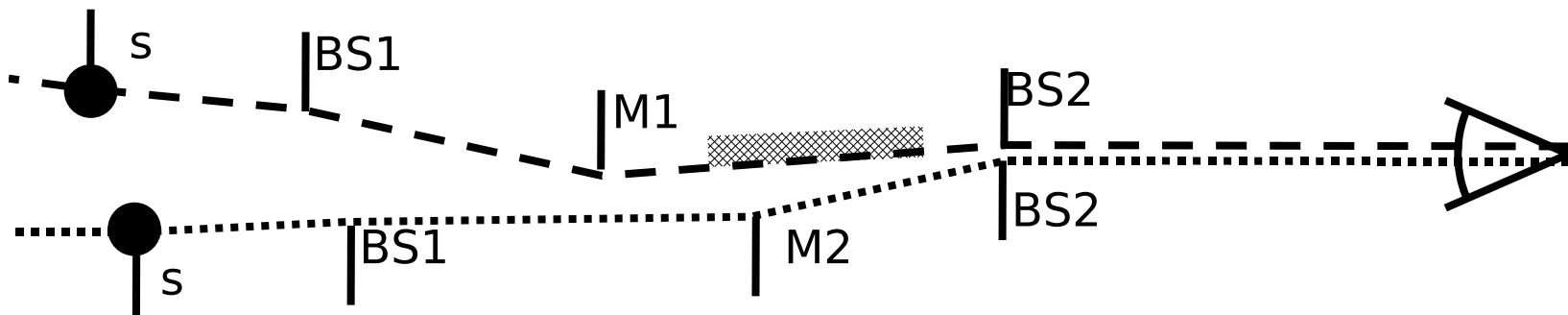
- Same Fringe Pattern

Alignment (1)

- Goal is to Match the Two Point–Source Locations
- Match at Two Points to Make Beams Coaxial
- Small Angular Error Produces Straight Fringes
  - Separation

$$d_{fringe} = \frac{\lambda}{\delta\theta}$$

  - Example: $\lambda = 500$nm with One Fringe over $d = 1$cm

$$\delta\theta = \frac{\lambda}{d} = \frac{500 \times 10^{-9}\text{m}}{0.01\text{m}} = 5 \times 10^{-5}\text{radians}$$

Alignment (2)

- Starting Point
  - Stable Environment (Floating Table?) for Interferometer
  - Adjustments in the Right Places (See Drawing, Previous Page)

- Transverse Alignment
  - Bring Point Sources Close Together
  - Iterate at Two Distances
  - Get it Close Enough to Make Fringe Spacing Visible

- Axial Alignment
  - OPL Mismatch May Lead to Loss of Coherence (Ch. 10)
  - Geometric Mismatch Makes Fringes too Close to See

- Fine Tuning
  - Adjust Position and Angle to Achieve Largest Fringes

Balanced Mixing (1)
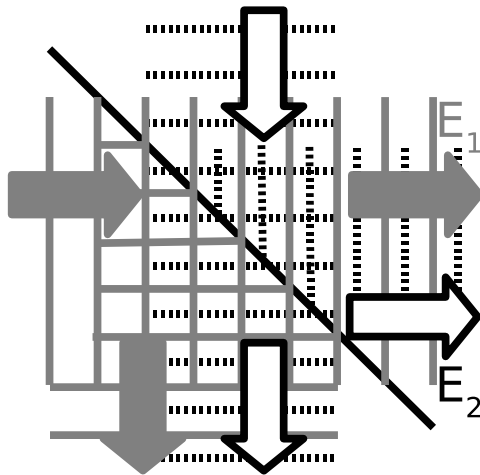
- Right Interference Pattern

$$I = I_0 \left(R_1 T_2 + T_1 R_2 + 2\sqrt{R_1 T_2 T_1 R_2} \cos \delta\phi \right)$$

- Down (Alternate) Intereference Pattern

$$I_{alt} = I_0 \left(T_1 T_2 + R_1 R_2 - 2\sqrt{T_1 T_2 R_1 R_2} \cos \delta\phi \right)$$

- Conservation of Energy Explains the Minus Sign

$$I_{total} = I + I_{alt} = \ldots$$

$$I_0 \left(R_1 T_2 + T_1 R_2 + T_1 T_2 + R_1 R_2 \right) =$$

$$I_0 \left[R_1 \left(1 - R_2 \right) + \left(1 - R_1 \right) R_2 + \ldots \right.$$

$$\left. \left(1 - R_1 \right) \left(1 - R_2 \right) + R_1 R_2 \right] = I_0$$

- Right Output

$$E_1 = E_0 \rho_1 \tau_2 e^{jknz_1}$$

$$E_2 = E_0 \tau_1 \rho_2 e^{jknz_2}$$

- Down Output

$$E_{1alt} = E_0 \rho_1 \rho_2 e^{jknz_1}$$

$$E_{2alt} = E_0 \tau_1 \tau_2 e^{jknz_2}$$

E_1

E_2

Balanced Mixing (2)

- Subtract Two Outputs

$$I_{dif} = I - I_{alt} = \ldots$$

$$I_0 \left[R_1 \left(T_2 - R_2 \right) + T_1 \left(R_2 - T_2 \right) + 4\sqrt{T_1 T_2 R_1 R_2} \cos \delta\phi \right]$$

- Lossless Beamsplitters $(T + R = 1)$

$$I_{dif} = I_0 \left[\left(1 - 2R_1 \right) \left(1 - 2R_2 \right) - 4\sqrt{\left(1 - R_1 \right) \left(1 - R_2 \right) R_1 R_2} \cos \delta\phi \right]$$

- One 50/50 Beamsplitter $(R_1 = 50\%$ or $R_2 = 50\%)$

$$I_{dif} = I_0 \cos \delta\phi$$

Balanced Mixing Comparison

- Setup: $P_0 = 1\text{mW}$, $\delta P_0 = 1\mu\text{W}$ (Noise), $P = IA$,
  $\delta\phi$ at Quadrature Point $\pm 1\mu$Rad, $R_1 = 0.01$, $R_2 = 0.5$

- Right Output Alone

$$P = P_0 \left(R_1 T_2 + T_1 R_2 + 2\sqrt{R_1 T_2 T_1 R_2} \cos\delta\phi \right)$$

  - Signal: $2P_0\sqrt{R_1 T_2 T_1 R_2} = 99\text{pW}$
  - DC: $P_0 \left(R_1 T_2 + T_1 R_2 \right) = 500\mu\text{W}$
  - Noise (on DC): $\delta P_0 \left(R_1 T_2 + T_1 R_2 \right) = 500\text{nW}$

- Balanced Mixing

$$I_{dif} = I_0 \left[\left(1 - 2R_1\right)\left(1 - 2R_2\right) - 4\sqrt{\left(1 - R_1\right)\left(1 - R_2\right) R_1 R_2} \cos\delta\phi \right]$$

  - Signal: $P_0 \times 4\sqrt{\left(1 - R_1\right)\left(1 - R_2\right) R_1 R_2} = 200\text{pW}$
  - DC: Zero for Perfect Balance (May Need to Adjust Gain)
  - Noise (On Mix): $\delta P_0 \times 4\sqrt{\left(1 - R_1\right)\left(1 - R_2\right) R_1 R_2} = 200\text{fW}$

Coherent Laser Radar (Lidar)

- Similar to Mach–Zehnder (Modified Mach–Zehnder?)

- Common Transmit/Receive Aperture: Use T/R Switch (Ch. 6)
  - Transmitter Polarization: P at Beamsplitters
  - Receiver Polarization: S: Need HWP in Reference (LO)
  - Ideally QWP Between Telescope and Target to Reduce Narcissus (Not Practical)

- BS1 and Recombining Beamsplitter High Reflectivity
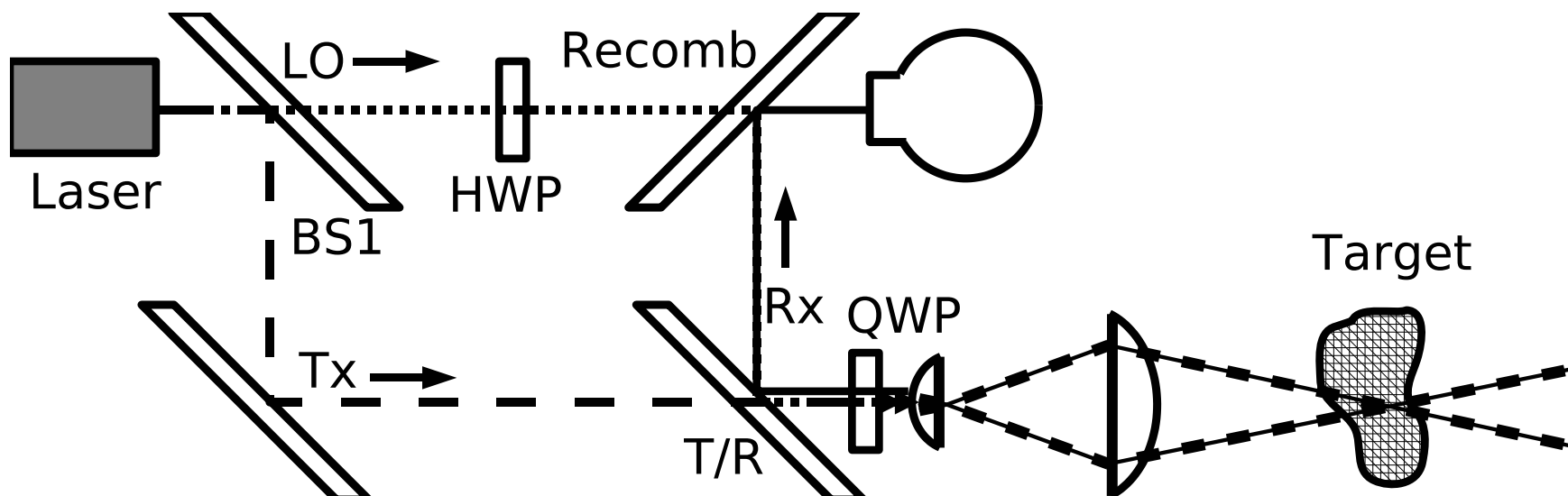
Laser Radar Equations

