EECE5646 Second Exam

with Solutions

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1 Lyot Filter

Here we consider a Lyot Filter, or liquid–crystal tunable filter, often used in hyperspectral imaging. A single "stage" of the Lyot filter consists of a pair of crossed polarizers with a liquid crystal sandwiched between them. A voltage is applied to the liquid crystal layer to vary its birefringence. The axes of the liquid crystal layer are oriented at 45 degrees relative to those of the polarizers. If the birefringence is set so that the layer is a half–wave plate for a wavelength, λ_{set}, then this wavelength is transmitted through the device with the only loss being from absorption and scattering in the materials. At other wavelengths, the stage will transmit less light. By adjusting the voltage, we can tune the filter.

However, there are ambiguities. If the device is a half–wave plate for λ_{set} then it is a 3/2 waveplate for $\lambda_{set}/3$ and so forth. Therefore, multiple stages are needed with different thicknesses. At λ_{set} these stages have birefringence equal to odd half–multiples of the wavelength, $OPD = N\lambda_{set}/2$ where $N = 1, 3, 5\ldots$.

1.1 Individual Stages

Plot the transmission for unpolarized light as a function of wavelength from 420 to 730 nm with the set wavelength at $\lambda_{set} = 650$ and 480 nm, for stages, $N = 1, 3, 5, 7, 9$. For my solution, I used coherency matrices.

For one stage,

$$\mathcal{J}_N = \begin{pmatrix} \cos 45^\circ & \sin 45^\circ \\ -\sin 45^\circ & \cos 45^\circ \end{pmatrix} \begin{pmatrix} e^{Nj2\pi OPD_1/\lambda} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos 45^\circ & -\sin 45^\circ \\ \sin 45^\circ & \cos 45^\circ \end{pmatrix} \quad (1)$$

The stages alternate with different polarizers,

$$\mathcal{J} = \ldots \mathcal{P}_x mx J_{n+2} \mathcal{P}_y mx J_{n+1} \mathcal{P}_x mx J_n \mathcal{P}_\dagger \quad (2)$$

The output is the trace of the output coherency matrix

$$\mathcal{C}_{out} = \mathcal{J} \frac{1}{2} \mathcal{C}_{in} \mathcal{J}^\dagger \quad (3)$$

where

$$\mathcal{C}_{in} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \quad (4)$$

1.2 Performance

Plot the overall transmission of the five–stage device ($N = 1, 3, 5, 7, 9$) on the same figure. What is the linewidth (full–width at half–maximum, FWHM) for each set wavelength?

What is the maximum transmission within the plotted band at a wavelength other than the desired one?

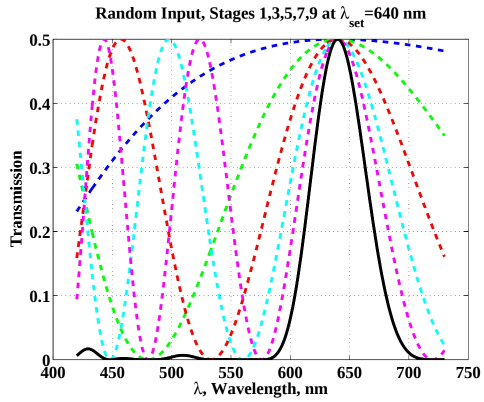
Solution below.

1.3 Improvement

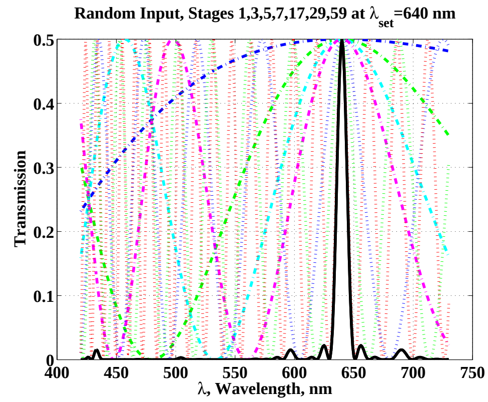
Using up to seven stages (not necessarily consecutive), see how narrow you can make the filter, while keeping the "leakage" of unwanted wavelengths low.

We don't need every stage. We need the first three to eliminate peaks within the band of interest. The rest of the peaks can be removed with an anti–aliasing filter. We need some high–order stages to narrow the passband.

