Optics for Engineers
Chapter 12

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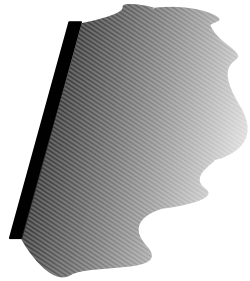
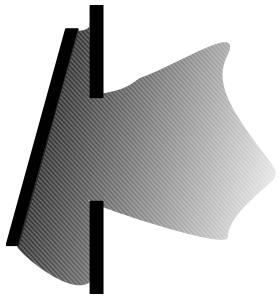
Radiometry and Photometry

Note: "Spectral X:" X_ν, units /Hz or X_λ, /μm.
Rad. to Phot:
683 lm/W $\times\, y\,(\lambda)$.
$y\,(555\text{nm}) \approx 1$.



Radiant Flux, Watt = W
Luminous Flux, lumen = lm

$\partial/\partial A \rightarrow$



Radiant Exitance, W/m^2
Luminous Exitance, lm/m^2 = lux = lx

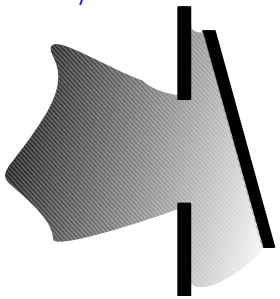
$\downarrow \partial/\partial\Omega \downarrow$

$\downarrow \partial/\partial\Omega \downarrow$

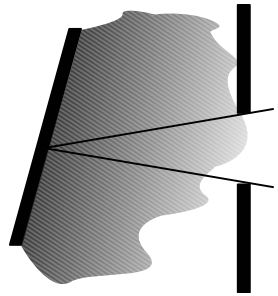
$\leftarrow \cdot/R^2$

$\partial/\partial A \rightarrow$

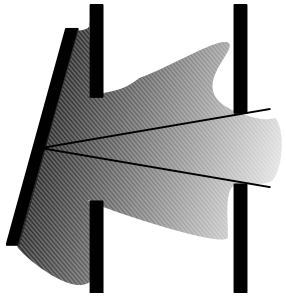
Irradiance, W/m^2
Illuminance, lm/m^2 = lux



1Ft.Candle = 1lm/ft^2



Radiant Intensity, W/sr
Luminous Intensity, lm/sr = cd



Radiance, W/m^2/sr
Luminance, nit = lm/m^2/sr

1Lambert = 1lm/cm^2/sr/π
1mLambert = 1lm/m^2/sr/π
1FtLambert = 1lm/ft^2/sr/π

Radiometric Quantities

Quantity	Symbol	Equation	SI Units
Radiant Energy	Q		Joules
Radiant Energy Density	w	$w = \frac{d^3Q}{dV^3}$	Joules/m^3
Radiant Flux or Power	P or Φ	$\Phi = \frac{dQ}{dt}$	W
Radiant Exitance	M	$M = \frac{d\Phi}{dA}$	W/m^2
Irradiance	E	$E = \frac{d\Phi}{dA}$	W/m^2
Radiant Intensity	I	$I = \frac{d\Phi}{d\Omega}$	W/sr
Radiance	L	$lm = \frac{d^2\Phi}{dA\cos\theta d\Omega}$	W/m^2
Fluence	Ψ	$\frac{dQ}{dA}$	J/m^2
Fluence Rate	F	$\frac{d\Psi}{dt}$	J/m^2
Emissivity	ϵ	$\epsilon = \frac{M}{M_{bb}}$	Dimensionless
Spectral ()	$()_\nu$ or $()_\lambda$	$\frac{d()}{d\nu}$ $\frac{d()}{d\lambda}$	()/Hz ()/μm
Luminous Flux or Power	P or Φ		lm
Luminous Exitance	M	$M = \frac{d\Phi}{dA}$	lm/m^2
Illuminance	E	$E = \frac{d\Phi}{dA}$	lm/m^2
Luminous Intensity	I	$I = \frac{d\Phi}{d\Omega}$	lm/sr
Luminance	L	$L = \frac{d^2\Phi}{dA\cos\theta d\Omega}$	lm/m^2
Spectral Luminous Efficiency	$V(\lambda)$		Dimensionless
Color Matching Functions	$\bar{x}(\lambda)$ $\bar{y}(\lambda)$ $\bar{z}(\lambda)$		Dimensionless

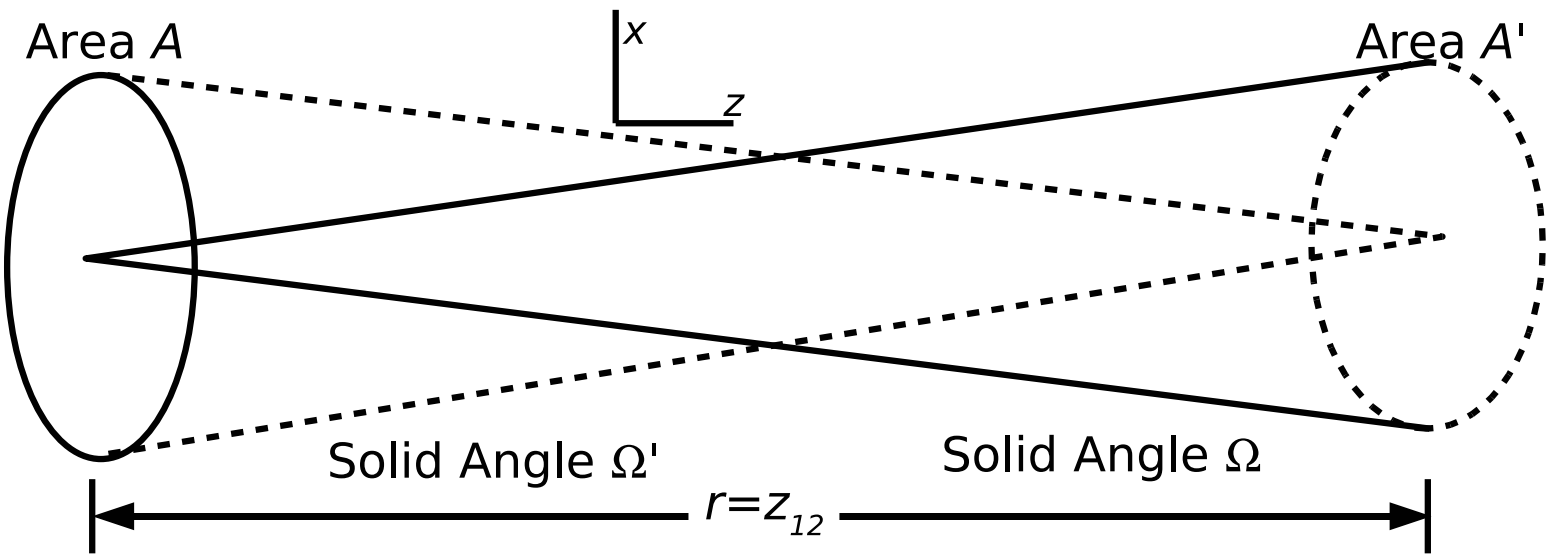
Irradiance

- Poynting Vector for Coherent Wave

$$d\mathbf{S} \approx \frac{dP}{4\pi r^2}\left(\sin\theta\cos\zeta\widehat{x} + \sin\theta\sin\zeta\widehat{y} + \cos\theta\widehat{z}\right)$$

- Irradiance (Projected Area, $A_{proj} = A\cos\theta$)

$$dE = \frac{d^2P}{dA'} = \frac{dP}{\left(4\pi r^2\right)}$$



Irradiance and Radiant Intensity

- Solid Angle

$$\Omega = \frac{A'}{r^2}$$

- Intensity from Irradiance (Unresolved Source, A)

$$\boxed{I = Er^2 \qquad E = \frac{I}{r^2}}$$

$$dI = \frac{d^2P}{d\Omega} = \frac{d^2P}{d\frac{A'}{r^2}} = \frac{dP}{\left(4\pi r^2\right)}$$

Intensity and Radiance

- Resolved Source: Many Unresolved Sources Combined

$$I = \int dI = \int \frac{\partial I}{\partial A} dA$$

- Radiance

$$\boxed{L(x, y, \theta, \zeta) = \frac{\partial I(\theta, \zeta)}{\partial A}}$$

$$\boxed{I(\theta, \zeta) = \int_A L(x, y, \theta, \zeta)\, dx dy}$$

- On Axis ($x' = y' = 0$)

$$L(x, y, \theta, \zeta) = \frac{\partial I(\theta, \zeta)}{\partial A} = \frac{\partial^2 P}{\partial A \partial \Omega}$$

- Off Axis (Projected Area)

$$L = \frac{\partial I}{\partial A \cos\theta} = \frac{\partial^2 P}{\partial A \partial \Omega \cos\theta}$$

Radiant Intensity and Radiant Flux

- Flux or Power, P or Φ
- Integrate Intensity

$$P = \Phi = \int\int I(\theta, \phi) \sin\theta d\theta d\zeta$$

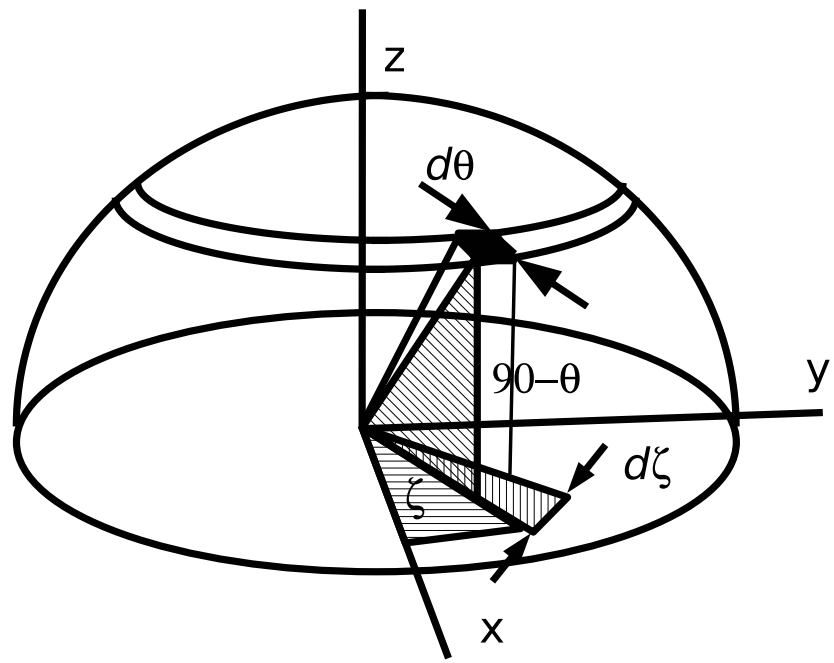
- Constant Intensity

$$P = \Phi = I\Omega$$

$$\Omega = \int\int \sin\theta' d\theta' d\zeta$$

- Solid Angle

$$\Omega = 2\pi\left(1 - \sqrt{1 - \left(\frac{NA}{n}\right)^2}\right)$$



$$\Omega = \int_0^{2\pi}\int_0^{\Theta} \sin\theta' d\theta' d\zeta = 2\pi(1 - \cos\Theta)\, 2\pi\left(1 - \sqrt{1 - \sin^2\Theta}\right)$$

Radiance and Radiant Exitance

- Radiant Exitance (Same Units as Irradiance)

$$M(x,y) = \int\int \left[\frac{\partial^2 P}{\partial A \partial d\Omega}\right] \sin\theta d\theta d\zeta$$

- Radiant Exitance from Radiance

$$M(x,y) = \int\int L(x,y,\theta,\zeta) \cos\theta \sin\theta d\theta d\zeta$$

- Radiance from Radiant Exitance

$$L(x,y,\theta,\zeta) = \frac{\partial M(x,y)}{\partial \Omega} \frac{1}{\cos\theta}$$

Radiance and Radiant Exitance: Special Cases

- Constant L and Small Solid Angle

$$M(x,y) = L\Omega\cos\theta$$

$$\Omega = 2\pi\left(1-\sqrt{1-\left(\frac{NA}{n}\right)^2}\right) \qquad \text{or} \qquad \Omega \approx \pi\left(\frac{NA}{n}\right)^2$$