- Reference or Local Oscillator (LO) Power (mW Typ.)

$$P_{LO} = P_0 T_{BS1} T_{Recomb}.$$

- Power to Target (W or More Typ.)

$$P_t = P_0 R_{BS1} T_{T/R}$$

- Power to Detector (F Accounts for Target and Geometry)

$$P_{sig} = P_t F R_{T/R} R_{Recomb} \qquad \text{(fW or More)}$$

$$F = \frac{\pi D^2}{4z^2} \times \rho(\pi) \times 10^{-2\alpha z/10}$$

- Mixing Signal: nW or More

Mixing Efficiency

- Mixing Power Detected

$$P_{mix} = \int I_{mix} dA = \int \frac{E_{Sig}\left(x_d, y_d\right) E_{LO}^*\left(x_d, y_d\right)}{Z} dA$$

- Mixing Efficiency Defined

$$P_{mix} = \eta_{mix}\sqrt{P_{sig}P_{LO}}$$

$$\eta_{mix} = \int \frac{E_{sig}\left(x_d, y_d\right) E_{LO}^*\left(x_d, y_d\right)}{\sqrt{\int E_{sig}\left(x_d, y_d\right) E_{sig}^*\left(x_d, y_d\right) dA \times \int E_{LO}\left(x_d, y_d\right) E_{LO}^*\left(x_d, y_d\right) dA}} dA$$

- Matched LO and Signal: $\eta_{mix} = 1$
- Losses for Different Shape, Random Scatter, *etc.*

Doppler Velocity

- Target: Dust, Fo, Rain, Snow, Smoke, *etc.*

- Doppler Equation from Source to Target

$$2\pi f_d = \mathbf{k} \cdot \mathbf{v} \qquad (> 0 \text{ for Approaching Velocities})$$

$$f_d = \frac{v_{parallel}}{\lambda} \qquad (\text{Moving Source or Detector})$$

- Doppler Lidar (or Radar) on Round Trip

$$f_{DR} = 2\frac{v_{parallel}}{\lambda}$$

$$(f_{DR} = 100\text{kHz for } v_{parallel} = 0.54\text{m/s at } \lambda = 10.59\mu\text{m})$$

Ranging Lidar

- Ranging by Focusing (Like a Confocal Microscope)
  - In Focus (High Mixing Efficiency)

$$I_{mix} = \frac{E_{sig}E_{LO}^*}{Z}$$

  - Out of Focus (Low Mixing Efficiency)

$$E_{sig} = \left| E_{sig} \right| e^{jk\sqrt{x^2+y^2+s'^2}}$$

  - Range Resolution Lmited by Aperture Size
  - Limited to Short Range by Diffraction (Ch. 8, 9)
- Ranging by Transit Time (Pulsed Lidar)
  - Round–Trip Time $2R/c$
  - Limted to Longer Ranges
  - Range Resolution Interacts with Velocity Resolution

Pulsed Laser Radar

- Range Resolution and Velocity Resolution

$$\delta r = \frac{c\tau}{2} \qquad \delta f_{DR} \approx \frac{1}{\tau} \qquad \delta v_{parallel} \approx \frac{\lambda}{2\tau}$$

- Hundreds of Meters and m/s Typical at 10.59μm
- Average Power

$$P_{avg} = P_{laser} \times \tau \times PRF$$

- Range Ambiguity May Limit PRF
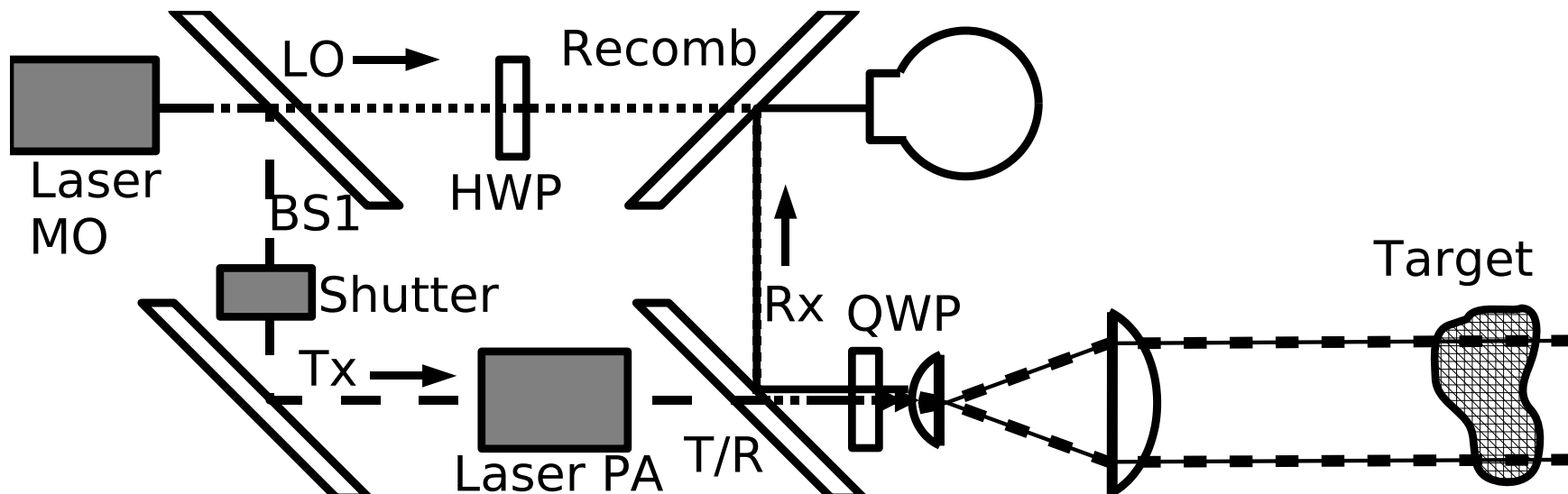
Ambiguities in Interferometry

- Variations in Amplitude and Phase May Be Confused
  - Important for Phase Measurement
  - But Lidar Signal Goes through Many Cycles
  - Solutions: Phase–Shifting and Quadrature
- Quadrature Point is Hard to Maintain
  - Important for Measuring Phase Shifts
  - Not Needed for Doppler Lidar
  - Solutions: Phase–Shifting and Quadrature
- Cosine is an Even Function
  - Particularly Significant for Doppler Lidar
  - Sign of Velocity Component is Lost
  - Solutions: Offset LO and Quadrature
- Cosine is a Periodic Function
  - Important for Absolute Distance Measurement
  - Not Important for Velocity Measurement
  - Solutions: Multiple Wavelengths, Counting

Frequency Offset

- Shift the Reference Frequency
  - Shift by More than Maximum Doppler
  - Negative Shift f_o In Figures
- Zero Doppler Corresponds to RF Frequency
- Positive or Negative Doppler Shifts Measurable
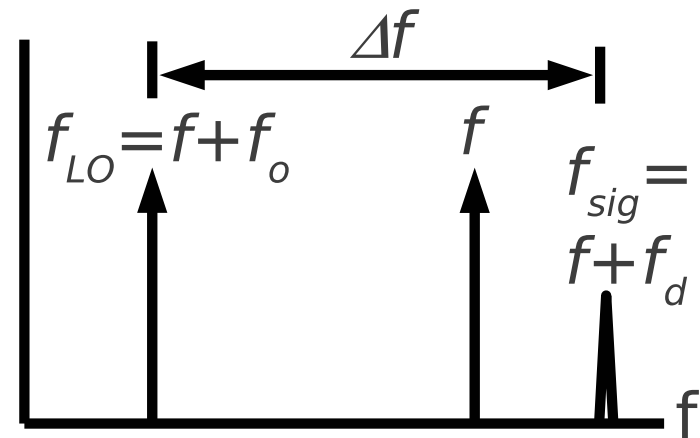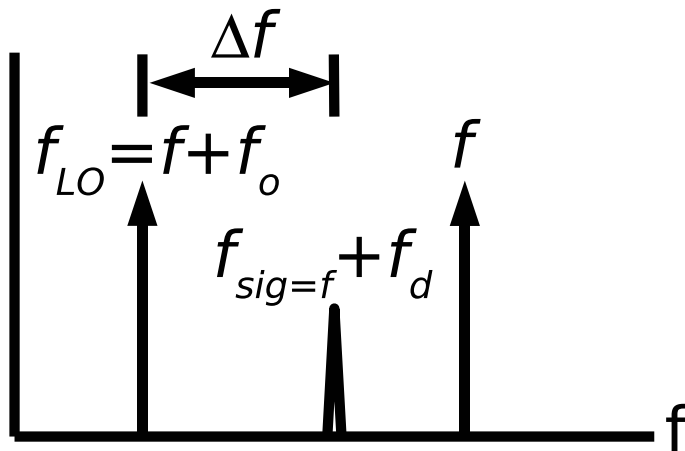
Negative Doppler Shift, $f_d < 0$ Positive Doppler Shift, $0 < f_d$

Right Figure Updated

Quadrature Detection

- Reference Wave Circularly Polarized
  (sin and cos Components)

- Signal Wave Linearly Polarized at 45° (Equal Components)