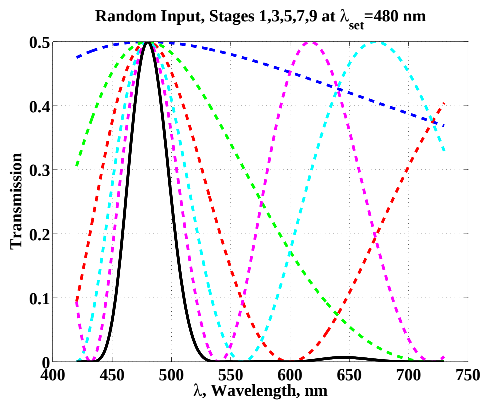
See Matlab file m11700fx1.m for solutions. Figures follow. This is just my solution. Others used different combinations with similarly good results.



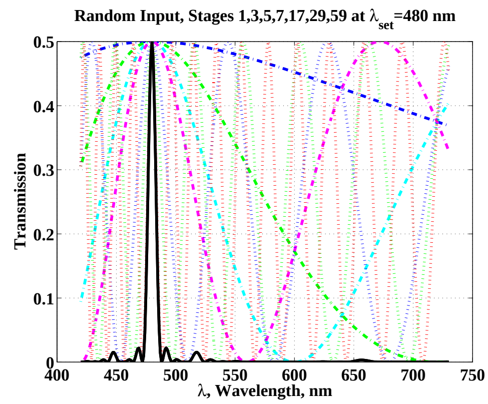
$FWHM = 52$ nm



$FWHM = 9$ nm



$FWHM = 40$ nm



$FWHM = 7$ nm

```
% m11700fx1.m % Lyot Filter Problem.
lambdaset=650e-9;close all;
for lambdaset=[640,480]*1e-9;

opd1=lambdaset/2; % First stage Half--Wave plate at set wavelength

r45=[1,1;-1,1]/sqrt(2);  % 45 degree rotation
rm45=[1,-1;1,1]/sqrt(2);  % -45 degree rotation
px=[1,0;0,0];
py=[0,0;0,1];

lambda=[300:1:1000]*1e-9;
coh_in=[1,0;0,1]/2;polstate='Random Input';
%coh_in=[1,0;0,0];polstate='X Input');

for n=1:length(lambda)
  J1=rm45*[exp(1i*2*pi*opd1/lambda(n)),0;0,1]*r45;
  J3=rm45*[exp(3*1i*2*pi*opd1/lambda(n)),0;0,1]*r45;
  J5=rm45*[exp(5*1i*2*pi*opd1/lambda(n)),0;0,1]*r45;
  J7=rm45*[exp(7*1i*2*pi*opd1/lambda(n)),0;0,1]*r45;
```

```
  J9=rm45*[exp(9*1i*2*pi*opd1/lambda(n)),0;0,1]*r45;
  J11=rm45*[exp(11*1i*2*pi*opd1/lambda(n)),0;0,1]*r45;
  J13=rm45*[exp(13*1i*2*pi*opd1/lambda(n)),0;0,1]*r45;
  J15=rm45*[exp(15*1i*2*pi*opd1/lambda(n)),0;0,1]*r45;
  J17=rm45*[exp(17*1i*2*pi*opd1/lambda(n)),0;0,1]*r45;
   JA=py*J9*px*J7*py*J5*px*J3*py*J1*px;lyotstagesA='Stages 1,3,5,7,9';
  coh_out=JA*coh_in*JA';
  TA(n)=trace(coh_out);
  coh_out=py*J1*px*coh_in*px'*J1'*py';
  T1(n)=trace(coh_out);
  coh_out=py*J3*px*coh_in*px'*J3'*py';
  T3(n)=trace(coh_out);
  coh_out=py*J5*px*coh_in*px'*J5'*py';
  T5(n)=trace(coh_out);
  coh_out=py*J7*px*coh_in*px'*J7'*py';
  T7(n)=trace(coh_out);
  coh_out=py*J9*px*coh_in*px'*J9'*py';
  T9(n)=trace(coh_out);
end;

figure;plot(lambda*1e9,TA,'k-',...
            lambda*1e9,T1,'b--',...
            lambda*1e9,T3,'g--',...
            lambda*1e9,T5,'r--',...
            lambda*1e9,T7,'c--',...
            lambda*1e9,T9,'m--',...
    lambda*1e9,TA,'k-');grid on;
xlabel('\lambda, Wavelength, nm');
ylabel('Transmission');
title([polstate,', ',lyotstagesA,...
      ' at \lambda_{set}=',num2str(lambdaset*1e9),' nm']);

for n=1:length(lambda)
  J1=rm45*[exp(1i*2*pi*opd1/lambda(n)),0;0,1]*r45;
  J3=rm45*[exp(3*1i*2*pi*opd1/lambda(n)),0;0,1]*r45;
  J5=rm45*[exp(5*1i*2*pi*opd1/lambda(n)),0;0,1]*r45;
  J7=rm45*[exp(7*1i*2*pi*opd1/lambda(n)),0;0,1]*r45;
  J9=rm45*[exp(9*1i*2*pi*opd1/lambda(n)),0;0,1]*r45;
  J11=rm45*[exp(11*1i*2*pi*opd1/lambda(n)),0;0,1]*r45;
  J13=rm45*[exp(13*1i*2*pi*opd1/lambda(n)),0;0,1]*r45;
  J15=rm45*[exp(15*1i*2*pi*opd1/lambda(n)),0;0,1]*r45;
  J17=rm45*[exp(17*1i*2*pi*opd1/lambda(n)),0;0,1]*r45;
  J29=rm45*[exp(29*1i*2*pi*opd1/lambda(n)),0;0,1]*r45;
  J59=rm45*[exp(59*1i*2*pi*opd1/lambda(n)),0;0,1]*r45;
   JB=py*J59*px*J29*py*J17*px*J7*py*J5*px*J3*py*J1*px;
```

```
   lyotstagesB='Stages 1,3,5,7,17,29,59';
  coh_out=JB*coh_in*JB';
  TB(n)=trace(coh_out);
  coh_out=py*J1*px*coh_in*px'*J1'*py';
  T1(n)=trace(coh_out);
  coh_out=py*J3*px*coh_in*px'*J3'*py';
  T3(n)=trace(coh_out);
  coh_out=py*J5*px*coh_in*px'*J5'*py';
  T5(n)=trace(coh_out);
  coh_out=py*J7*px*coh_in*px'*J7'*py';
  T7(n)=trace(coh_out);
  coh_out=py*J17*px*coh_in*px'*J17'*py';
  T17(n)=trace(coh_out);
  coh_out=py*J29*px*coh_in*px'*J29'*py';
  T29(n)=trace(coh_out);
  coh_out=py*J59*px*coh_in*px'*J59'*py';
  T59(n)=trace(coh_out);
end;

figure;plot(lambda*1e9,TB,'k-',...
            lambda*1e9,T1,'b-.',...
            lambda*1e9,T3,'g-.',...
            lambda*1e9,T5,'c-.',...
            lambda*1e9,T7,'m-.',...
            lambda*1e9,T17,'b:',...
            lambda*1e9,T29,'g:',...
            lambda*1e9,T59,'r:',...
    lambda*1e9,TB,'k-');grid on;
xlabel('\lambda, Wavelength, nm');
ylabel('Transmission');
title([polstate,', ',lyotstagesB,...
      ' at \lambda_{set}=',num2str(lambdaset*1e9),' nm']);


end;  % End of lambdaset loop

figure(1);print -depsc 11700fx1a.eps
figure(2);print -depsc 11700fx1b.eps
figure(3);print -depsc 11700fx1c.eps
figure(4);print -depsc 11700fx1d.eps
```