- Constant L, over Hemisphere

$$M(x,y) = \int_0^{2\pi}\int_0^{\pi/2} L\cos\theta\sin\theta d\theta d\zeta = 2\pi L\frac{\sin^2\frac{\pi}{2}}{2}.$$

$$\boxed{M(x,y) = \pi L \qquad \text{(Lambertian Source)}}$$

Radiant Exitance and Flux

- Power or Flux from Radiant Exitance

$$P = \Phi = \int\int M\left(x, y\right) dxdy,$$

- Radiant Exitance from Power

$$M\left(x, y\right) = \frac{\partial P}{\partial A}$$

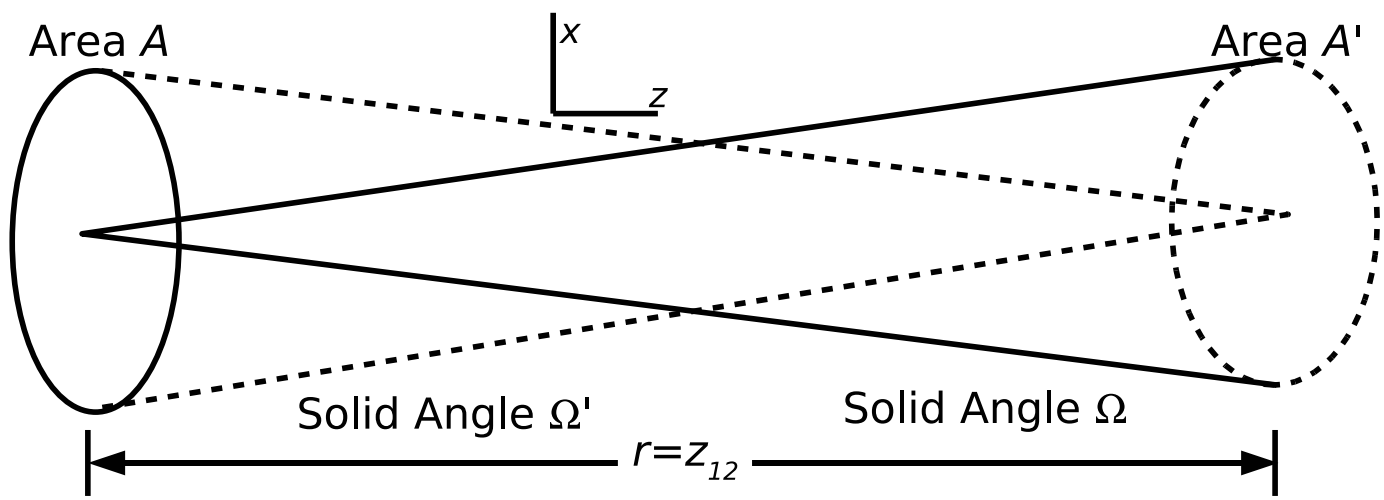
The Radiance Theorem

- Solid Angle two ways

$$\Omega = \frac{A'}{r^2} \qquad \Omega' = \frac{A}{r^2}$$

- Power Increment

$$dP = LdAd\Omega = LdA\frac{dA'}{r^2}\cos\theta \qquad dP = LdA'd\Omega' = LdA'\frac{dA}{r^2}\cos\theta$$



Using the Radiance Theorem: Examples Later

- Radiance is Conserved in a Lossless System
- Losses Are Multiplicative
  - Fresnel Reflections and Absorption
- Radiance Theorem Simplifies Calculation of Detected Power
  - Determine Object Radiance
  - Multiply by Scalar, T_{total}, for Loss
  - Find Exit Window (Of a Scene or a Pixel)
  - Find Exit Pupil
  - Compute Power

$$P = L_{object} T_{total} A_{exit\ window} \Omega_{exit\ pupil}$$

Etendue

- Abbe Invariant:

$$n'x'd\alpha' = nxd\alpha$$

- Etendue

$$n^2 A\Omega = \left(n'\right)^2 A'\Omega'$$

- Power Conservation

$$\int\int\int\int Ld^2Ad^2\Omega = \int\int\int\int L'd^2A'd^2\Omega'$$

$$\int\int\int\int \frac{L}{n^2}n^2d^2Ad^2\Omega = \int\int\int\int \frac{L'}{n^2}\left(n'\right)^2 d^2A'd^2\Omega'$$

$$\boxed{\frac{L}{n^2} = \frac{L'}{(n')^2}}$$

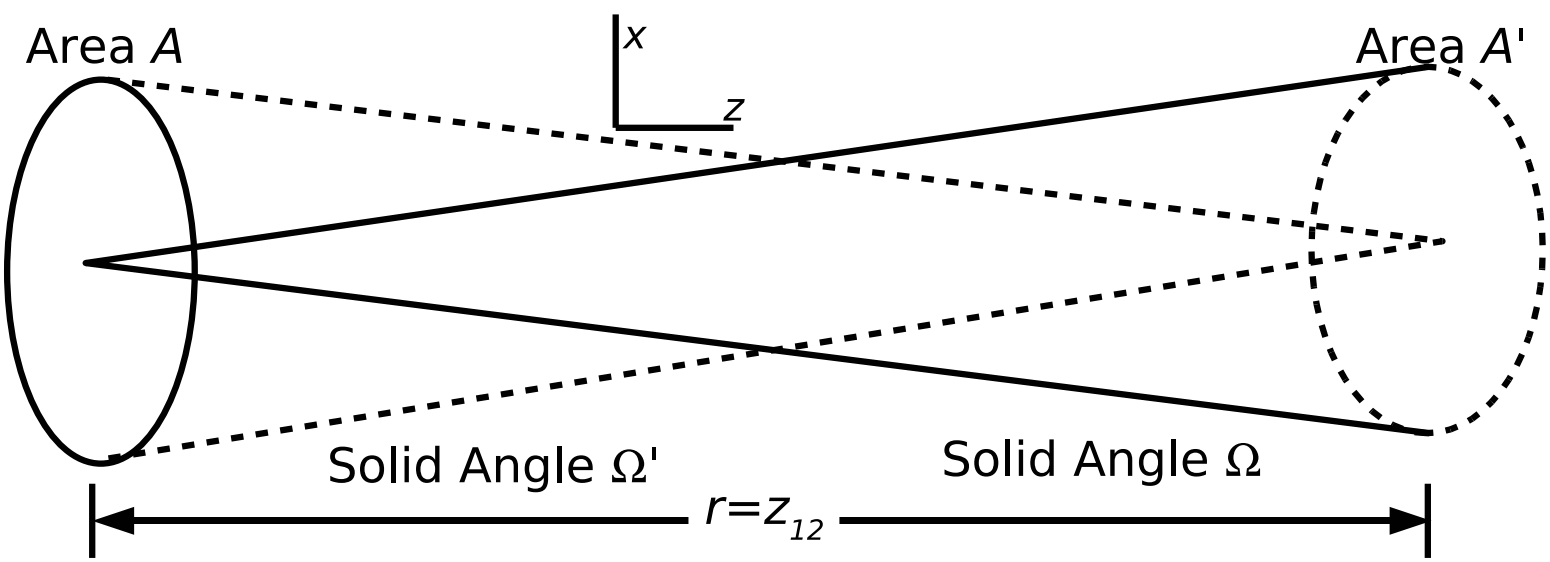
- Basic Radiance, L/n^2, Conserved (Thermodynamics Later)

Radiance Theorem Example: Translation

- Translation Matrix Equation

$$\begin{pmatrix} dx_2 \\ d\alpha_2 \end{pmatrix} = \begin{pmatrix} 1 & z_{12} \\ 0 & 1 \end{pmatrix} \left[\begin{pmatrix} x_1 + dx_1 \\ \alpha_1 + d\alpha_1 \end{pmatrix} - \begin{pmatrix} x_1 \\ \alpha_1 \end{pmatrix} \right] = \begin{pmatrix} dx_1 \\ d\alpha_1 \end{pmatrix}$$

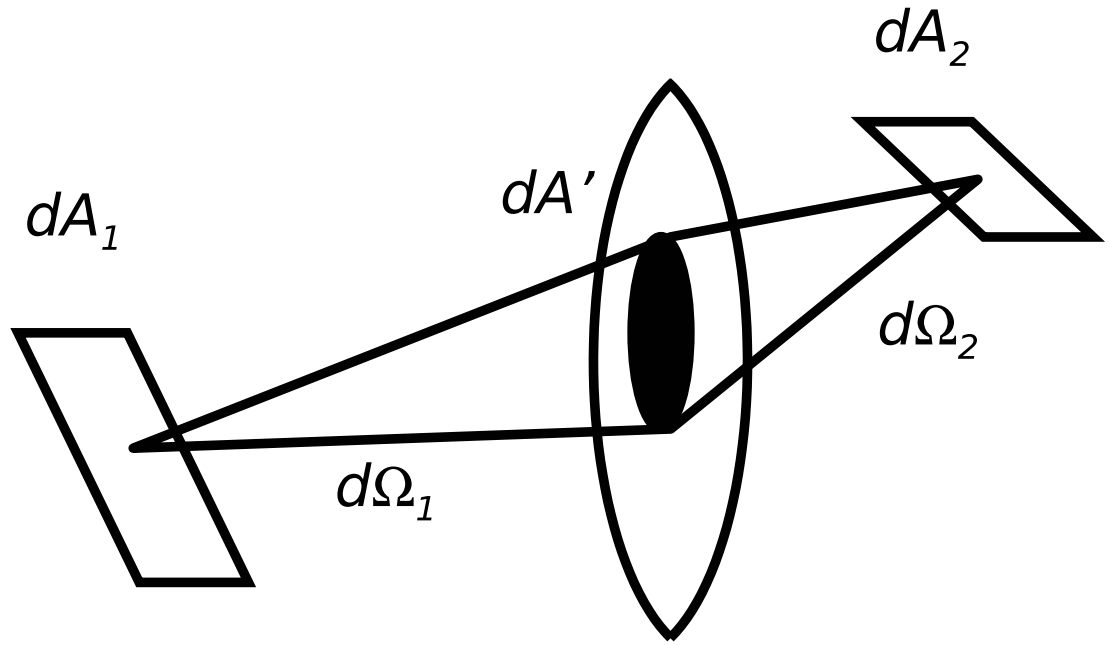
$$dx_2 d\alpha_2 = dx_1 d\alpha_1$$



Radiance Theorem Example: Imaging

- Imaging Matrix Equation

$$\mathcal{M}_{SS'} = \begin{pmatrix} m & 0 \\ 0 & \frac{n}{n'm} \end{pmatrix}$$



Radiance Theorem Example: Dielectric Interface

- Dielectric Interface Matrix Equation

$$\begin{pmatrix} 1 & 0 \\ 0 & \frac{n}{n'} \end{pmatrix}$$

- Power Increment

$$d^2\Phi = L_1 dA \cos\theta_1 d\Omega_1 = L_2 dA \cos\theta_2 d\Omega_2$$

$$L_1 dA \cos\theta_1 \sin\theta_1 d\theta_1 d\zeta_1 = L_2 dA \cos\theta_2 \sin\theta_2 d\theta_2 d\zeta_2$$

- Snell's Law and Derivative

$$n_1 \sin\theta_1 = n_2 \sin\theta_2 \qquad n_1 \cos\theta_1 d\theta_1 = n_2 \cos\theta_2 d\theta_2$$

$$\frac{\sin\theta_1}{\sin\theta_2} = \frac{n_2}{n_1} \qquad \frac{d\theta_1}{d\theta_2} = \frac{n_2 \cos\theta_2}{n_1 \cos\theta_1}$$

- Radiance Theorem

$$\frac{L}{n^2} = \frac{L'}{(n')^2}$$

Radiance Theorem Example: Telecentric Relay

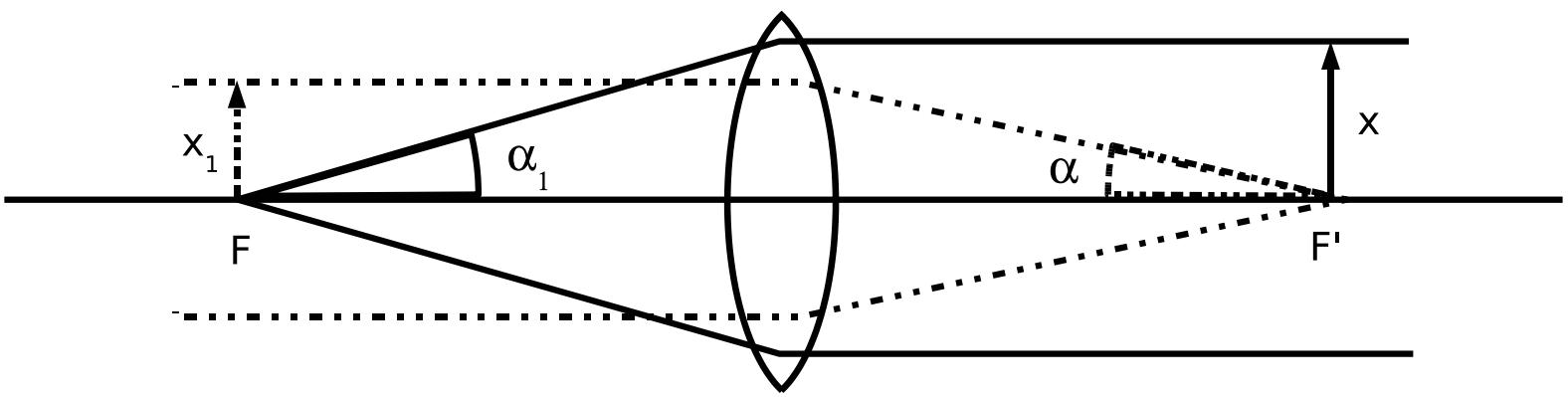
- Fourier–Optics Matrix Equation

$$\mathcal{M}_{FF'} = \begin{pmatrix} 0 & f \\ \frac{1}{f} & 0 \end{pmatrix}$$

$$\det \mathcal{M}_{FF'} = 1 = n'/n \qquad n^2 A\Omega = \left(n'\right)^2 A'\Omega'$$

- Radiance Theorem

$$\frac{L}{n^2} = \frac{L'}{(n')^2}$$



Idealized Example: 100W Lamp

- $P = 100\text{W}$, Uniform in Angle
- Tungsten Filament: Area $A = (5\text{mm})^2$, Distance, R
- Radiant Exitance

$$M = 100\text{W}/\left(5\text{mm}\right)^2 = 4\text{W}/\text{mm}^2$$

- Radiance (Uniform in Angle)

$$L = M/\left(4\pi\right) \approx 0.32\text{W}/\text{mm}^2/\text{sr}$$

- Intensity

$$I = 100\text{W}/\left(4\pi\right) \approx 7.9\text{W}/\text{sr}$$

- Reciever: Area A': Received Power and Irradiance

$$PA'/\left(4\pi r^2\right) = IA'/r^2 \qquad E = \frac{I}{r^2} \approx 8\text{W}/\text{sr}/\left(10m\right)^2 = 0.08\text{W}/\text{m}^2$$

100W Lamp at the Receiver

- Using Source Radiance at Receiver

$$E = L\frac{A}{r^2} = L\Omega'$$

- Irradiance Same as Previous Page

$$E \approx 0.32\text{W/mm}^2/\text{sr} \times \frac{(5\text{mm})^2}{(10\text{m})^2} \approx 0.08\text{W/m}^2$$

- Useful if R and A are Not Known
  - Only Depends on Ω' (Measured at Reciever)

Practical Example: Imaging the Moon

- Known Moon Radiance $L_{moon} = 13\text{W/m}^2\text{/sr}$
- Calculation
  - Jones matrices for Transmission (0.25), other multiplicative losses (12 Lenses: 0.94^{12}).