- Two Detectors (or Four with Balanced Mixing)
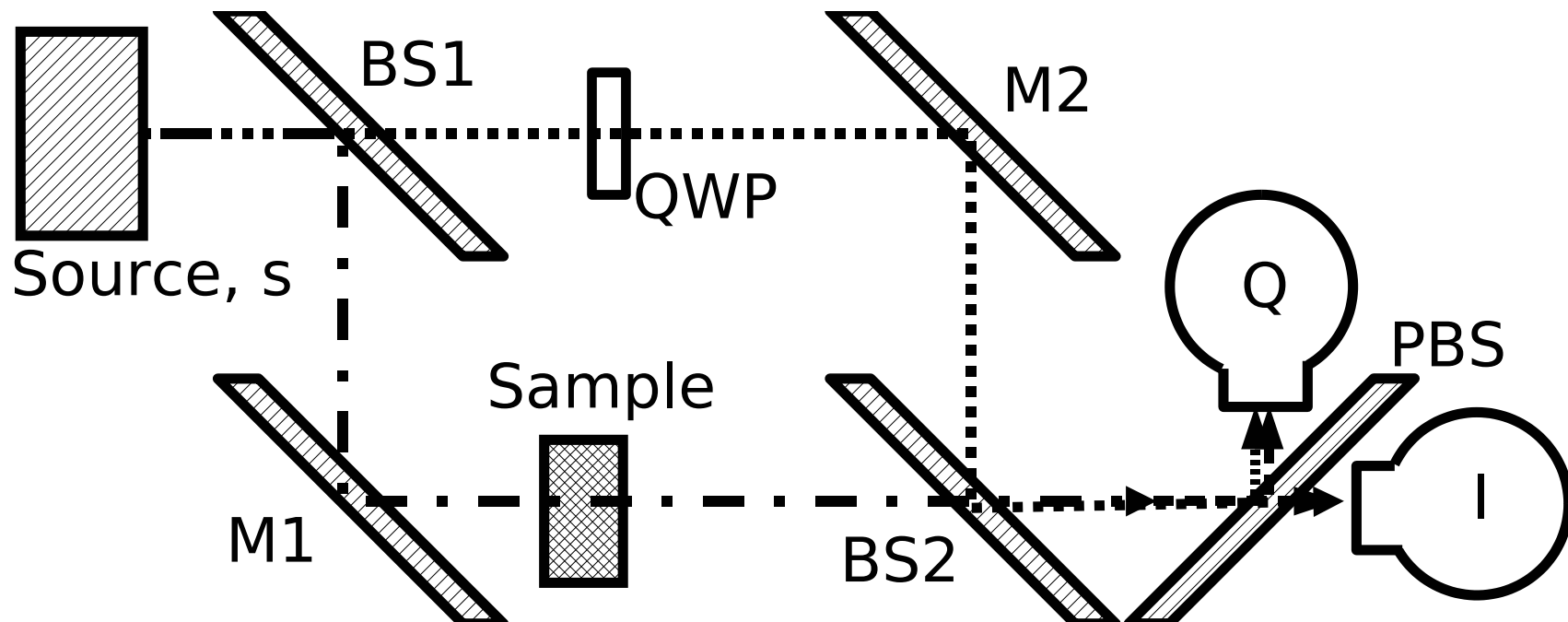
Quadrature Equations (1)

- Jones Vector Sum

$$\mathbf{E} = \mathbf{E}_{sig} + \mathbf{E}_{LO}$$

- Unit Laser Input at 45°

$$\mathbf{E}_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \times \frac{1}{\sqrt{2}}$$

- Reference or LO

$$E_{LO} = \mathcal{R}_2 \mathcal{Q} \mathcal{T}_1 \mathbf{E}_0$$

$$\mathcal{Q} = \begin{pmatrix} 1 & 0 \\ 0 & j \end{pmatrix}$$

- Signal

$$E_{sig} = \mathcal{T}_2 \mathcal{J}_{sample} \mathcal{R}_1 \mathbf{E}_0$$

- At "I" Detector

$$\mathbf{E}_I = \mathcal{P}_x \left(\mathbf{E}_{sig} + \mathbf{E}_{LO} \right)$$

$$P_I = P_0 \times \ldots$$

$$\left(\mathbf{E}_{sig}^{\dagger} + \mathbf{E}_{LO}^{\dagger} \right) \mathcal{P}_x^{\dagger} \mathcal{P}_x \left(\mathbf{E}_{sig} + \mathbf{E}_{LO} \right)$$

- At "Q" Detector

$$\mathbf{E}_Q = \mathcal{P}_y \left(\mathbf{E}_{sig} + \mathbf{E}_{LO} \right)$$

$$P_Q = P_0 \times \ldots$$

$$\left(\mathbf{E}_{sig}^{\dagger} + \mathbf{E}_{LO}^{\dagger} \right) \mathcal{P}_y^{\dagger} \mathcal{P}_y \left(\mathbf{E}_{sig} + \mathbf{E}_{LO} \right)$$

Quadrature Equations (2)

- No Depolarization at Target

$$\mathcal{J}_{sample} = Ae^{i\phi}$$

- Beamsplitters

$$\tau_1 = \sqrt{T_1} \qquad \tau_2 = \sqrt{T_2} \qquad \rho_1 = \sqrt{R_1} \qquad \rho_2 = \sqrt{R_2}$$

- "I" Detector Power

$$P_I = A^2 T_2 R_1 P_0/2 + R_2 T_1 P_0/2 + \ldots$$

$$\sqrt{T_2 R_1 R_2 T_1} Ae^{j\phi} P_0/2 + \sqrt{T_2 R_1 R_2 T_1} Ae^{-j\phi} P_0/2$$

- "Q" Detector Power

$$P_Q = A^2 T_2 R_1 P_0/2 + R_2 T_1 P_0/2 - \ldots$$

$$j\sqrt{T_2 R_1 R_2 T_1} Ae^{j\phi} P_0/2 + j\sqrt{T_2 R_1 R_2 T_1} Ae^{-j\phi} P_0/2$$

Quadrature Equations (3)

- Sum Signals From Detector (in Computer)

- 3rd Terms in Previous Eqns. (Complex Mixing Terms) Add

- 4th (Conjugate) Terms Cancel

$$P_I + jP_Q = (1 + j)\left(A^2 T_2 R_1 P_0/2 + R_2 T_1 P_0/2\right) + \sqrt{T_2 R_1 R_2 T_1}\, Ae^{j\phi}P_0$$

- But We're Stuck with the DC Terms Too

  – Need at Least One More Measurement

Quadrature and Balanced Mixing

- Alternate Output Terms

$$P_{-I} = T_2 R_1 P_0/2 + R_2 T_1 P_0/2 - \ldots$$

$$\sqrt{T_2 R_1 R_2 T_1} A e^{j\phi} P_0/2 - \sqrt{T_2 R_1 R_2 T_1} A e^{-j\phi} P_0/2$$

$$P_{-Q} = T_2 R_1 P_0/2 + R_2 T_1 P_0/2 + \ldots$$

$$j\sqrt{T_2 R_1 R_2 T_1} A e^{j\phi} P_0/2 - j\sqrt{T_2 R_1 R_2 T_1} A e^{-j\phi} P_0/2$$

- Complex Signal (DC & Conjugate Terms Cancel)

$$P_I + j P_Q + j^2 P_{-I} + j^3 P_{-Q} = 2\sqrt{T_2 R_1 R_2 T_1} A e^{j\phi} P_0$$