2 Laser Cavity

Let's look at an Argon ion laser designed for operation on the 514 nm line. Now, let's design this cavity so that the minimum beam diameter (Gaussian $1/e^2$ in the cavity is 3 mm, and the beam achieves a focus outside the cavity in front of the laser, with a waist diameter of 500 μm. The cavity must be 30 cm long.

2.1 Cavity Design

Determine

- the distance from the front mirror to the waist,
- the beam diameter at the rear mirror, and
- the radii of curvature of the two mirrors.

Is the cavity stable or unstable? Make a qualitative comment about tolerances for this design, based on stability.

Let's call the distance from the front mirror to the waist z_1. Then,

$$d_{front} = d_0 \sqrt{1 + \left(\frac{z_{01}}{b}\right)^2} \tag{5}$$

$$z_{01} = b\sqrt{\frac{d_{front}^2}{d_0^2} - 1} = 2.26 \text{ m} \tag{6}$$

$$d_{back} = d_0 \sqrt{1 + \left(\frac{z_{12} + z_{01}}{b}\right)^2} = 3.4 \text{ mm}. \tag{7}$$

$$\rho_1 = z_{01} + \frac{b^2}{z_{01}} = 2.3245 \text{ m} \tag{8}$$

$$\rho_2 = z_{12} + z_{01} + \frac{b^2}{z_{12} + z_{01}} = 2.6170 \text{ m} \tag{9}$$

Stability:

$$f_1 = -\rho_1/2 \qquad f_2 = \rho_2/2 \tag{10}$$

$$\mathcal{M}_1 = \begin{pmatrix} 1 & 0 \\ -1/f1 & 1 \end{pmatrix} \qquad \mathcal{M}_2 = \begin{pmatrix} 1 & 0 \\ -1/f2 & 1 \end{pmatrix} \qquad \mathcal{T}_{12} = \begin{pmatrix} 1 & z_{12} \\ 0 & 1 \end{pmatrix} \tag{11}$$

$$\mathcal{M}_{roundtrip} = \mathcal{T}_{12}\mathcal{M}_1\mathcal{T}_{12}\mathcal{M}_2 = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{12}$$

$$(A + D)^2 = 3.99 \tag{13}$$

The required number for stability is 4. The cavity is pretty close to unstable. Mirror tolerances will be very tight.