    $L = 13\text{W/m}^2\text{/sr} \times 0.25 \times 0.92^{12} = 13\text{W/m}^2\text{/sr} \times 0.25 \times 0.367 = 1.2\text{W/m}^2\text{/sr}$
  - Exit Pupil ($NA = 0.1$) and exit window (Pixel: $d = 10\mu\text{m}$).
  - Irradiance, $E = L\Omega$ and Power on a Pixel, $P = EA = LA\Omega$

$$P = 1.2\text{W/m}^2\text{/sr} \times 2\pi\left(1 - \sqrt{1 - NA^2}\right) \times \left(10 \times 10^{-6}\text{m}\right)^2$$

$$P = 1.2\text{W/m}^2\text{/sr} \times 0.315\text{sr} \times 10^{-10}\text{m}^2 = 3.8 \times 10^{-12}\text{W}$$

  - If desired, multiply by time (1/30sec) to obtain energy.
    - About One Million Photons (and Electrons)
  - Alternative to Solve in Object Space (Need Pixel Size on Moon)

Radiometry Summary

- Five Radiometric Quantities: Radiant Flux Φ or Power P, Radiant Exitance, M, Radiant Intensity, I, Radiance, L, and Irradiance, E, Related by Derivatives with Respect to Projected Area, $A\cos\theta$ and Solid Angle, Ω.

- Basic Radiance, L/n^2, Conserved, with the Exception of Multiplicative Factors.

- Power Calculated from Numerical Aperture and Field Of View in Image (or Object) Space, and the Radiance.

- Losses Are Multiplicative.

- Finally "Intensity" is Not "Irradiance."

Spectral Radiometry Definitions

- Any Radiometric Quantity Resolved Spectrally
  - Put the Word Spectral in Front
  - Use a Subscript for Wavelength or Frequency
  - Modify Units

- Example: Radiance, L, Spectral Radiance (Watch Units)

$$L_\nu = \frac{dL}{d\nu} \text{ W/m}^2\text{/sr/THz} \qquad \text{or} \qquad L_\lambda = \frac{dL}{d\lambda} \text{ W/m}^2\text{/sr/}\mu\text{m}$$

- Spectral Fraction

$$f_\lambda(\lambda) = \frac{X_\lambda(\lambda)}{X} \qquad \text{for} \qquad X = \Phi, M, I, E, \text{ or } L$$

Spectral Radiometric Quantities

Quantity	**Units**	$d/d\nu$	**Units**	$d/d\lambda$	**Units**
Radiant Flux, Φ Power, P	W	Spectral Radiant Flux, Φ_ν	W/Hz	Spectral Radiant Flux, Φ_λ	W/μm
Radiant Exi-tance, M	W/m^2	Spectral Radiant Exi-tance, M_ν	W/m^2/Hz	Spectral Radiant Exi-tance, M_λ	W/m^2/μm
Radiant Inten-sity, I	W/sr	Spectral Radiant Inten-sity, I_ν	W/sr/Hz	Spectral Radiant Inten-sity, I_λ	W/sr/μm
Radiance, L	W/m^2/sr	Spectral Radi-ance, L_ν	W/m^2/sr/Hz	Spectral Radi-ance, L_λ	W/m^2/sr/μm

Spectral Fraction: Sunlight on Earth

- 500K Black Body (Discussed Later) Defines M_λ

$$f_\lambda(\lambda) = \frac{M_\lambda(\lambda)}{M} \qquad M = \int_0^\infty M_\lambda d\lambda$$

- Compute Spectral Irradiance with Known $E = 1000\text{W/m}^2$

$$E_\lambda(\lambda) = E f_\lambda(\lambda), \qquad \text{with} \qquad f_\lambda(\lambda) = \frac{M_\lambda(\lambda)}{M} = \frac{M_\lambda(\lambda)}{\int_0^\infty M_\lambda(\lambda)\, d\lambda}$$

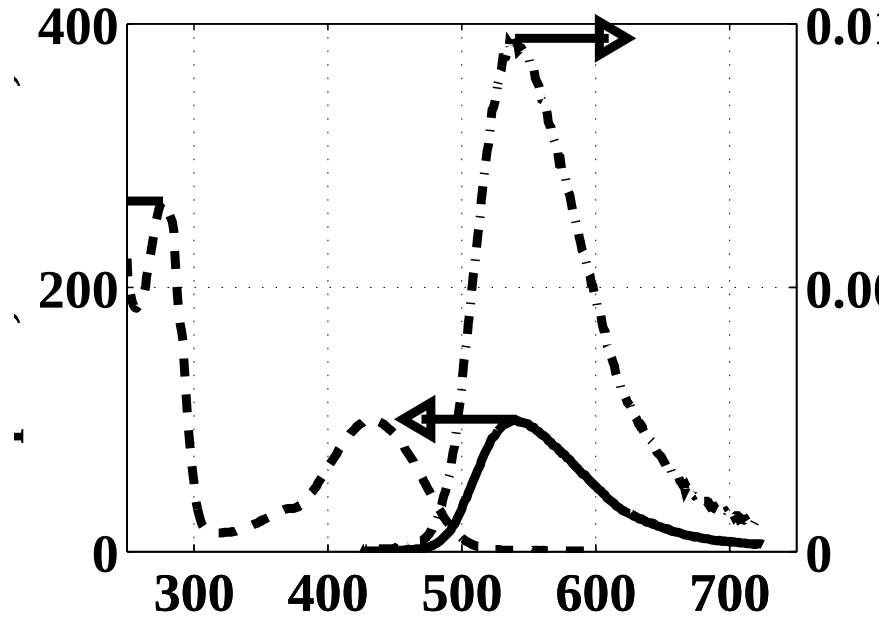
Spectral Fraction: Molecular Tag

- Excitation: Argon 488nm
- Green Emission Power, Φ

$$\Phi_\lambda(\lambda) = \Phi f_\lambda(\lambda),$$

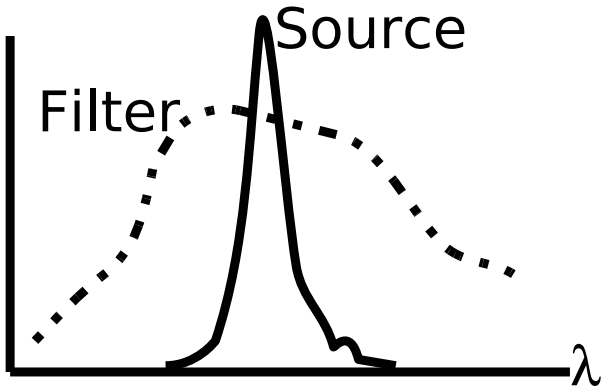
$$f_\lambda(\lambda) = \frac{S_\lambda(\lambda)}{\int_0^\infty S_\lambda(\lambda)\, d\lambda}$$

  - Dash–Dot on Plot
  - Obtained from Experiment (Radiometric Calibration not Needed)
  - Available from Vendor

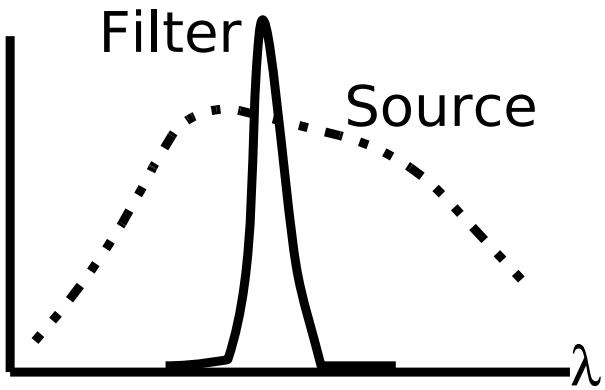


- Fluorescein Spectra
  - Dash: Absorption Spectrum
  - Solid: Emission Spectrum
  - Dash–Dot: Spectral Fraction for Emission

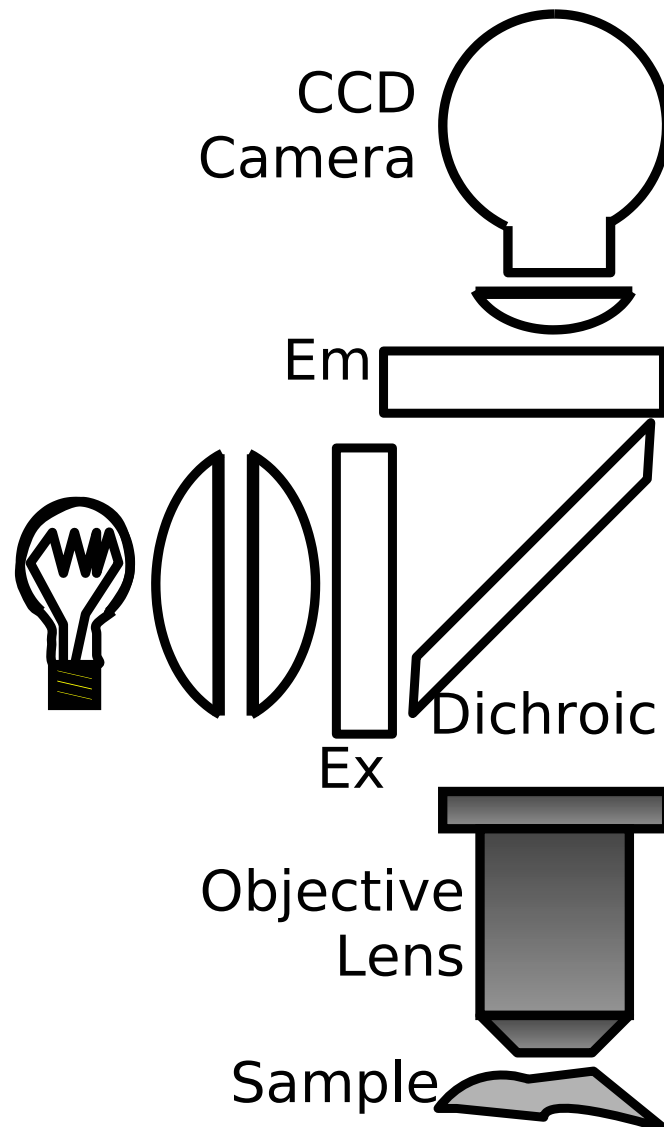
Molecular Tag in Epi–Fluorescence



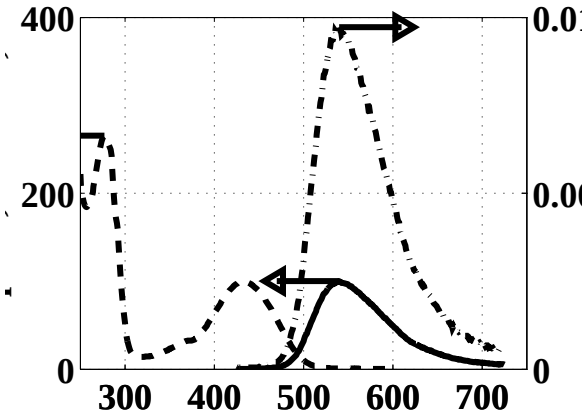
Narrow Signal
Bad Rejection



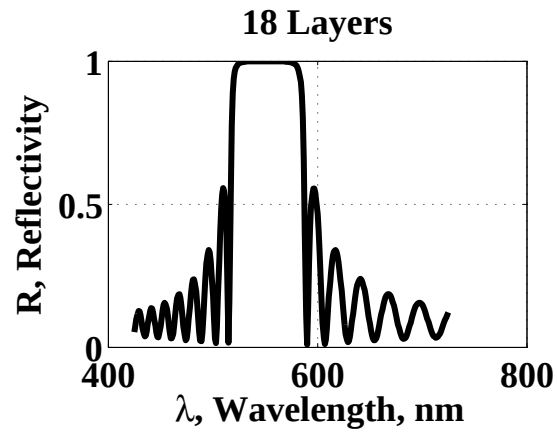
Narrow Filter
Lost Light



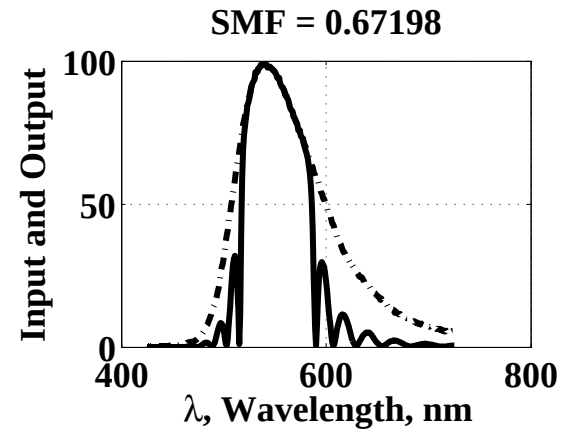
Spectral Matching Factor (1)