Periodicity Issues

- One Solution: Use Multiple Wavelengths
- Another Solution: Unwrap the Phase

Phase Unwrapping. The object here is a prism with an index of refraction 0.04 above that of the background medium. The left panel shows the phase (solid line), the phase with noise added to the field (dashed line), and the unwrapped phase (dash–dot). The right panel shows the original prism thickness and the reconstructed value obtained from the unwrapped phase.
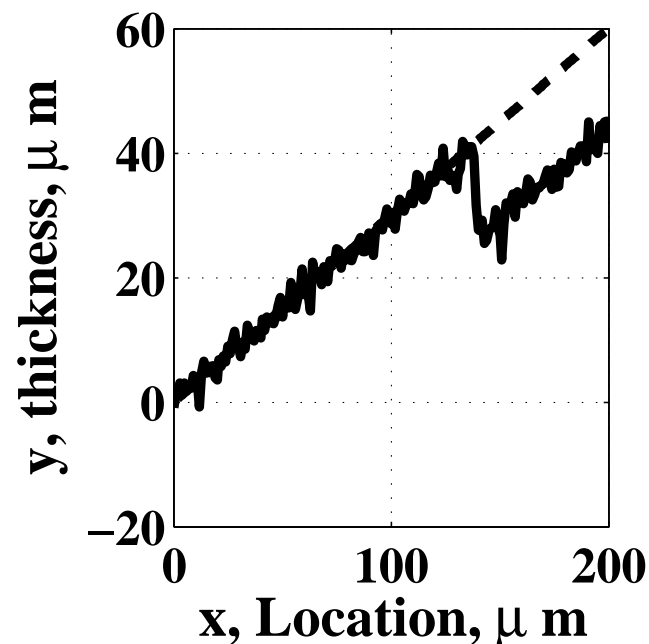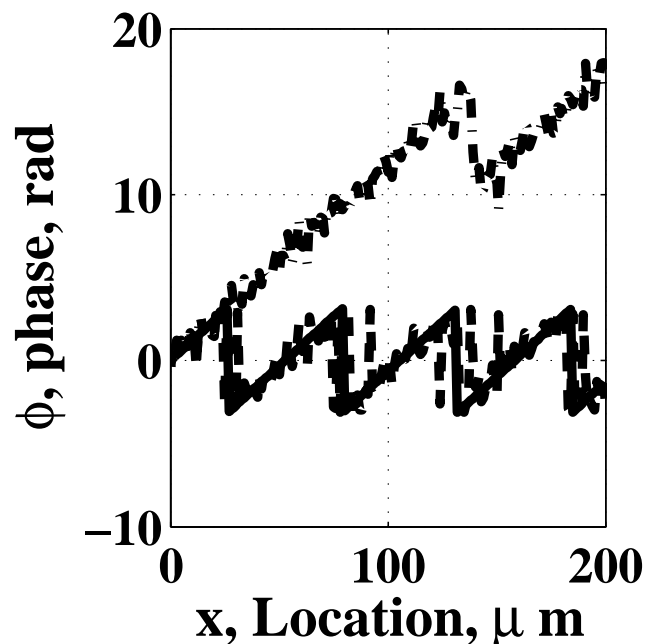
Michelson Interferometer
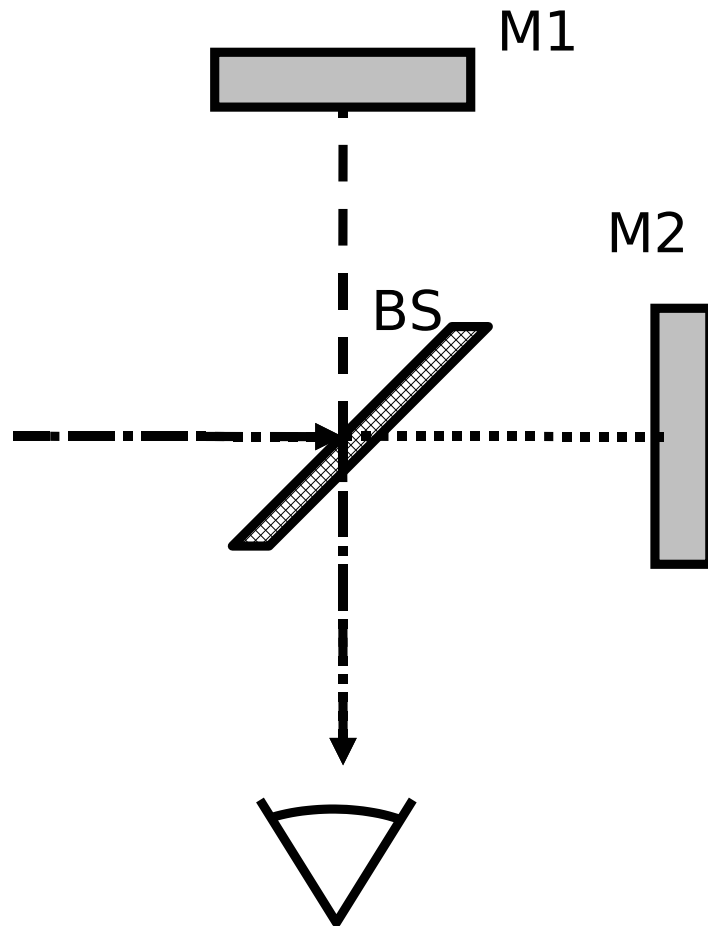
$$I = I_0 \left[R_{BS} R_{M1} T_{BS} + T_{BS} R_{M2} R_{BS} + \right.$$

$$\left. \sqrt{R_{BS}^2 R_{M1} R_{M2} T_{BS}^2} \cos(\delta\phi) \right]$$

M1

M2

BS

Straight−Line Layout

Transit Time

s   BS   M1   BS

s   BS   M2   BS

Geometric Optics

s   BS   M1   BS
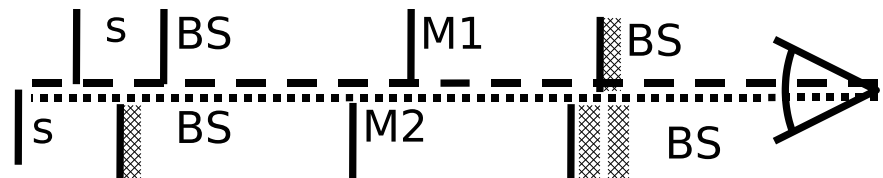
s   BS   M2   BS
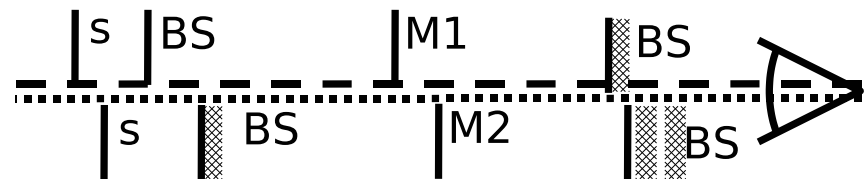
Time Match is Important
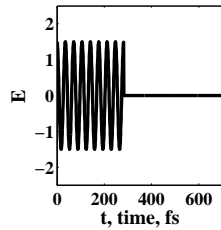
In Phase	Quadrature	Out of Phase	Delayed
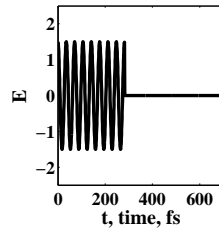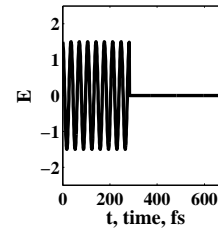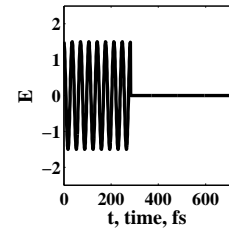
Reference Field   Amplitude $= 1.5$ Energy $= 1.5^2$
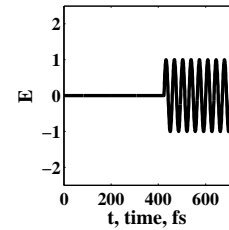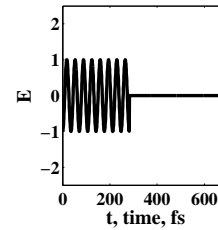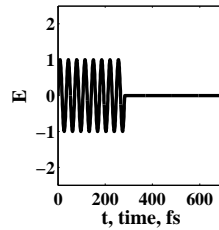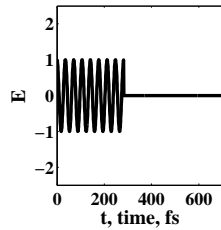
Signal Field   Amplitude $= 1$ Energy $= 1^2$
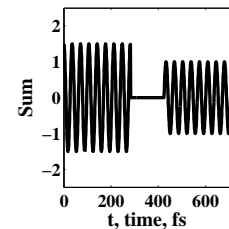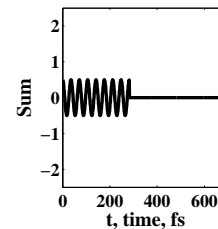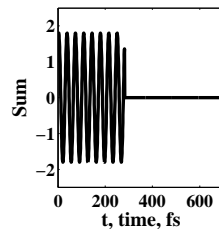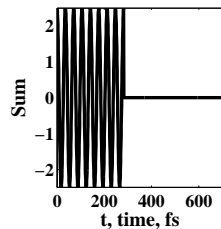
Coherent Sum
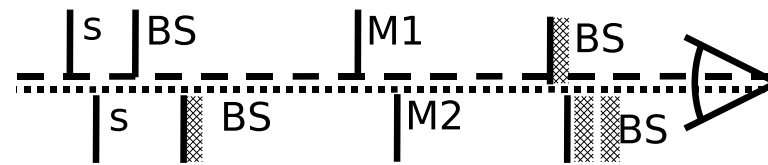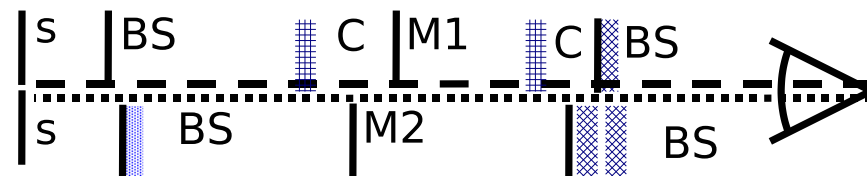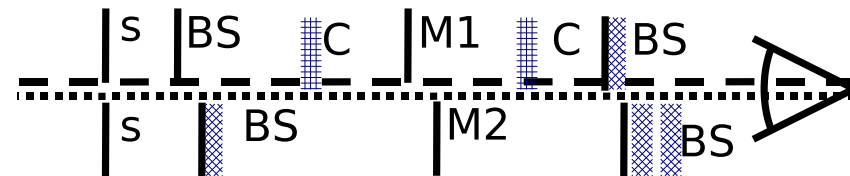Energy 2.5^2 $1.5^2 + 1$ 0.5^2 $1.5^2 + 1$