Fluorescein
$L_\lambda(\lambda)$



Emission Filter
$R(\lambda)$



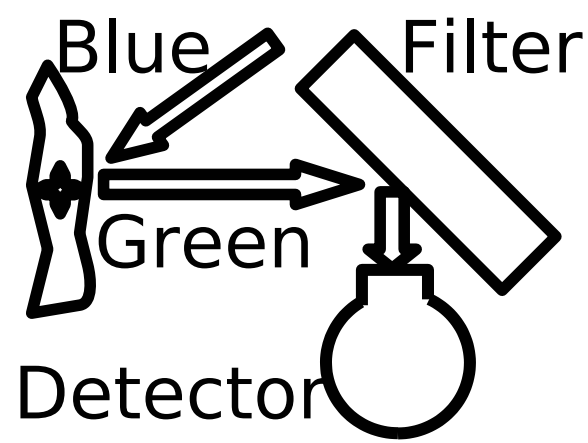
Spectral Matching
$L_\lambda(\lambda)\,R(\lambda)$

- Reflective Filter (Ch. 8)
- Spectral Radiance Output of Filter

$$L'_\lambda(\lambda) = L_\lambda(\lambda)\,R(\lambda)$$

- Total Radiance Output

$$L' = \int_0^\infty L'_\lambda(\lambda)\,d\lambda = \int_0^\infty L_\lambda(\lambda)\,R(\lambda)\,d\lambda$$



Spectral Matching Factor (2)

- Total Radiance (Previous Page)

$$L' = \int_0^\infty L'_\lambda(\lambda)\, d\lambda = \int_0^\infty L_\lambda(\lambda)\, R(\lambda)\, d\lambda$$

- Define Reflectance for This Filter; $R = L'/L$
  - Relate to Transmission at Spectral Maximum
  - Characterize SMF for Given Input Spectrum

$$\boxed{R = R_{max} \int_0^\infty \frac{R(\lambda)}{R_{max}} f_\lambda(\lambda)\, d\lambda = R_{max} SMF}$$

$$SMF = \frac{\delta\lambda_f}{\delta\lambda_s} \qquad \delta\lambda_f \ll \delta\lambda_s$$

Summary of Spectral Radiometry

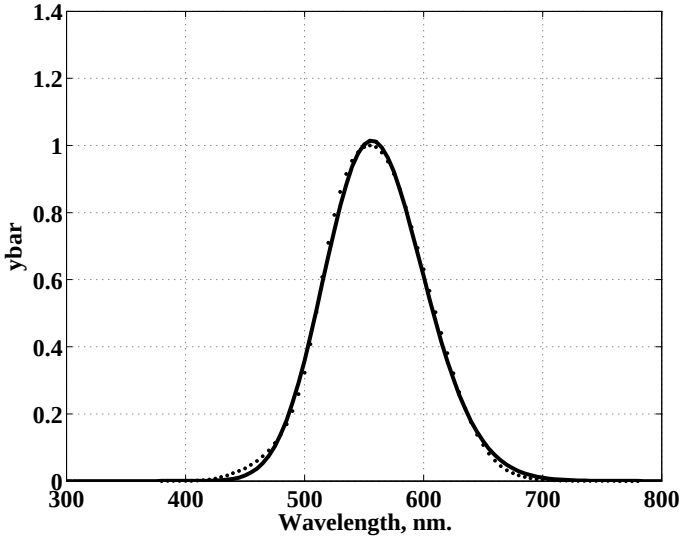
- For every radiometric quantity, there exists a spectral radiometric quantity. In the name, no distinction is made between frequency and wavelength derivatives.
- The notation: subscript ν for frequency or λ for wavelength.
- The units are the original units divided by frequency or wavelength units.
- Spectral Fraction, f_λ can be applied to any of the radiometric quantities.
- The spatial derivatives (area and angle) are valid for the spectral quantities, wavelength–by–wavelength.
- The behavior of filters is more complicated, and is usually treated with the spectral matching factor.

Photometry and Colorimetry

- Spectral Luminous Efficiency, $\bar{y}(\lambda)$
- Source Spectral Radiance, $L_\lambda(\lambda, x, y)$
- Eye Response

$$Y(x,y) = \int_0^\infty \bar{y}(\lambda)\, L_\lambda(\lambda, x, y)\, d\lambda$$

- Four LEDs: Equal Radiance
  - Blue, 400 Appears Weak
  - Green, 550 Appears Strong
  - Red, 630 Moderately Weak
  - IR, 980 Invisible



Lumens

- Power or Radiant Flux (Watts)

$$P = \int_0^\infty P_\lambda(\lambda)$$

- Eye Response

$$Y = \int_0^\infty \bar{y}(\lambda) P_\lambda(\lambda) d\lambda$$

- Luminous Flux (Lumens, Subscript $_V$ for Clarity)

$$P_{(V)} = \frac{683 \text{ lumens/Watt}}{\max(\bar{y})} \int_0^\infty \bar{y}(\lambda) P_\lambda(\lambda) d\lambda$$

- Luminous Efficiency

$$\frac{P_{(V)}}{P} = 683 \text{ lumens/Watt} \int_0^\infty \frac{\bar{y}(\lambda)}{\max(\bar{y})} \frac{P_\lambda(\lambda)}{P} d\lambda$$

Some Typical Radiance and Luminance Values

Object	$W/m^2/sr$		nits $= lm/m^2/sr$	Footlamberts	lm/W
Minimum Visible	7×10^{-10}	Green	5×10^{-7}	1.5×10^{-7}	683
Dark Clouds	0.2	Vis	40	12	190
Lunar disk	13	Vis	2500	730	190
Clear Sky	27	Vis	8000	2300	300
Bright Clouds	130	Vis	2.4×10^4	7×10^3	190
	300	All			82
Solar disk	4.8×10^6	Vis	7×10^8	2.6×10^7	190
	1.1×10^7	All		$\times 10^7$	82

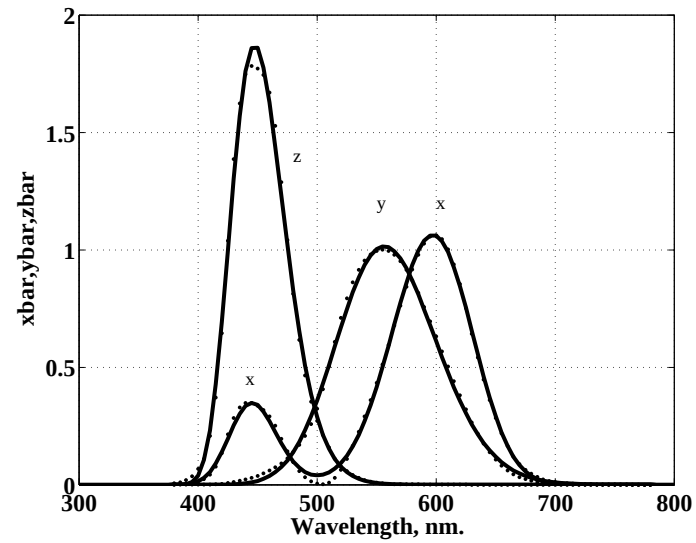
Tristimulus Values: Three is Enough

- X, Y, Z

$$X = \int_0^\infty \bar{x}(\lambda) L_\lambda(\lambda) d\lambda$$

$$Y = \int_0^\infty \bar{y}(\lambda) L_\lambda(\lambda) d\lambda$$

$$Z = \int_0^\infty \bar{z}(\lambda) L_\lambda(\lambda) d\lambda$$



- Example: 3 Lasers

Krypton $\lambda_{red} = 647.1\text{nm}$

$$X_R = \int_0^\infty \bar{x}(\lambda) \delta(\lambda - 647.1\text{nm}) d\lambda = \bar{x}(647.1\text{nm})$$

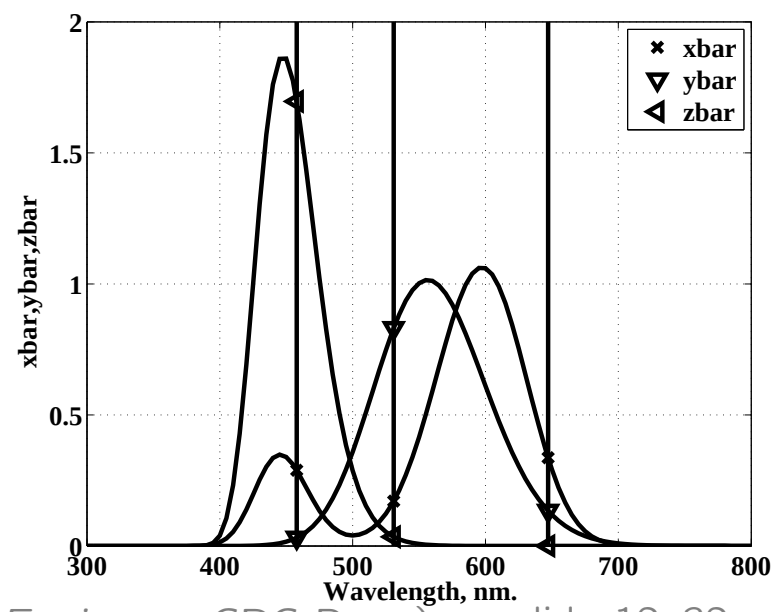
$X_R = 0.337 \quad Y_R = 0.134 \quad Z_R = 0.000$

Krypton $\lambda_{green} = 530.9\text{nm}$

$X_G = 0.171 \quad Y_G = 0.831 \quad Z_G = 0.035$

Argon $\lambda_{blue} = 457.9\text{nm}$

$X_B = 0.289 \quad Y_B = 0.031 \quad Z_B = 1.696$



Chromaticity Coordinates (1)

- Three Laser Powers, R, G, and B, Watts
- Tristimulus Values

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} X_R & X_G & X_B \\ Y_R & Y_G & Y_B \\ Z_R & Z_G & Z_B \end{pmatrix} \begin{pmatrix} R \\ G \\ B \end{pmatrix}$$

- Chromaticity Coordinates (Normalized X, Y

$$x = \frac{X}{X+Y+Z} \qquad y = \frac{Y}{X+Y+Z}$$

- Monochromatic Light

$$x = \frac{\bar{x}(\lambda)}{\bar{x}(\lambda)+\bar{y}(\lambda)+\bar{z}(\lambda)} \qquad y = \frac{\bar{y}(\lambda)}{\bar{x}(\lambda)+\bar{y}(\lambda)+\bar{z}(\lambda)}$$

Chromaticity Coordinates (2)

Monochromatic Light (Boundary)

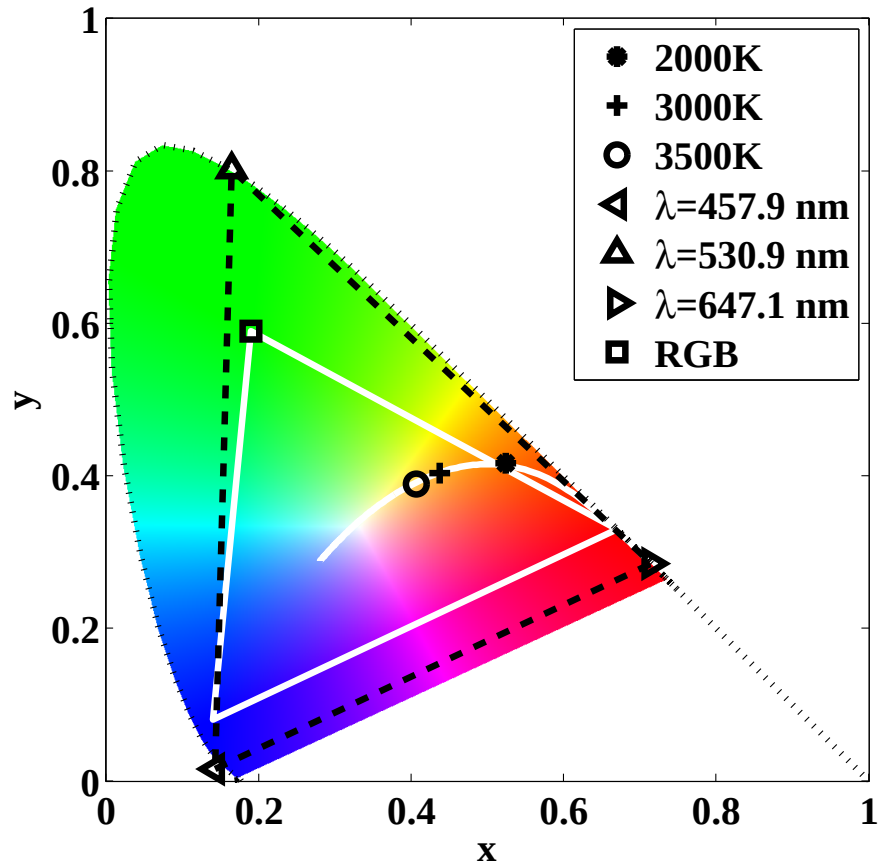
$$x = \frac{\bar{x}(\lambda)}{\bar{x}(\lambda) + \bar{y}(\lambda) + \bar{z}(\lambda)}$$

$$y = \frac{\bar{y}(\lambda)}{\bar{x}(\lambda) + \bar{y}(\lambda) + \bar{z}(\lambda)}$$

3 Lasers (Triangles) and Their Gamut (Black - -)

Phosphors (White)

Thermal Sources (Later)



Generating Colored Light

- Given $P_{(V)}$, x, and y
- Required Tristimulus Values

$$Y = \frac{P_L}{y \times 683\text{lm/W}}$$

$$X = \frac{xP_L}{y \times 683\text{lm/W}}$$

$$Z = \frac{(1 - x - y)\, P_L}{y \times 683\text{lm/W}}$$

- Powers of Three Lasers

$$\begin{pmatrix} R \\ G \\ B \end{pmatrix} = \begin{pmatrix} X_R & X_G & X_B \\ Y_R & Y_G & Y_B \\ Z_R & Z_G & Z_B \end{pmatrix}^{-1} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}$$

- Projector: 100lux "White"
  - $x = y = 1/3$
  - 400lm on $(2\text{m})^2 = 4\text{m}^2$
  - $X = Y = Z =$ 400lm/(1/3)/683lm/W
  - $R = 1.2\text{W}$, $G = 0.50\text{W}$, $B = 0.33\text{W}$

Another Example: Simulating Blue "Laser" Light

- Monochromatic Light from Blue Argon Laser Line

  $\lambda = 488 unnm \qquad X = 0.0605 \qquad Y = 0.2112 \qquad Z = 0.5630$

- Solution with Inverse Matrix

  $$R = -0.2429 \qquad G = 0.2811 \qquad B = 0.3261$$

- Old Bear–Hunters' Saying:
  - Sometimes You Eat the Bear, . . .
  - and Sometimes the Bear Eats You.

- Best Compromise (Right Hue, Insufficient Saturation)

  $$R = 0 \qquad G = 0.2811 \qquad B = 0.3261$$

Summary of Color

- Three color–matching functions $\bar{x}$, $\bar{y}$, and $\bar{z}$
- Tristimulus Values: 3 Integrals X, Y, and Z Describe Visual Response
- Y Related to Brightness. 683lm/W, Links Radiometric to Photometric Units
- Chromaticity Coordinates, x and y, Describe Color. Gamut of Human Vision inside the Horseshoe Curve.
- Any 3 sources R, G, B, Determine Tristimulus Values.
- Sources with Different Spectra May Appear Identical.
- Light Reflected or Scattered May Appear Different Under Different Sources Of Illumination.
- R, G, and B, to X, Y, Z Conversion Can Be Inverted over Limited Gamut.

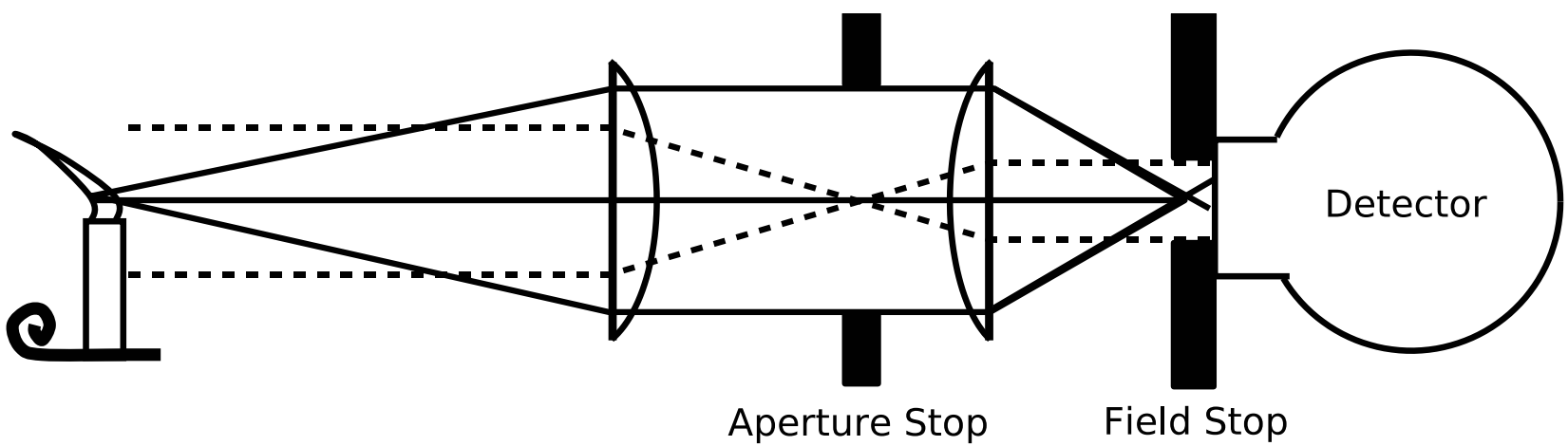
The Radiometer or Photometer

- Aperture Stop
- Field Stop
- Measured Power

$$P = LA\Omega = LA'\Omega'$$

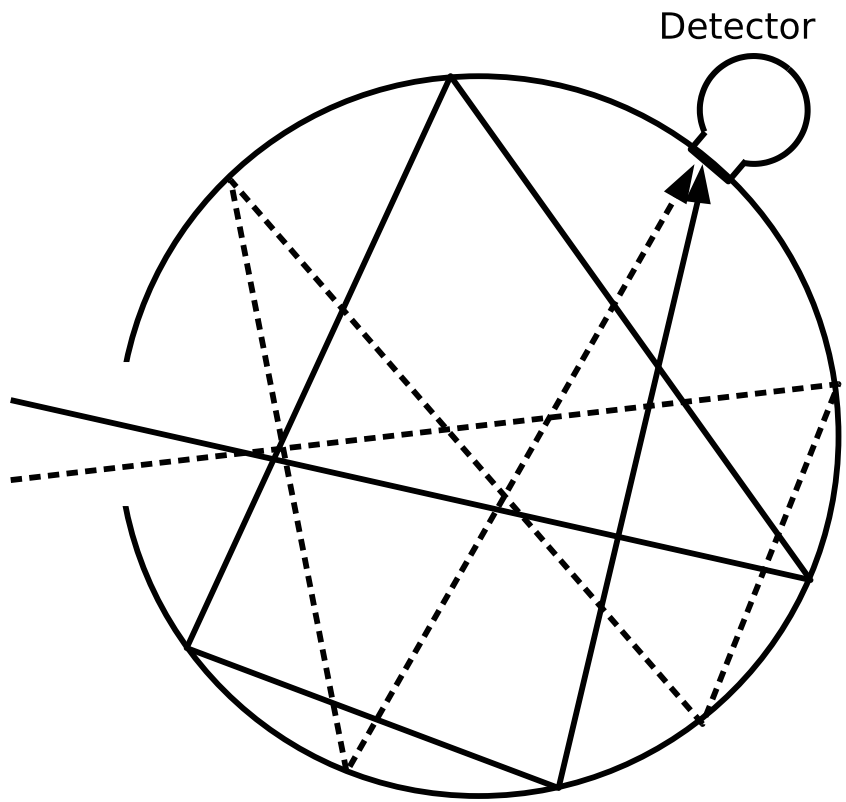
- Adjustable Stops (Match FOV)
- Sighting Scope?

- Calibration Required
- Spectrometer on Output?
- Spectral Filter?
- Photometric Filter?
- Computer
  - Radiance
  - Luminance (All Units)
  - Spectral Quantities



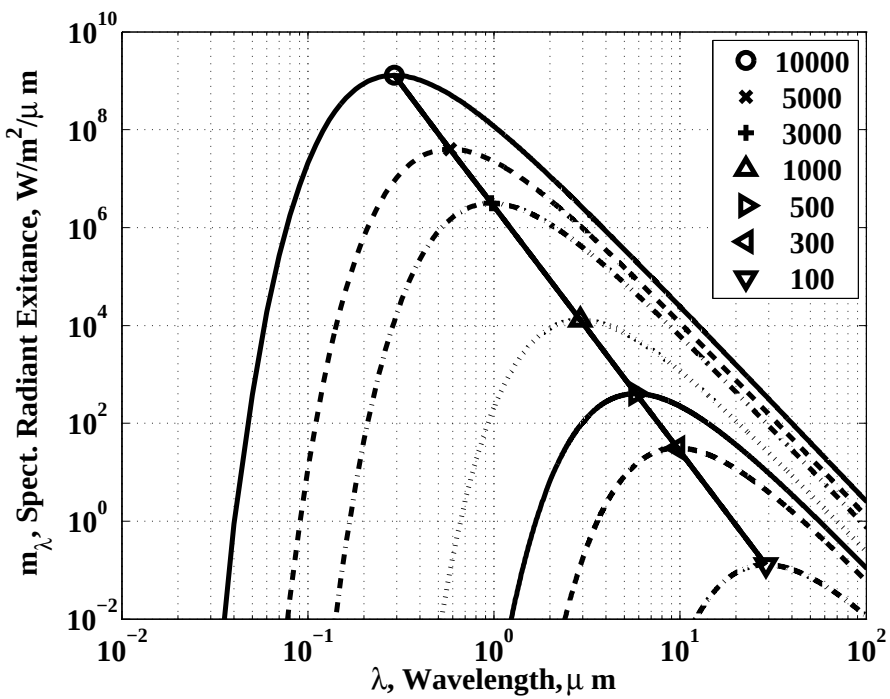
Integrating Sphere

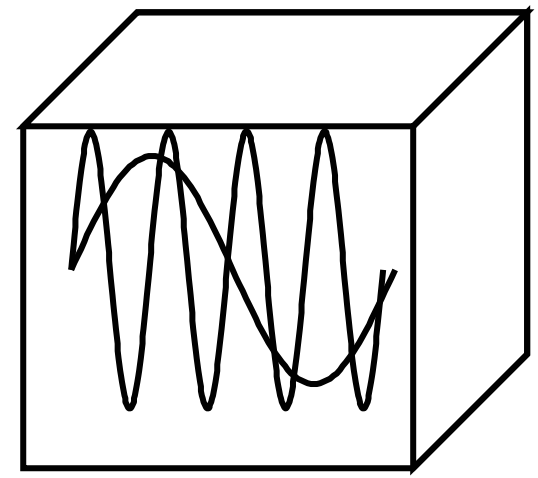
- Power from Intensity
- Integrate over Solid Angle
  - Goniometry
    - Information–Rich
    - Time–Intensive
    - Integrating Sphere
      - Easy
      - Single Measurement
- Applications
  - Wide–Angle Sources
  - Diffuse Materials
- Variations
  - Two Spheres
  - Spectroscopic Detector
  - More

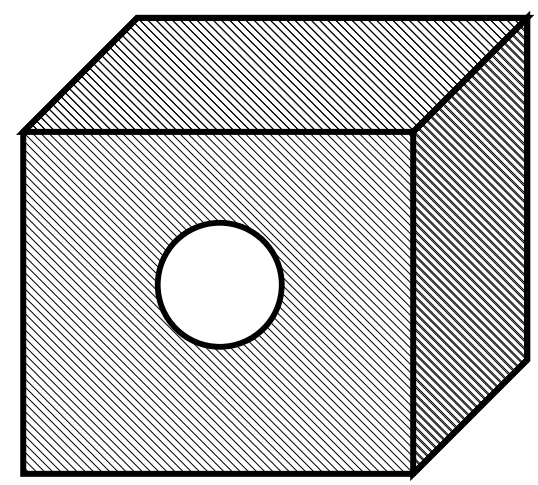


Blackbody Radiation Outline

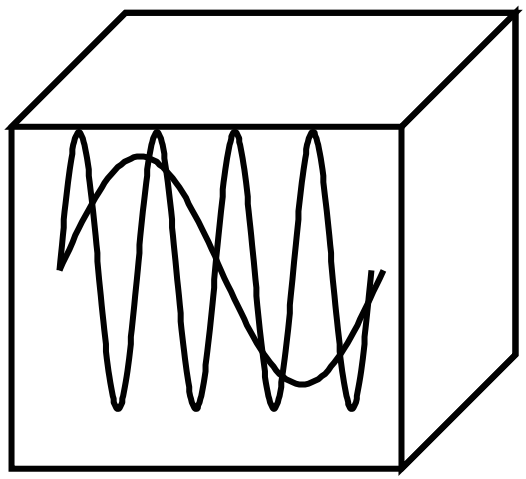
- Background
- Equations, Approximations
- Examples
- Illumination
- Thermal Imaging
- Polar Bears, Greenhouses