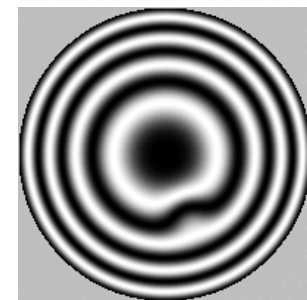# Compensator Plate
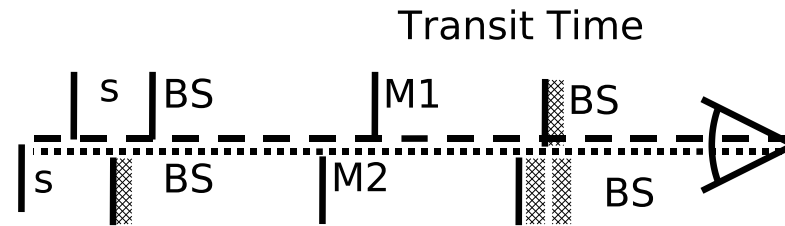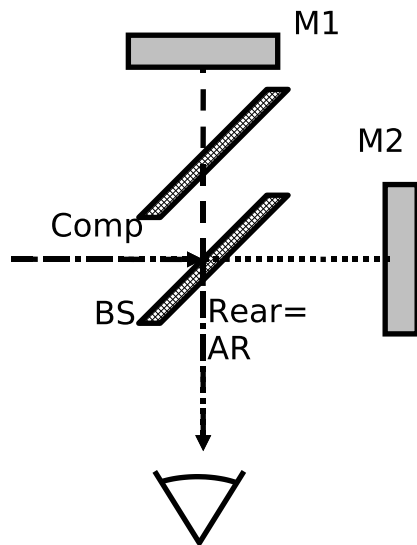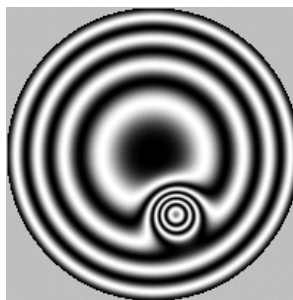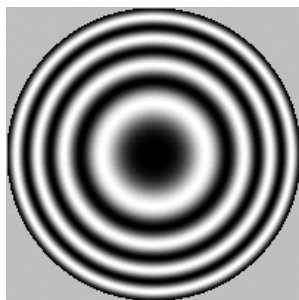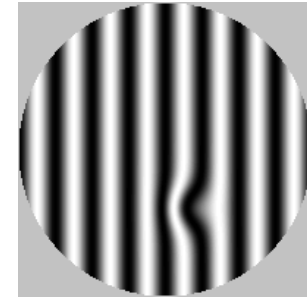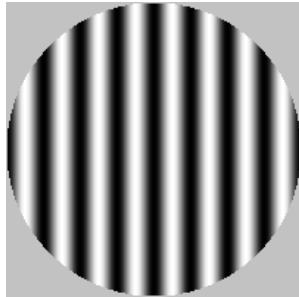

Above Right: Michelson Straight–Line Layout
—

Below Right: with Compensator (Corrects both Time and Geometry)
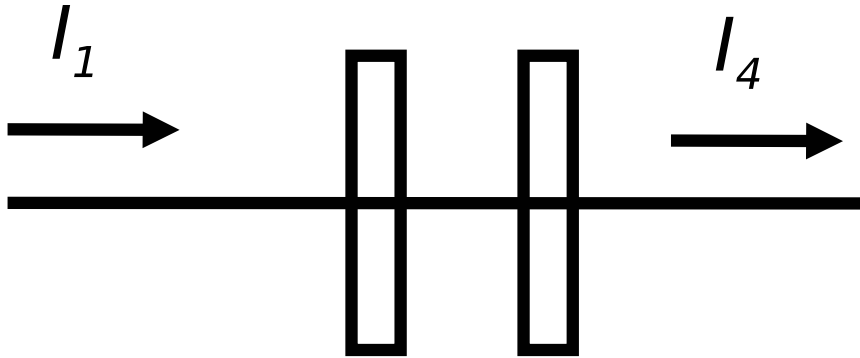
Optical Testing

Synthetic Results Illustrating Fringe Patterns

Perfect Surface   2–Wavelength Bump   0.2–Wavelength Bump
                           (4 Fringes)   (0.4 Fringes)

Fabry–Perot

I_1 →

I_4 →

- Resonant Frequencies and Free Spectral Range

$$f = N f_0$$

$$FSR = f_0 = \frac{c}{2\ell}$$

- Recirculating Power

$$P_{recirculating} = \frac{P_{out}}{T_2} =$$

$$\frac{P_{out}}{1 - R_2} = \frac{P_0}{1 - R_2}$$

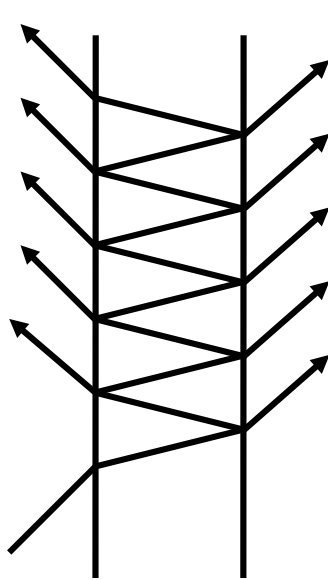- N Round Trips

$$P_N = (R_1 R_2)^N$$

- 50% Probability

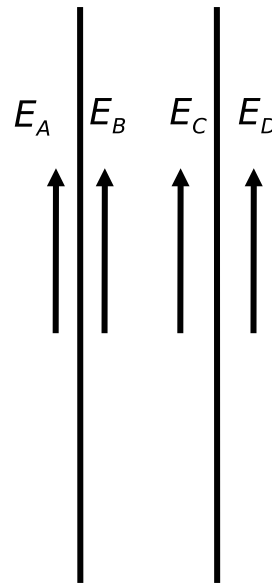$$N = -\log 2 / \log (R_1 R_2)$$

e.g. $R_1 = R_2 = 0.999$:
$N = 346$
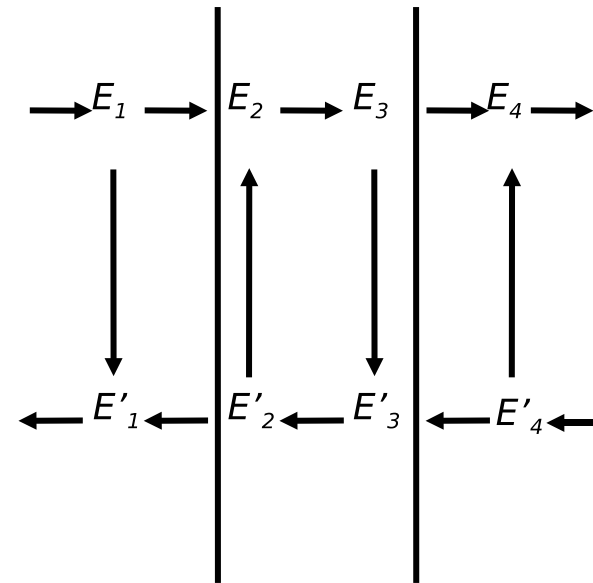
- Resolution of a Longer Interferometer

Fabry–Perot Equations
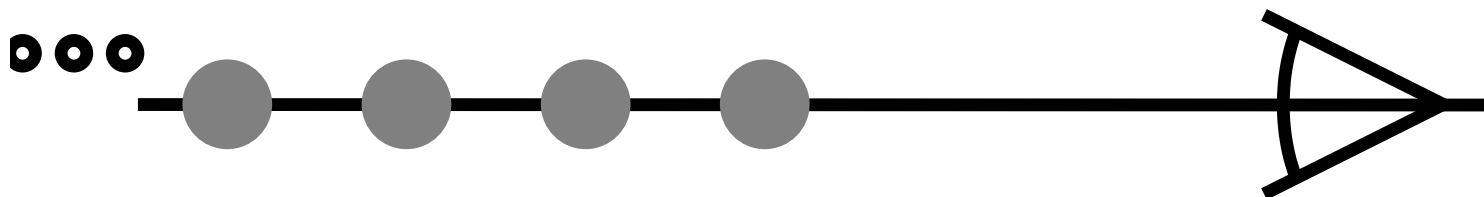

A. Infinite–Sum Solution

B. E/M Fields BC Solution

C. Network Approach with ρ and τ

The Infinite Sum, or "Barber's Chair" Approach

Fabry–Perot Infinite–Sum Equations

Definitions ρ_1 for light incident from the left, and ρ_1'

Plane Waves (Otherwise Include Wavefront Effects)

$$E_t = E_0 \left[\tau_1 e^{jk\ell} \tau_2 + \tau_1 e^{jk\ell} \rho_2 e^{jk\ell} \rho_1' e^{jk\ell} \tau_2 + \ldots \right]$$

$$E_t = E_0 \tau_1 \tau_2 e^{jk\ell} \sum_{m=0}^{\infty} \left(\rho_1' \rho_2 e^{2jk\ell} \right)^m$$

$$E_r = E_0 \left[\rho_1 + \tau_1 e^{jk\ell} \rho_2 e^{jk\ell} \tau_1' + \tau_1 e^{jk\ell} \rho_2 e^{jk\ell} \rho_1' e^{jk\ell} \rho_2 e^{jk\ell} \tau_1' + \ldots \right]$$

$$E_r = E_0 \rho_1 + \tau_1 \tau_1' e^{2jk\ell} \sum_{m=0}^{\infty} \left(\rho_1' \rho_2 e^{2jk\ell} \right)^m$$

$$\text{For } 0 < x < 1, \qquad \sum_{m=0}^{\infty} x^m = \frac{1}{1-x}$$

Computing the Sums

$$E_t = E_0 \tau_1 \tau_2 e^{jk\ell} \frac{1}{1 - \rho_1' \rho_2 e^{2jk\ell}}$$

$$E_r = E_0 \rho_1 + \tau_1 \tau_1' e^{2jk\ell} \frac{1}{1 - \rho_1' \rho_2 e^{2jk\ell}}$$

$$T = \left| \frac{E_t}{E_0} \right|^2 \qquad T = \tau_1 \tau_2 \tau_1^* \tau_2^* \frac{1}{1 - \rho_1' \rho_2 e^{2jk\ell}} \frac{1}{1 - (\rho')_1^* \rho_2^* e^{-2jk\ell}}$$

$$T = T_1 T_2 \frac{1}{1 - 2\sqrt{R_1 R_2} \cos(2k\ell) + R_1 R_2}$$

Including Loss

- Define

$$\ell = n\ell_p$$

- Replace $R_1 R_2$ by ...

$$\eta_{rt} = R_1 R_2 e^{-2\Im nk\ell_p}$$

$$T = T_1 T_2 \frac{1}{1 - 2\sqrt{\eta_{rt}} + 4\sqrt{\eta_{rt}} \sin^2(k\ell) + \eta_{rt}}$$