Blackbody Background



- Cavity Modes

$$\frac{2\pi\nu_x\ell}{c} = N_x\pi$$

$$\frac{2\pi\nu_x\ell}{c} = N_y\pi$$

$$\frac{2\pi\nu_x\ell}{c} = N_z\pi$$

- Mode Frequencies

$$\nu = \sqrt{\nu_x^2 + \nu_y^2 + \nu_z^2} \qquad \nu_0 = \frac{c}{2\ell}$$

$$\nu = \nu_0\sqrt{N_x^2 + N_y^2 + N_z^2}$$

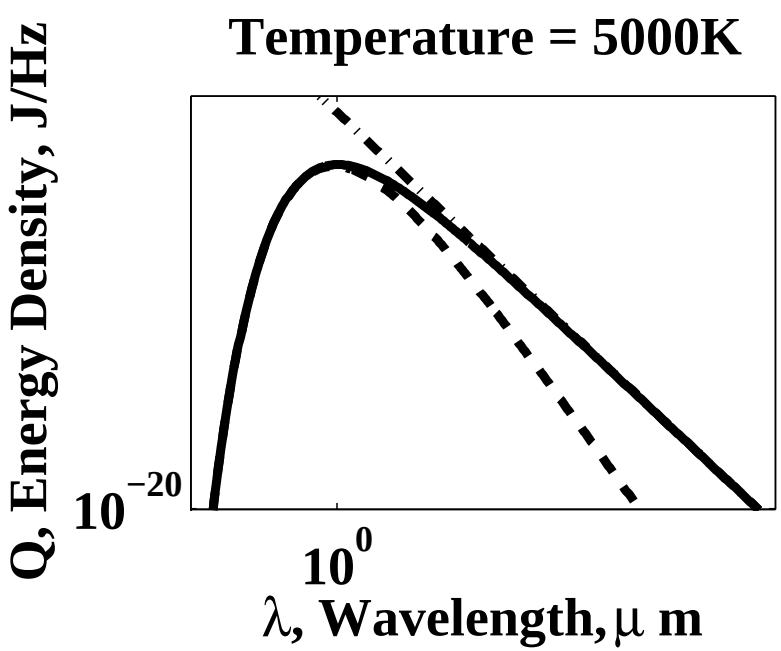
- Waves in a Cavity

$$\frac{\partial^2 E}{\partial x^2} + \frac{\partial^2 E}{\partial y^2} + \frac{\partial^2 E}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2 E}{\partial t^2}$$

- Boundary Conditions

$$E\left(x, y, z, t\right) = \sin\frac{2\pi\nu_x x}{c}\sin\frac{2\pi\nu_y y}{c}\sin\frac{2\pi\nu_z z}{c}\sin 2\pi\nu t$$

Counting Modes



- Mode Density (Sphere)

$$dN = \frac{4\pi\nu^2 d\nu/8}{c^3/(8\ell^3)}$$

$$N_\nu = \frac{dN}{d\nu} = \frac{4\pi\nu^2\ell^3}{c^3}$$

- $\times$ 2 for Polarization

$$N_\nu = \frac{8\pi\nu^2\ell^3}{c^3}$$

- Energy Distribution
  - Rayleigh–Jeans: Equal Energy, $k_B T$ (Dash–Dot)

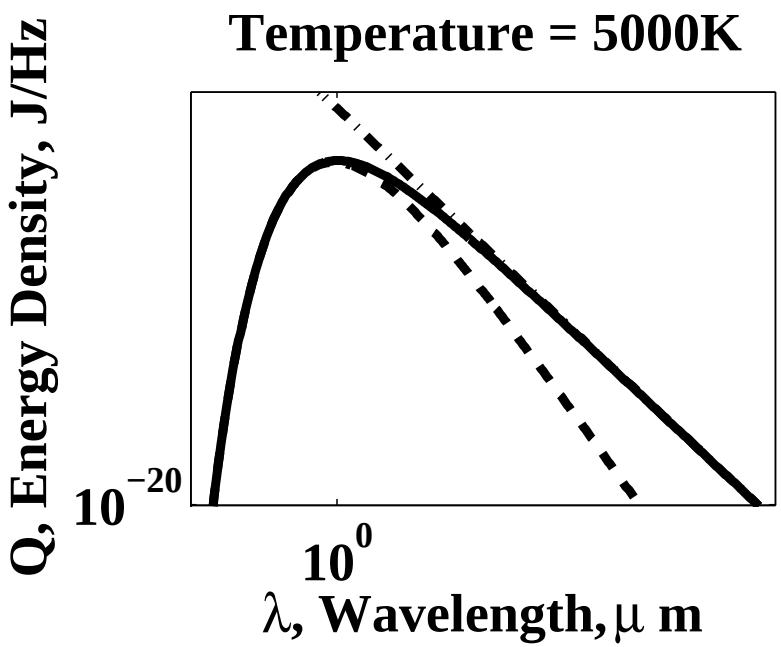
$$J_\nu = k_B T \frac{8\pi\nu^2\ell^3}{c^3}$$

    * "Ultraviolet Catastrophe"
    * Wein $h\nu \exp(h\nu/kT)$ per Mode

$$J_\nu = h\nu \exp(h\nu/kT) \frac{8\pi\nu^2\ell^3}{c^3}$$

    · Dashed

Planck Got it Right



- Planck
  - Mode Energy Quantized $J_N = Nh\nu$
  - N Random
  - Average Energy Density

$$\bar{J} = \frac{\sum_{N=0}^{\infty} J_N P(J_N)}{\sum_{N=0}^{\infty} P(J_N)}$$

- Boltzmann Distribution

$$P(J_N) = e^{\frac{-J_N}{k_B T}}$$

- Average Energy

$$\bar{J} = \frac{\sum_{N=0}^{\infty} Nh\nu e^{\frac{-Nh\nu}{k_B T}}}{\sum_{N=0}^{\infty} e^{\frac{-Nh\nu}{k_B T}}} =$$

$$h\nu \frac{\sum_{N=0}^{\infty} N e^{\frac{-Nh\nu}{k_B T}}}{\sum_{N=0}^{\infty} e^{\frac{-Nh\nu}{k_B T}}}$$

- With Some Work. . .

$$\bar{J} = h\nu \frac{\frac{e^{h\nu/k_B T}}{(1-e^{h\nu/k_B T})^2}}{\frac{1}{(1-e^{h\nu/k_B T})}} =$$

$$\frac{h\nu}{e^{h\nu/k_B T} - 1}$$

Planck's Result (1)

- Previous Page

$$\frac{h\nu}{e^{h\nu/k_BT}-1}$$

- $\times$ Mode Density

$$N_\nu \bar{J} = \frac{8\pi\nu^2\ell^3}{c^3}\frac{h\nu}{e^{h\nu/k_BT}-1}$$

- Solid Curve (Previous Page)
- Energy Per Volume

$$\frac{d\bar{w}}{d\nu} = \frac{N_\nu \bar{J}}{\ell^3} = \frac{8\pi\nu^2}{c^3}\frac{h\nu}{e^{h\nu/k_BT}-1}$$

- 1. Radiance Independent of Direction
- 2. Emssion Balanced by Absorption

$$\bar{M}(T)_{emitted} = \epsilon_a \bar{E}(T)_{incident}$$

- 3. Balance at Every λ
- 4. Light Moves at c

$$w_\lambda = \frac{1}{c}\int_{sphere} L d\Omega = \frac{4\pi l}{c}$$

- E from L

$$E_\lambda = \int\int L_\lambda \cos\theta \sin\theta d\theta d\zeta =$$

$$\pi L_\lambda$$

Planck's Result (2)

- E from w: $E_\lambda = 4cw_\lambda$
- Wall Balance & $\epsilon_a = 1$: $\bar{M}(T)_{bb} = E(\bar{T})_{incident}$
- Planck's Law

$$M_{\nu(bb)}(\lambda, T) = h\nu c \frac{2\pi\nu^2}{c^3} \frac{1}{e^{h\nu/k_B T} - 1} = \frac{2\pi h\nu^3}{c^2} \frac{1}{e^{h\nu/k_B T} - 1}$$

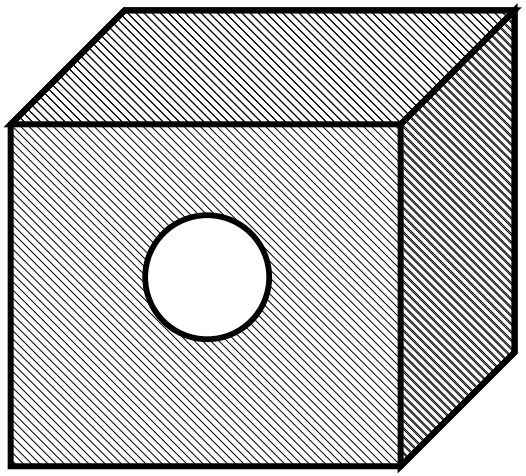
- Emissivity

$$M_\nu(\lambda, T) = \epsilon M_{\nu(bb)}(\lambda, T)$$

- Detailed Balance

$$\boxed{\epsilon(\lambda) = \epsilon_a(\lambda)}$$

The Planck Equation



- Fractional Linewidth

$$|d\nu/\nu| = |d\lambda/\lambda|$$

$$M_\lambda = \frac{dM}{d\lambda} = \frac{d\nu}{d\lambda}\frac{dM}{d\nu} = \frac{\nu}{\lambda}\frac{dM}{d\nu} = \frac{c}{\lambda^2}\frac{dM}{d\nu}$$

- Planck Law *vs.* Wavelength

$$M_\lambda(\lambda, T) = \frac{2\pi hc^2}{\lambda^5}\frac{1}{e^{h\nu/k_BT} - 1}$$

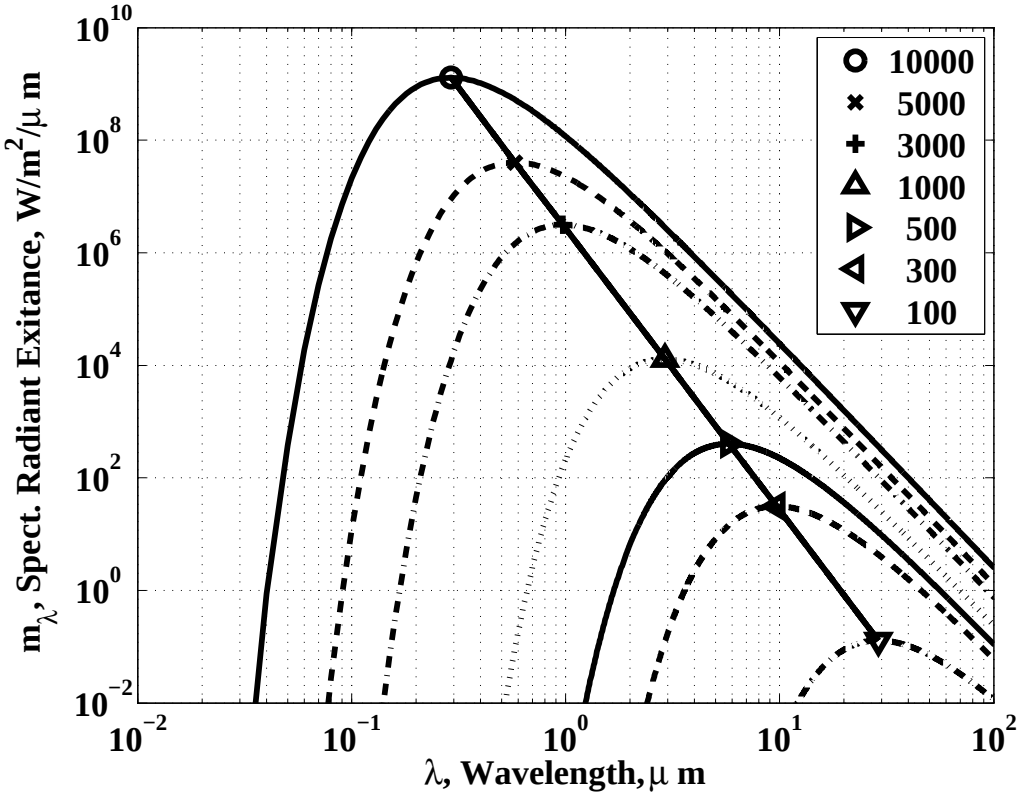
Useful Blackbody Equations

- Wien Displacement Law

$$\lambda_{peak} T = 2898 \mu\text{m} \cdot \text{K}$$

- Stefan–Boltzmann Law

$$M(T) = \frac{2\pi^5 k^4 T^4}{15 h^3 c^2} = \sigma_e T^4$$



- Stefan–Boltzmann Constant

$$\sigma_e = 5.67032 \times 10^{-12} \text{W/cm}^2/\text{K}^4 = 5.67032 \times 10^{-8} \text{W/m}^2/\text{K}^4$$

Some Examples: Thermal Equilibrium

- Earth Temperature
  - Heating
    - 1kw/m^2
    - Half of Surface
  - Cooling
    - Radiation

$$M(T) = \sigma_e T^4$$

  - Result

$$T \approx 306\text{K}$$

- Body Temperature
  - Cooling (310K)

$$M(T) = \sigma_e T^4 = 524\text{W/m}^2$$

  - Heating (Room=295K)

$$M(T) = \sigma_e T^4 = 429\text{W/m}^2$$

  - Excess Cooling

$$\Delta M A \approx 100\text{W}$$

$$\approx 2000\text{kCal/day}$$

(Many Approximations)

Because $d\left(T^4\right) = 4T^3 dT$, small changes in temperature have big effects on heating or cooling.

Temperature and Color

- Radiant Exitance

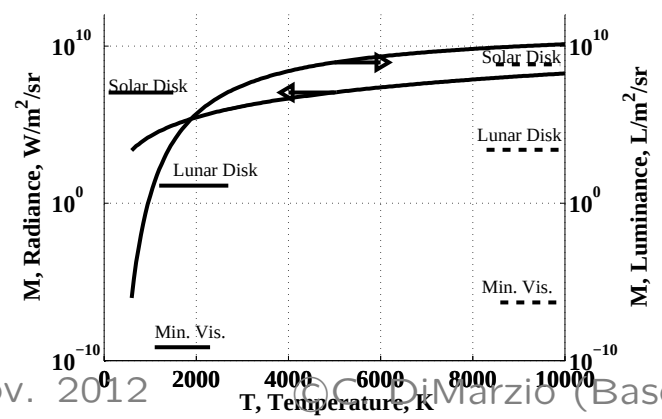
$$M_{(R)} = \int_0^\infty M_{(R)\lambda} d\lambda$$

- Luminous Exitance

$$M_{(V)} = \int_0^\infty M_{(V)\lambda} d\lambda$$

$$M_{(V)} = 683\text{lm/W} \times$$

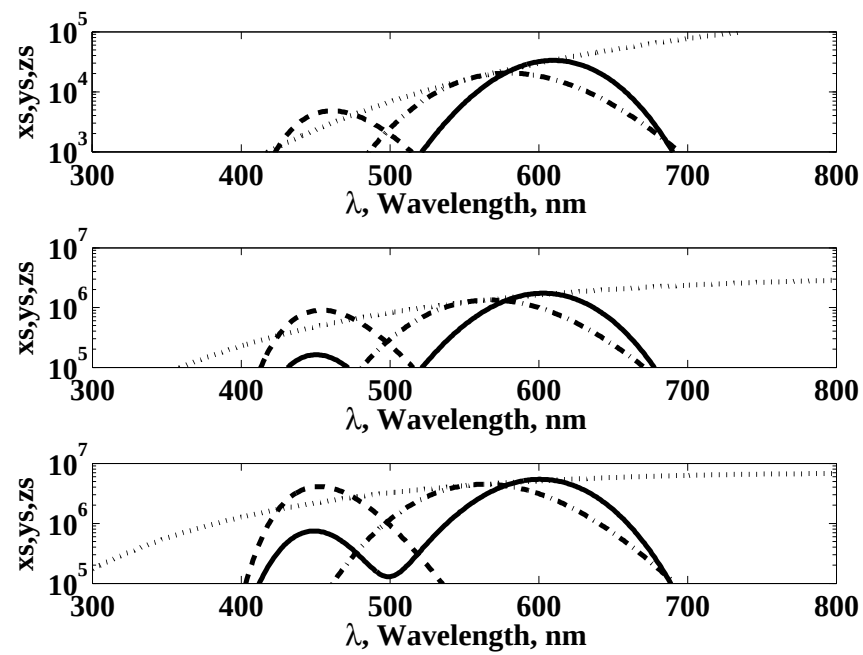
$$\int_0^\infty \bar{y} M_{(R)\lambda} d\lambda$$



- Tristimulus Values

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \int_0^\infty \begin{pmatrix} \bar{x} \\ \bar{y} \\ \bar{z} \end{pmatrix} M_{(R)\lambda} d\lambda$$

- Integrands for 2000 3000, 3500K

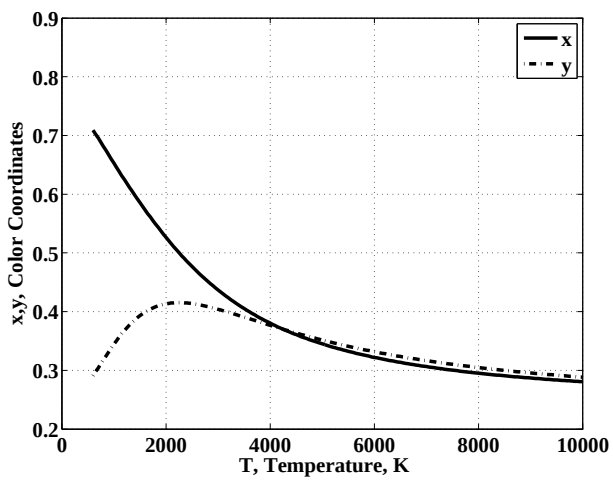


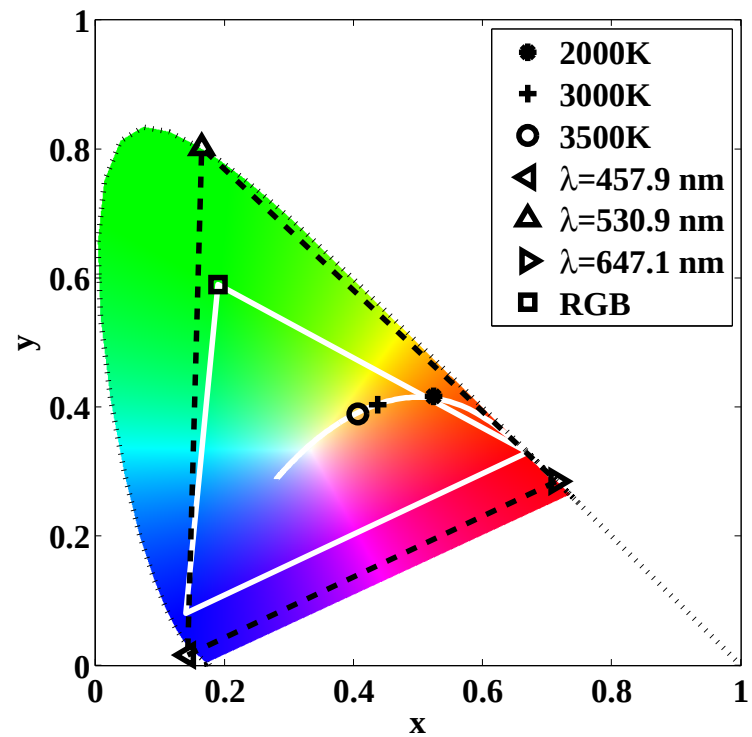
Chromaticity Coordinates of Blackbody

- Chromaticity

$$x = \frac{X}{X+Y+Z}$$

$$y = \frac{Y}{X+Y+Z}$$

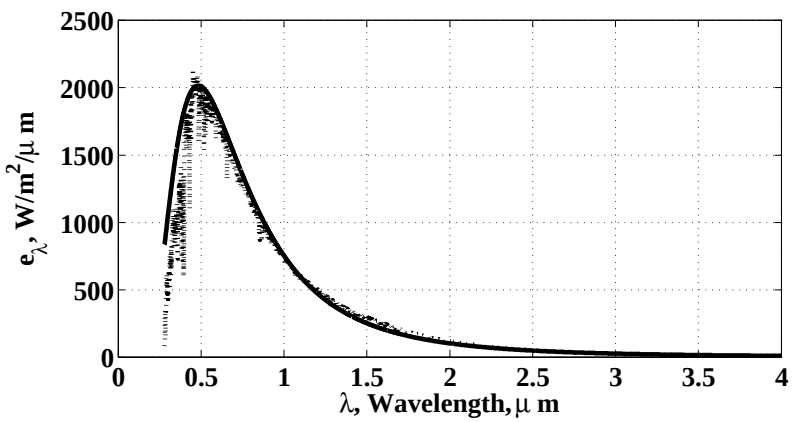




Solar Spectrum

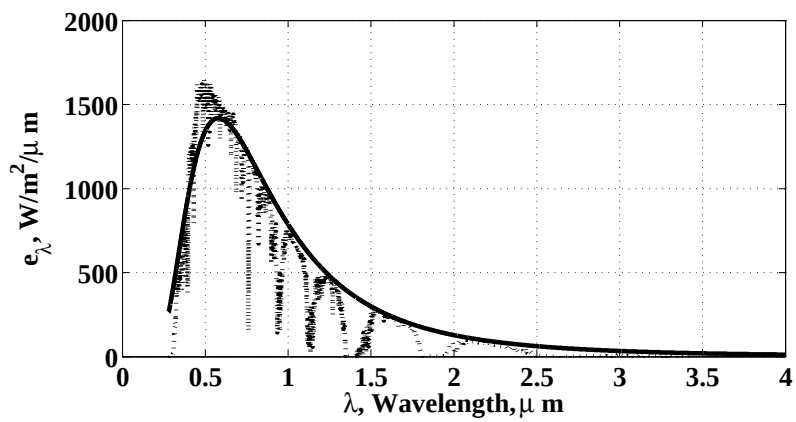
- Exo–Atmospheric
  - 6000K, 1480W/m^2

$$E_\lambda(\lambda) = 1560\text{W/m}^2 \times f_\lambda(\lambda, 6000\text{K})$$



- Sea Level
  - 5000K, 1000W/m^2

$$E_\lambda(\lambda) = 1250\text{W/m}^2 \times f_\lambda(\lambda, 5000\text{K})$$



Constants are higher than total irradiance to account for absorption in certain regions of the spectrum.

Outdoor Radiance

- Sunlit Cloud

$$E_{incident} =$$

$$1000\text{W/m}^2$$

$$5000\text{K}$$

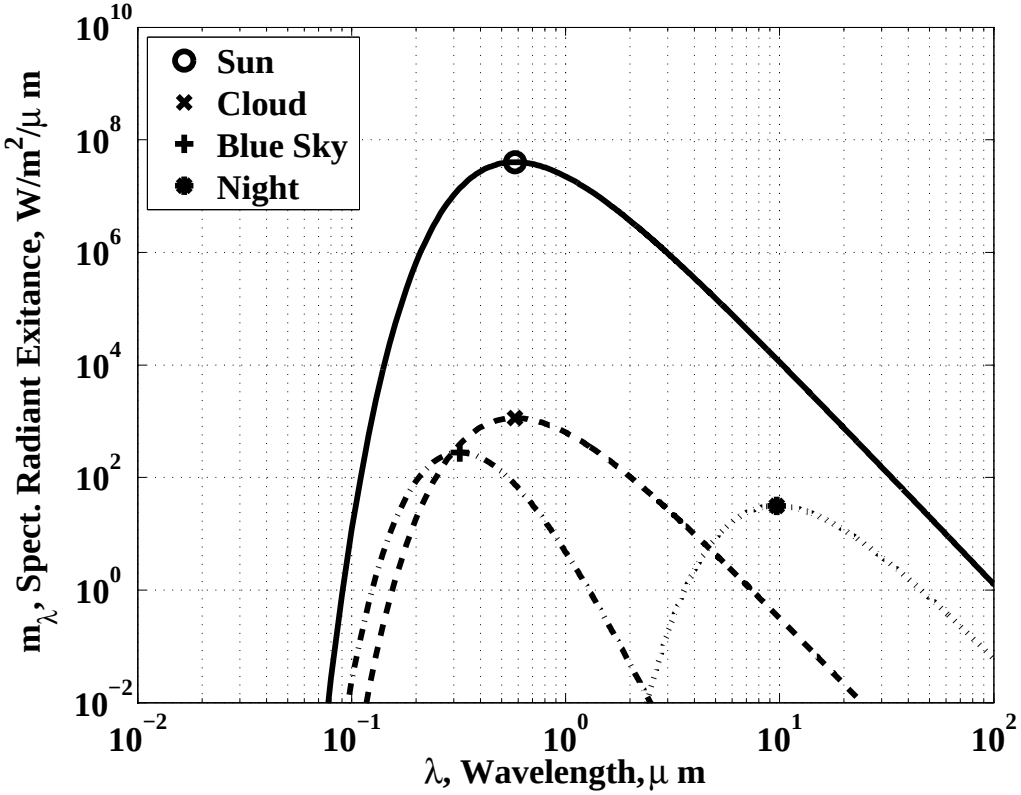
  - White Cloud

$$M_{cloud} = E_{incident}$$

  - Lambertian

$$L_{cloud} = M_{cloud}/\pi$$

- Blue Sky: $M_\lambda = M_{sky} f_\lambda(5000\text{K}) \times \lambda^{-4}$



- Radiant Exitance, M of Object Surface Illuminated with E

$$M_\lambda(\lambda) = R(\lambda) E_\lambda(\lambda)$$

Thermal Illumination

- Thermal Sources (*e*.g. Tungsten)

$$M_{(R)} = \int_0^\infty M_{(R)\lambda} d\lambda \qquad M_{(V)} = 683\text{lm/W} \int_0^\infty \bar{y} M_{(R)\lambda} d\lambda$$

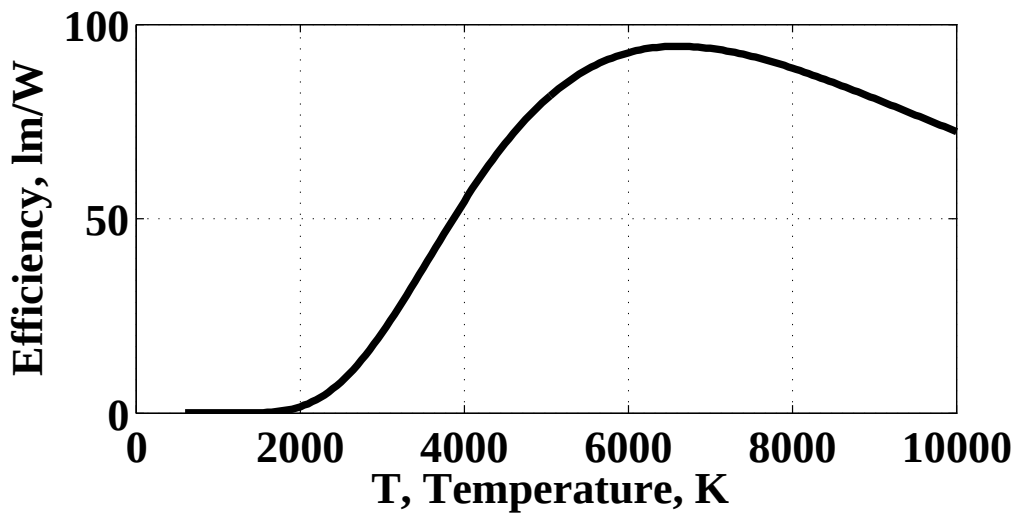
- Luminous Efficiency

$$M_{(V)}/M_{(R)} = 683\text{lm/W} \frac{\int_0^\infty \bar{y} M_{(R)\lambda} d\lambda}{\int_0^\infty M_{(R)\lambda} d\lambda}$$