- Finesse

$$F = \frac{4\sqrt{\eta_{rt}}}{1 - \eta_{rt}}$$

$$T = \frac{T_1 T_2}{R_1 R_2} \frac{\eta_{rt}}{\eta_{rt} - 1} \frac{1}{1 + F\sin^2(k\ell)}$$

Can Include Wavefronts, Alignment, *etc.*
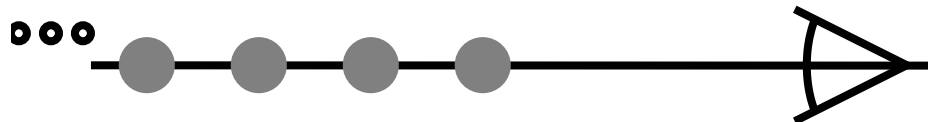
Linewidth

- Maximum Transmission

$$T_{max} = \frac{T_1 T_2}{R_1 R_2} \frac{\eta_{rt}}{\eta_{rt} - 1}$$

- Half of Maximum

$$F \sin^2 (nk\ell) = 1 \qquad k\ell = \frac{2\pi}{\lambda}\ell = \frac{2\pi f}{c}\ell = \arcsin \frac{1}{\sqrt{F}}$$

- FWHM

$$\delta f = 2\frac{c}{2\pi\ell} \arcsin \frac{1}{\sqrt{F}}$$

$$\frac{\delta f}{FSR} = \frac{2}{\pi} \arcsin \frac{1}{\sqrt{F}}$$

Linewidth in Wavelength

- Fractional Linewidth

$$\left|\frac{\delta\lambda}{\lambda}\right| = \left|\frac{\delta f}{f}\right| \qquad \frac{\delta\lambda}{\lambda} = \frac{\lambda}{\ell}\frac{\mathrm{arcsin}\frac{1}{\sqrt{F}}}{\pi}$$
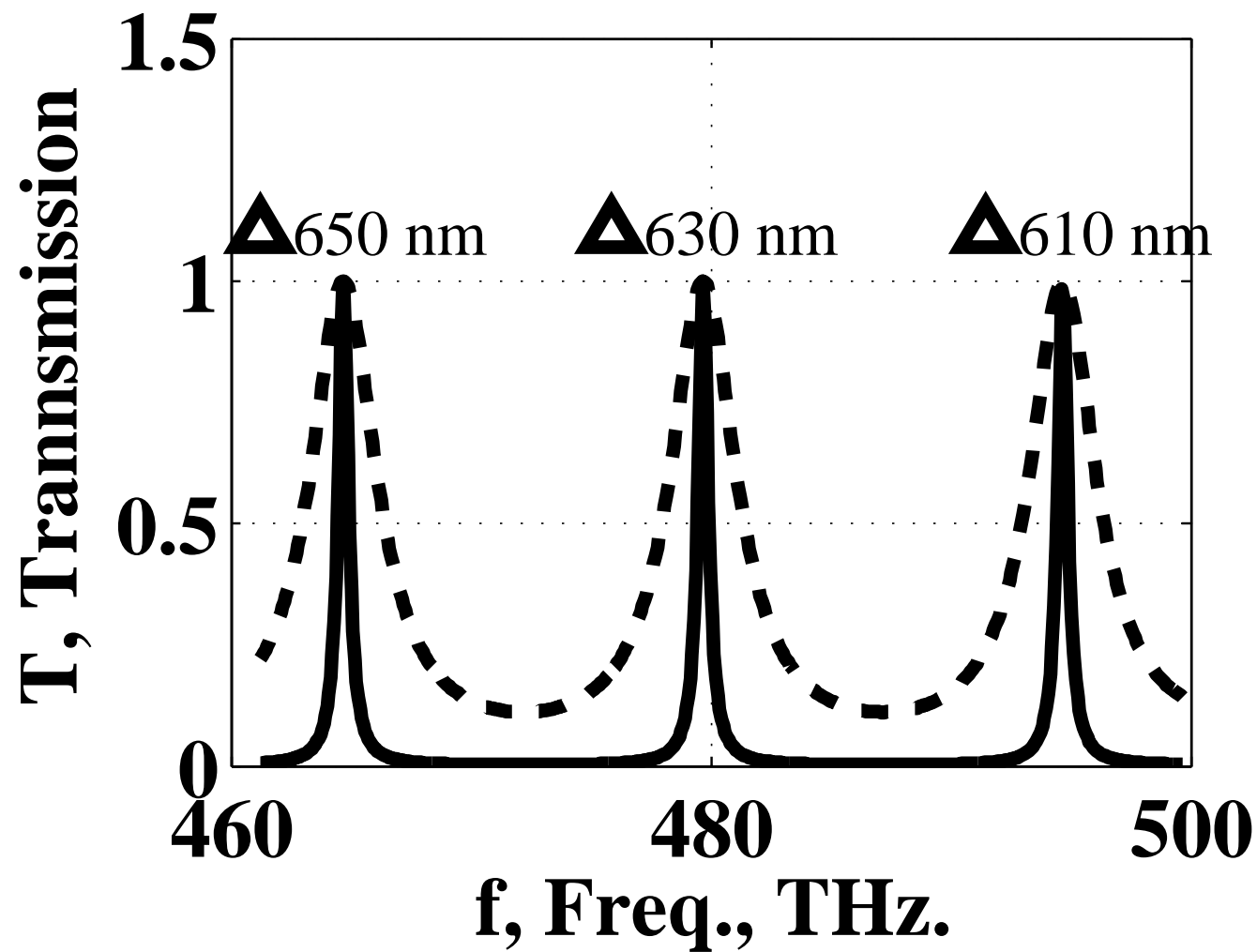
- FWHM

$$\frac{\delta f}{FSR} = \frac{2}{\pi}\arcsin\frac{1}{\sqrt{F}}$$

$$\frac{\delta\lambda}{\lambda} = \frac{\delta f}{f} = \frac{2}{\pi}\frac{FSR}{f}\arcsin\frac{1}{\sqrt{F}}$$

$$\frac{\delta\lambda}{\lambda} = \frac{\delta f}{f} = \frac{2}{\pi}\frac{c}{2\ell f}\arcsin\frac{1}{\sqrt{F}}$$

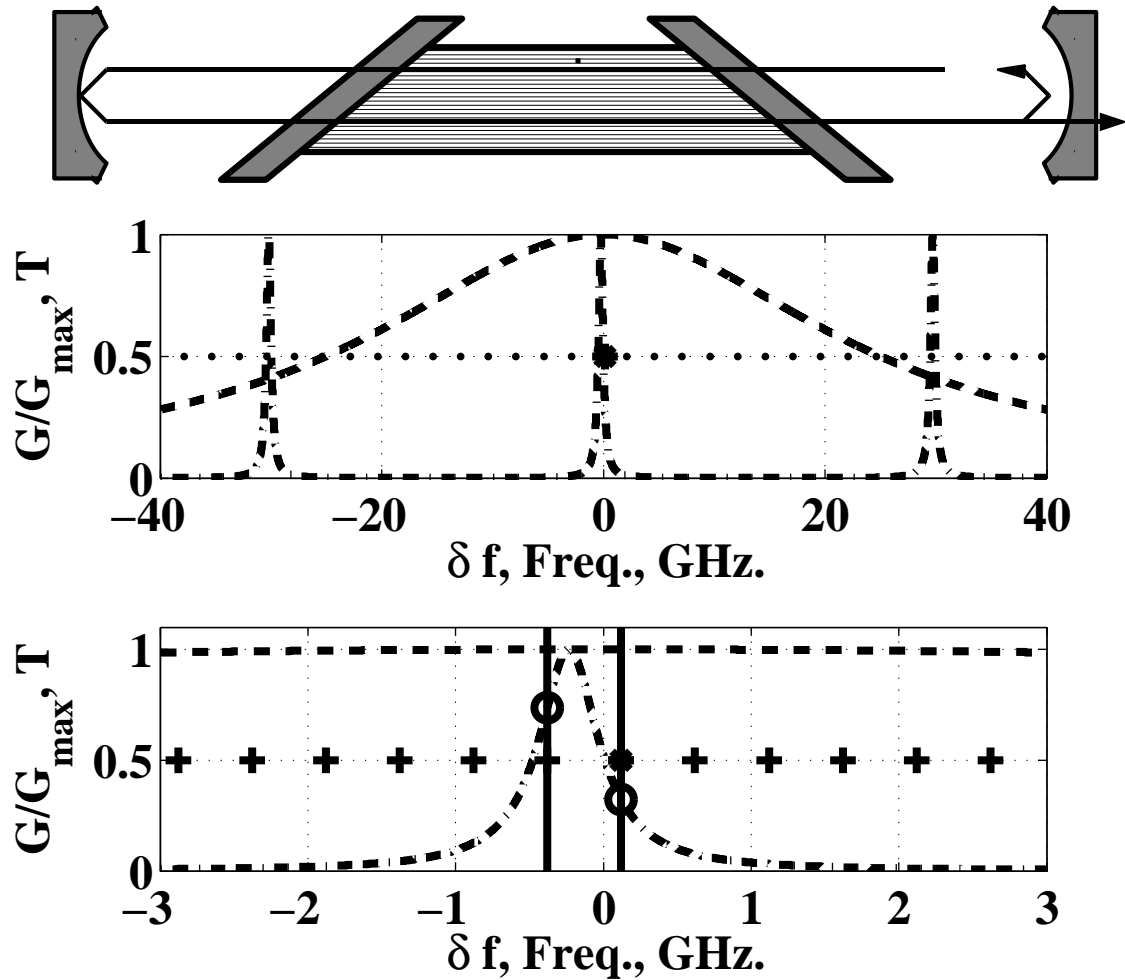$$\frac{\delta\lambda}{\lambda} = \frac{\lambda}{\pi\ell}\arcsin\frac{1}{\sqrt{F}}$$
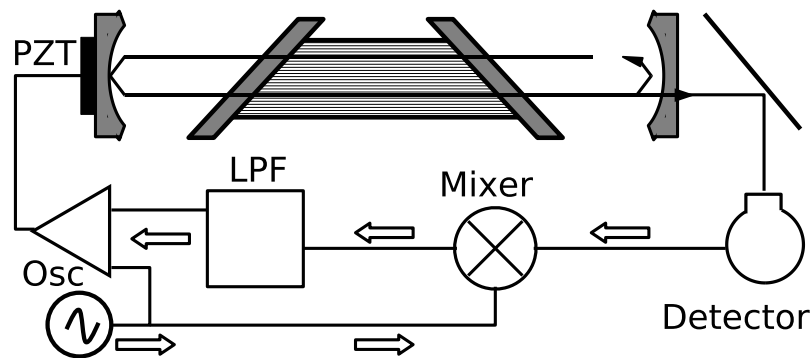
Etalon Transmission