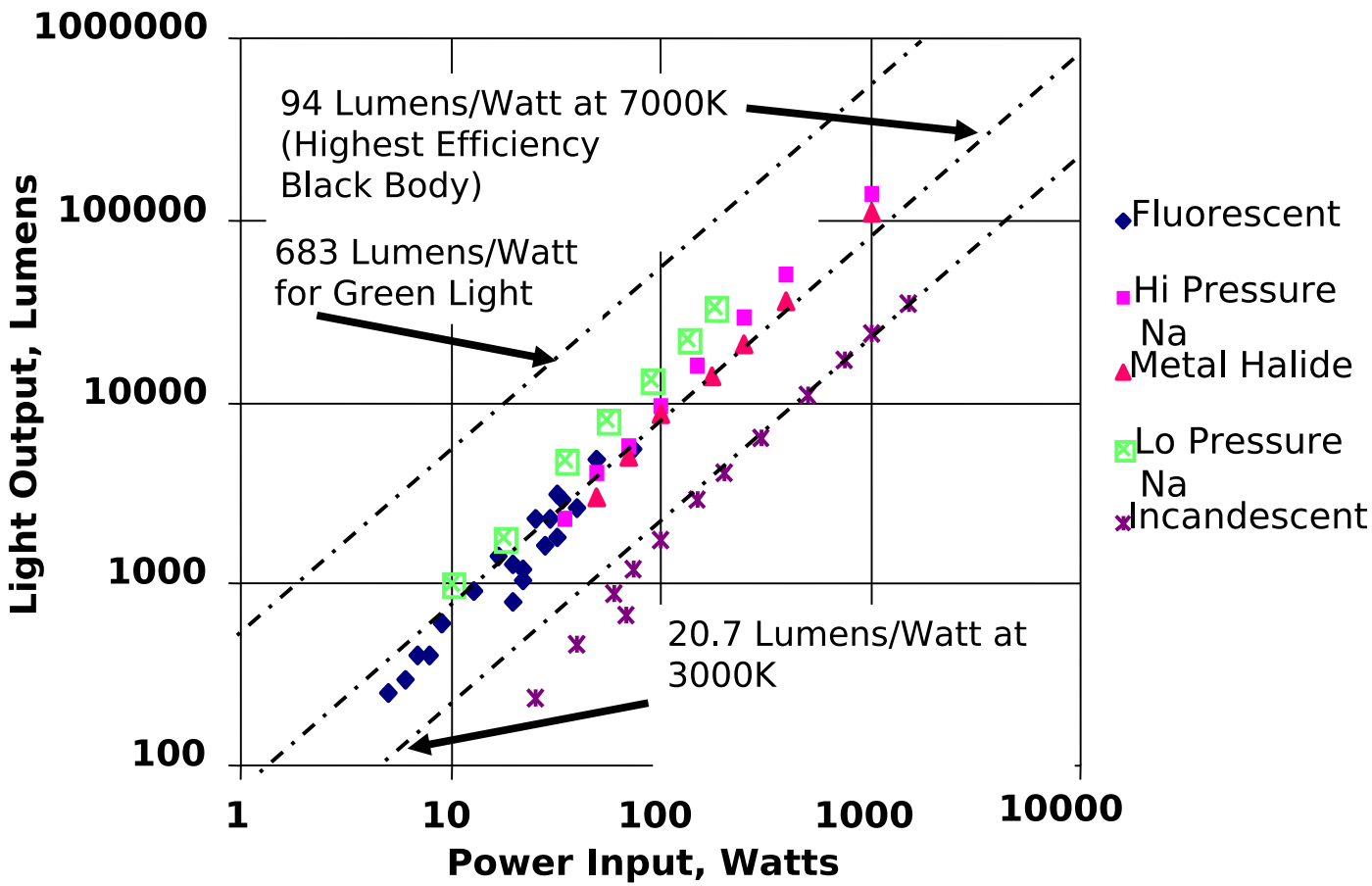
- Highest Efficiency

  $\lambda = 555\text{nm}$

- Bad Color



Illumination



Thanks to Dr. Joseph F. Hetherington

IR Thermal Imaging

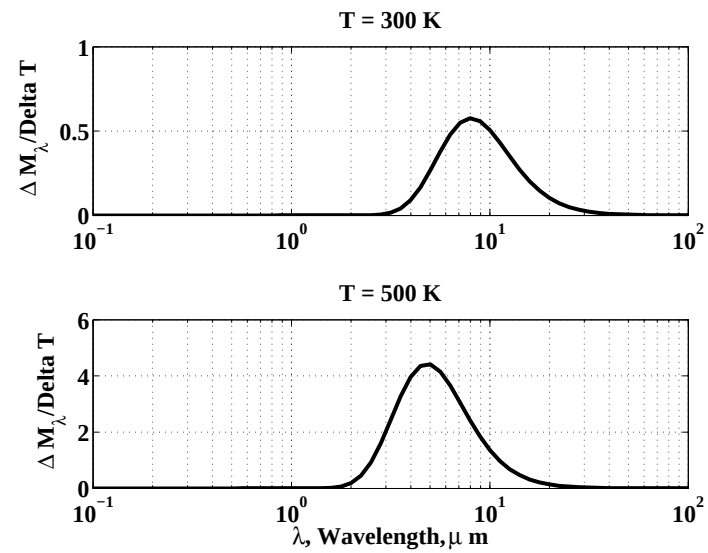
- Infrared Imaging

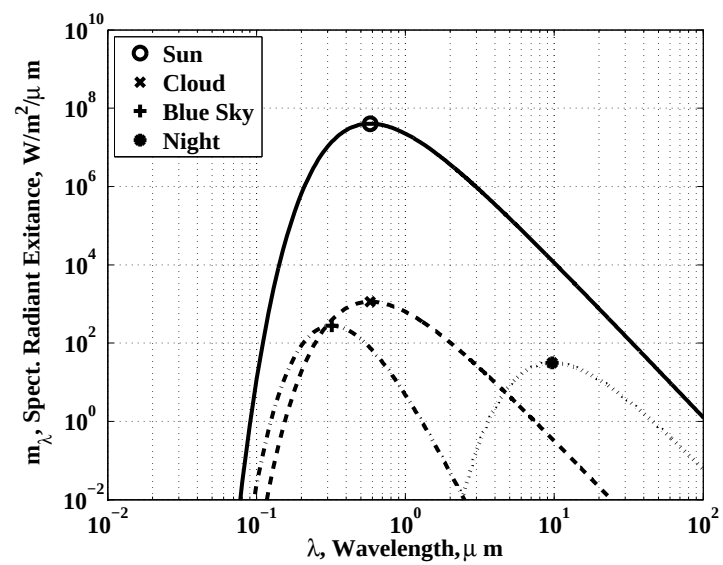
$$\int_{\lambda_1}^{\lambda_2} \rho(\lambda)\, \epsilon(\lambda)\, M_\lambda(\lambda, T)\, d\lambda$$

  - ρ is Responsivity
  - ϵ is Emissivity
- Maximize Sensitivity

$$\frac{\partial M_\lambda(\lambda, T)}{\partial T}$$

- Work in Atmospheric Pass Bands
- Shorter Wavelength for Higher Temperatures
- Hard to Calibrate (Emissivity, etc.)
- Reflectance and Emission

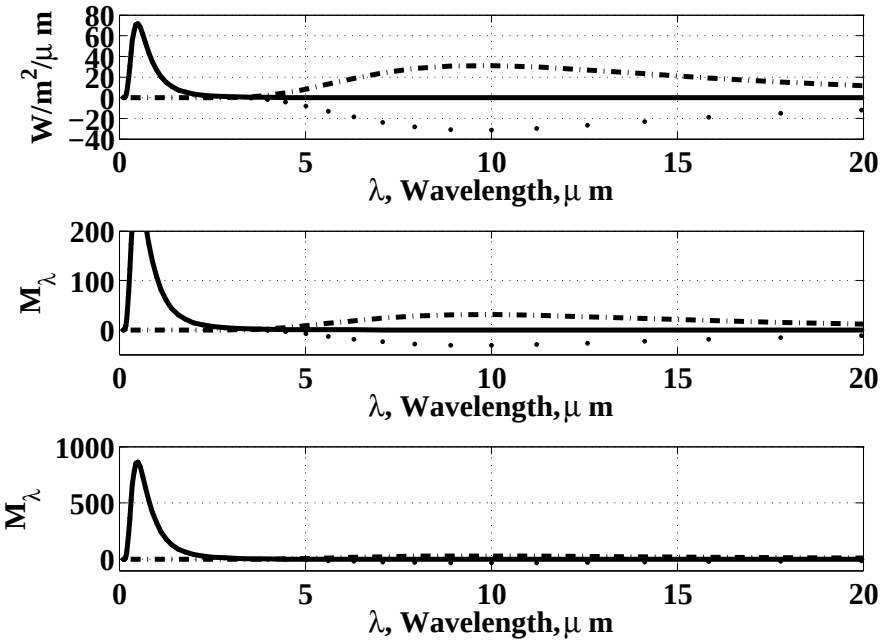




Polar Bears

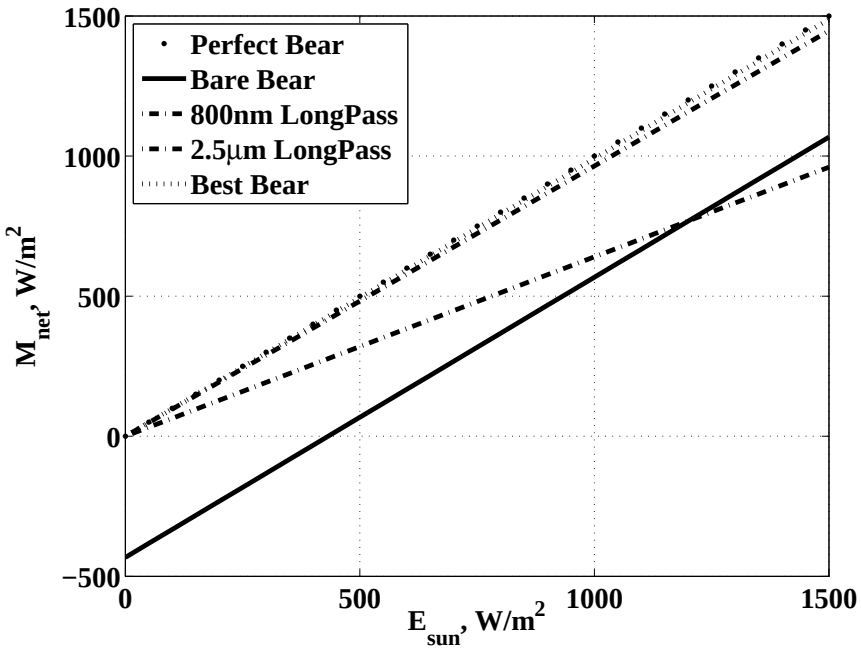
- Thermal Equilibrium
  - Heating by Sun
    - High Temperature
    - High Radiance
    - Small Solid Angle
  - Cooling to Surroundings
    - Body Temperature
    - Low Radiance
    - $\Omega = 2\pi$
  - Extra Heat from Metabolism
- Short–Pass Filter
  - Pass Visible

$E_{incident} = 50\text{W/m}^2$, 200W/m^2 600W/m^2, top to bottom



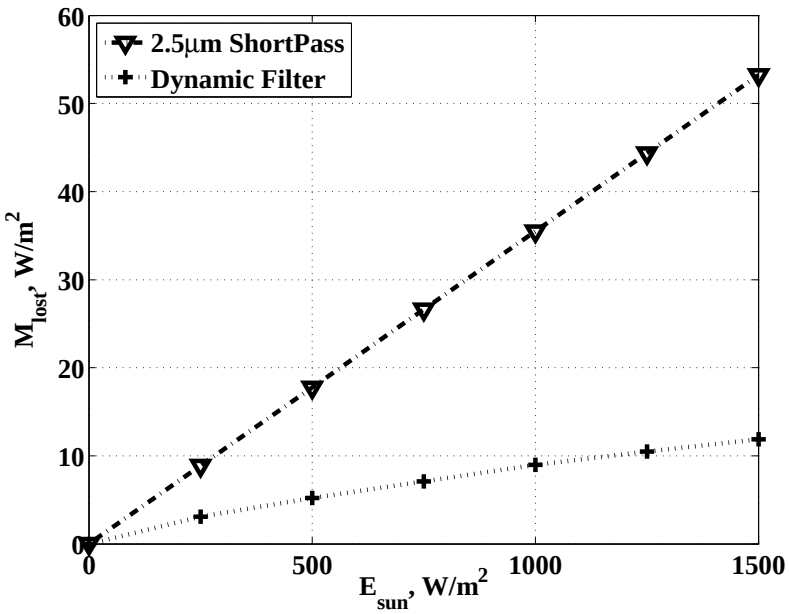
Heating (—), Cooling (- -), Net (··)

Wavelength Filtering



- Bare Bear: No Filter
- 800nm Short Pass
- 2.5nm Short Pass Like Glass
- Best Bear is a Dynamic Filter

- Net Cooling for Two Best



- Dynamic Filter Follows Zero Crossing of Net M_λ
- Perfect: No Cooling (Impossible: Nothing Out Means Nothing In)