Etalon

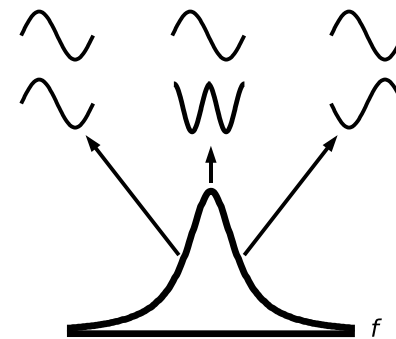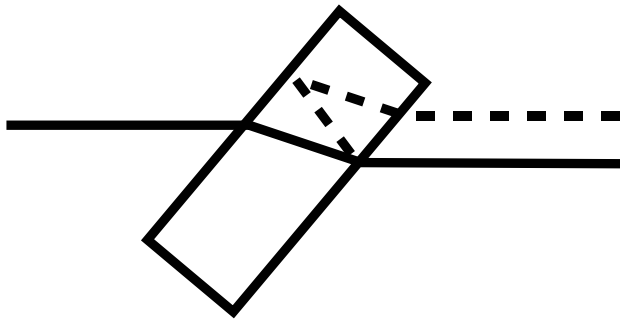Dash is gain, Dot-dash is Etalon, Dots at 0.5 every 5th cavity mode

Stabilization Loop

(A) Circuit Diagram   Signals

Beamsplitter

R_1, %	R_2, %	V_T, %	V_R, %
1	10	6.3	58.0
10	1	6.3	52.7
1	90	18.8	21.1
90	1	18.8	2.1

Beamsplitter Fringe Visibility.

Beamsplitter. Ideally we would like one reflected and one transmitted beam. However, there is always some reflection from the second surface. If this reflection adds coherently, undesirable fringes may result.

The goal is to keep the fringe visibility low. High reflectivity (90%)beamsplitters are better than low (10%), and beamsplitters with the reflective surface first are better than those with the anti–reflective (1%) side first.

Beamsplitter

- Reflection from the AR side may produce unwanted effects such as ghost images and interference patterns.

- Coherent sources exacerbate the problem.

- Make stronger beam the reflected one if possible.

- Reflective side toward the source.

- Tilt the second surface?

- Reflected wave will have weaker fringes than the transmitted one.

- Beamsplitter cube: Flat surface problem.

- Aberrations of converging/diverging waves (transmitted direction). Worse for a cube.

Thin Films: Approach
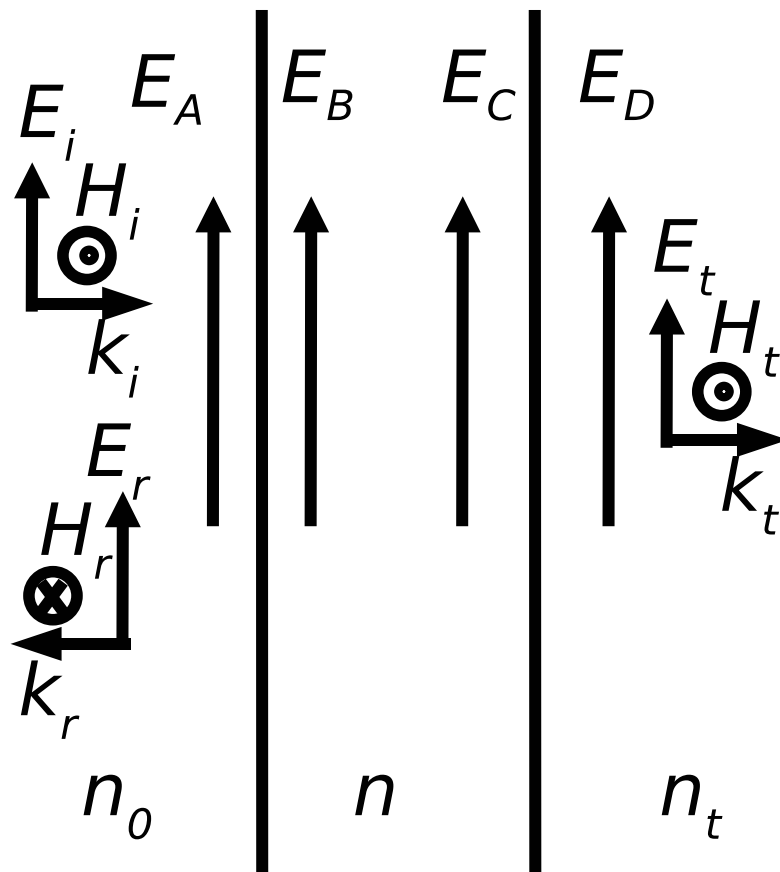
- Normal Incidence
- Wave from Left

$$E_A = E_i + E_r \qquad E_D = E_t$$

$$H_A = H_i - H_r \qquad H_D = H_t$$

- Electric Field BC

$$E_B = E_A \qquad E_C = E_D$$

- Magnetic Field BC

$$H_B = H_A \qquad H_C = H_D$$

$$\frac{E_B}{nZ_0} = \frac{E_A}{n_0 Z_0} \qquad \frac{E_c}{nZ_0} = \frac{E_D}{n_t Z_0}$$

$$\frac{E_B}{n} = \frac{E_A}{n_0} \qquad \frac{E_C}{n} = \frac{E_D}{n_t}$$

E_i
E_A
E_B
E_C
E_D
H_i
k_i
E_r
H_r
k_r
E_t
H_t
k_t
n_0
n
n_t

Thin Films: In the Medium

- Right–Propagating

$$E_{right}e^{inkz} \qquad \text{and} \qquad H_{right}e^{inkz} = \frac{1}{nZ_0}E_{right}e^{inkz}$$

- Left–Propagating

$$E_{left}e^{-inkz} \qquad \text{and} \qquad H_{left}e^{-inkz} = -\frac{1}{nZ_0}E_{left}e^{-inkz}$$

- Boundaries

$$E_B = E_{left} + E_{right} \qquad E_c = E_{left}e^{-jnk\ell} + E_{right}e^{jnk\ell}$$

- Result

$$E_i + E_r = E_t \cos{(nk\ell)} - E_t\frac{n_t}{n}\sin{(nk\ell)}$$

$$n_0 E_i - n_0 E_r = -in E_t \sin{(nk\ell)} + n_t E_t \cos{(nk\ell)}$$

Thin Films: Matrix Equation

- Previous Result

$$E_i + E_r = E_t \cos{(nk\ell)} - E_t \frac{n_t}{n} \sin{(nk\ell)}$$

$$n_0 E_i - n_0 E_r = -in E_t \sin{(nk\ell)} + n_t E_t \cos{(nk\ell)}$$

- Matrix Equation

$$\begin{pmatrix} 1 \\ n_0 \end{pmatrix} E_i + \begin{pmatrix} 1 \\ -n_0 \end{pmatrix} E_r = \begin{pmatrix} \cos{(nk\ell)} & -\frac{i}{n} \sin{(nk\ell)} \\ -in \sin{(nk\ell)} & \cos{(nk\ell)} \end{pmatrix} \begin{pmatrix} 1 \\ n_t \end{pmatrix} E_t$$
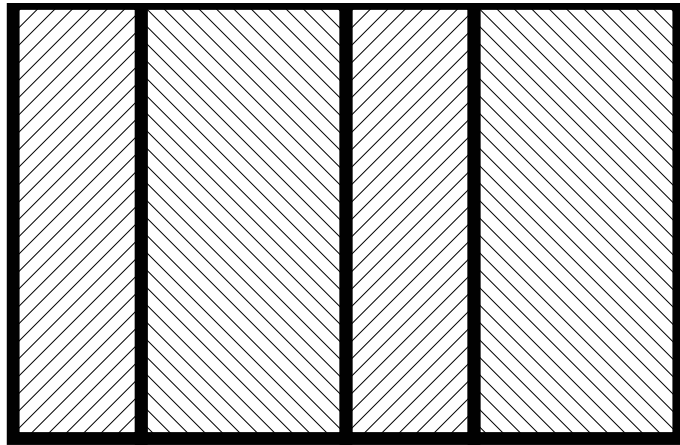
- Characteristic Matrix

$$\mathcal{M} = \begin{pmatrix} \cos{(nk\ell)} & -\frac{i}{n} \sin{(nk\ell)} \\ -in \sin{(nk\ell)} & \cos{(nk\ell)} \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ n_0 \end{pmatrix} + \begin{pmatrix} 1 \\ -n_0 \end{pmatrix} \rho = \mathcal{M} \begin{pmatrix} 1 \\ n_t \end{pmatrix} \tau$$

Multiple Layers

$$\mathcal{M} = \begin{pmatrix} \cos\left(nk\ell\right) & -\frac{i}{n}\sin\left(nk\ell\right) \\ -in\sin\left(nk\ell\right) & \cos\left(nk\ell\right) \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ n_0 \end{pmatrix} + \begin{pmatrix} 1 \\ -n_0 \end{pmatrix}\rho = \mathcal{M}\begin{pmatrix} 1 \\ n_t \end{pmatrix}\tau$$

$$\mathcal{M} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

M$_1$ **M$_4$**

$$\rho = \frac{An_0 + Bn_tn_0 - C - Dn_t}{An_0 + Bn_tn_0 + C + Dn_t}$$

$$\mathcal{M} = \mathcal{M}_1\mathcal{M}_2\mathcal{M}_3\ldots$$

$$\tau = \frac{2n_0}{An_0 + Bn_tn_0 + C + Dn_t}$$

Dielectric Stacks

- High Reflectivity (Often Better than Metal)

- Anti–Reflection "Coatings" or Stacks

- Narrow–Band Filters, Mirrors, *etc.*

- Bandpass Devices that Are Not Narrow–Band

- Hot Mirror or Cold Mirror

- Long–Pass Dichroic

- Short–Pass Dichroic

- Beamsplitters (Specific Reflectivity, Angle, Polarization, Wavelength Range, *etc.*)
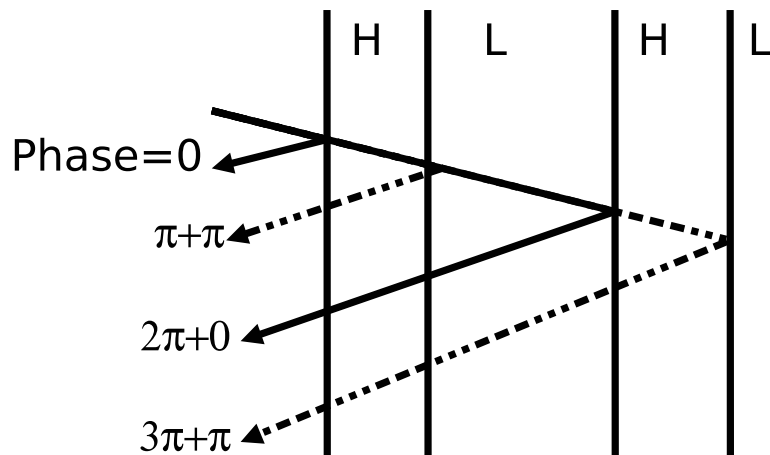
High–Reflectance Stack (1)
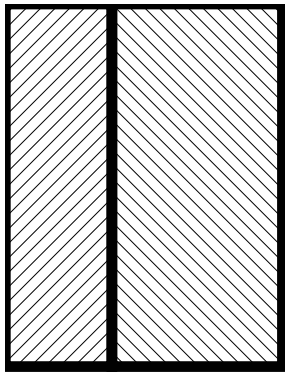
One Pair
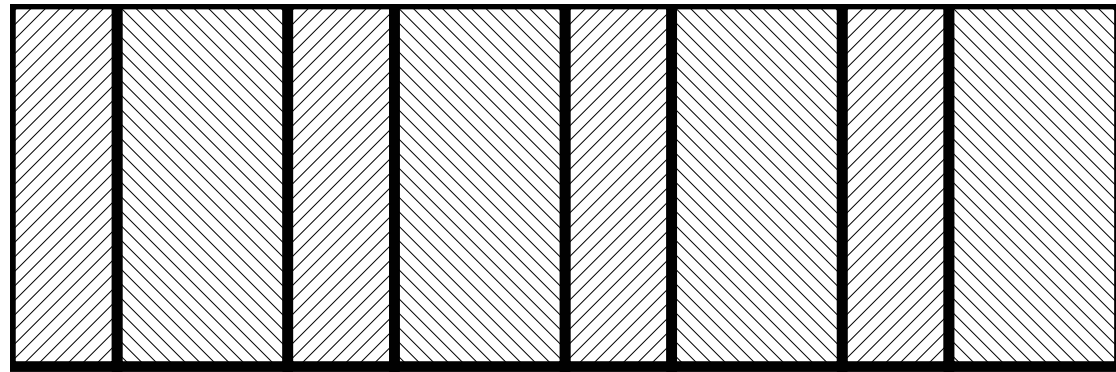
Multi–Layer Stack

- One Layer ($\lambda/4$)

$$\mathcal{M}_i = \begin{pmatrix} 0 & -i/n_i \\ in_i & 0 \end{pmatrix}$$

- One Pair

$$\mathcal{M}_p = \begin{pmatrix} 0 & -i/n_h \\ in_h & 0 \end{pmatrix} \begin{pmatrix} 0 & -i/n_\ell \\ in_\ell & 0 \end{pmatrix} = $$

$$\begin{pmatrix} -n_\ell/n_h & 0 \\ 0 & -n_h/n_\ell \end{pmatrix}$$

H   L   H   L

Phase=0

$\pi+\pi$

$2\pi+0$

$3\pi+\pi$

High–Reflectance Stack (2)

- One Pair of $\lambda/4$ Layers

$$\mathcal{M}_p = \begin{pmatrix} -n_\ell/n_h & 0 \\ 0 & -n_h/n_\ell \end{pmatrix}$$

- Multiple Pairs

$$\mathcal{M}_N = \begin{pmatrix} (-n_\ell/n_h)^N & 0 \\ 0 & (-n_h/n_\ell)^N \end{pmatrix}$$

- Reflectivity

$$R = \left(\frac{\left(\frac{n_\ell}{n_h}\right)^{2N} - \frac{n_t}{n_0}}{\left(\frac{n_\ell}{n_h}\right)^{2N} + \frac{n_t}{n_0}} \right)^2$$

  - Narrow Band ($\lambda/4$)
  - Almost Indep.of n_t, n_0

- Example
  - Zinc Sulfide, $n_h = 2.3$
  - Magnesium Fluoride,
    $n_\ell = 1.35$
  - 8 Layers

    $$N = 4 \rightarrow R = 0.97$$

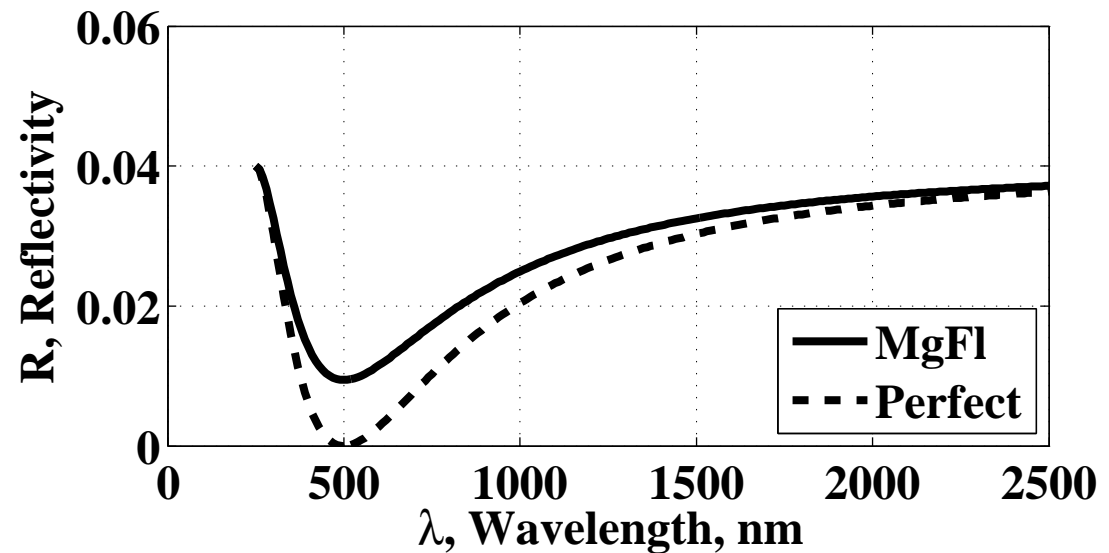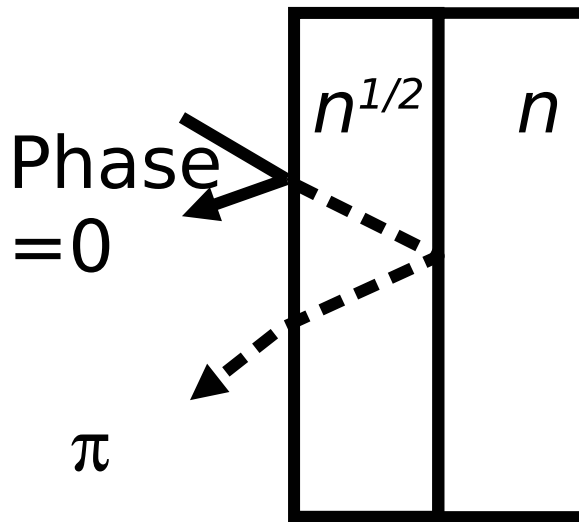 - 30 Layers

    $$N = 15 \rightarrow R = 0.999$$

  - For Lasers, *eg.* HeNe
- Compare Metal
  (≈ 0.96 Typical)
  - Near–Zero Heating

Anti–Reflection Coating

- Ideal AR Coating
  - Quarter–Wave Coating

$$n_{layer} = \sqrt{n_t}$$

- Other Issues
  - Durability
  - Cost
  - Safety

- Example
  - One Layer
  - Magnsium Fluoride

  $$n = 1.35$$

  - $R = 0.01$ at Design λ
  - $R = 0.04$ at Extremes
  - Looks Pink or Purple

Multi–Layer AR Stacks