2.2 Multi–Wavelength Operation

Now, suppose that the laser produces output on multiple gain lines of the argon ion. In particular, where is the waist and how large is it, for the 488 nm line?

$$\rho_1 = z_{01} + \frac{b^2}{z_{01}} \tag{14}$$

$$\rho_2 = z_{12} + z_{01} + \frac{b^2}{z_{12} + z_{01}}, \tag{15}$$

where z_{01} and b are unknown. Simplifying,

$$\rho_1 z_{01} = z_{01}^2 + b^2 \tag{16}$$

$$\rho_2 z_{12} + \rho_2 z_{01} = z_{12}^2 + 2z_{12}z_{01} + z_{01}^2 + b^2 \tag{17}$$

Subtract to eliminate b;

$$\rho_2 z_{12} + \rho_2 z_{01} - \rho_1 z_{01} = z_{12}^2 + 2z_{12}z_{01} + z_{01}^2 + b^2 - z_{01}^2 - b^2 \tag{18}$$

Once we see that we can eliminate b from the equation for z_{01}, we realize that the waist location is independent of wavelength. In any case, let's complete the last steps to obtain a general answer.

$$(\rho_2 - \rho_1 - 2z_{12})\, z_{01} = -\rho_2 z_{12} - z_{12}^2 \tag{19}$$

$$z_{01} = -\frac{\rho_2 z_{12} - z_{12}^2}{\rho_2 - \rho_1 - 2z_{12}} = 2.26 \text{ m} \tag{20}$$

What does change is b,

$$b = \sqrt{(\rho_{front} - z_{01})z_{01}} = 38.2 \text{ cm} \tag{21}$$

$$d_0 = \sqrt{4\lambda b/\pi} = 487\ \mu\text{m}. \tag{22}$$

3 Mode–Locked Pulse

Here we look at a mode–locked laser pulse from a titanium–doped sapphire laser, with a 100 fs pulse length and 80 MHz pulse repetition frequency (Free Spectral Range), and a center wavelength of 730 nm.

In class, we discussed the issue of dispersion through glass and its broadening of the pulse in time.. Suppose that we correct as well as possible for the glass in our microscope so that we deliver a transform–limited pulse to the sample. However, we still have to deliver the light to different depths in the sample. Let's see if this is an important consideration.

3.1 Linewidth

We first direct the laser to a grating spectrometer and measure the distribution of the light across wavelengths. What is the width of the spectrum that we expect to see?

$$df = 1/dt = 1/100 \text{ fs.} \tag{23}$$

$$\frac{d\lambda}{\lambda} = \frac{df}{f} \tag{24}$$

$$d\lambda = \lambda^2 \frac{df}{c} = 18 \text{ nm.} \tag{25}$$

3.2 Dispersion

Here we will assume the sample consists of water and we want to image over a range of 2 mm in depth (about the maximum possible with optical coherence tomography. Look at the website http://refractiveindex.info/ for refractive index information for water. Note that there is a link there to tabular data, which is probably the most useful for this calculation. You probably want to smooth the curve to obtain a single value for $dn/d\lambda$.

Is the peak irradiance reduced by the dispersion? *Hint:* Remember that we can write the field as

$$\sum_{m=-M}^{M} e^{j2\pi(f+m\times FSR)t+j\phi_m}, \tag{26}$$

where $2M \times FSR$ is the linewidth in frequency, M is large enough to account for the full spectrum of the pulse, and we've included a phase for each frequency, ϕ_m. If ϕ varies linearly in frequency, that causes a shift in time (as in increased OPL), but not a broadening. It is a change in the OPL with frequency that changes the irradiance.

From the data, $n = 1.3315$ at 640 nm and 1.3285 at 850 so

$$\frac{dn}{d\lambda} = \frac{0.0030}{210 \text{ nm}} = -1.43 \times 10^4 \text{ m}^{-1} \tag{27}$$

The optical path difference across the spectrum is

$$OPD = d\lambda \frac{dn}{d\lambda} \ell = 50n \text{ nm}, \tag{28}$$

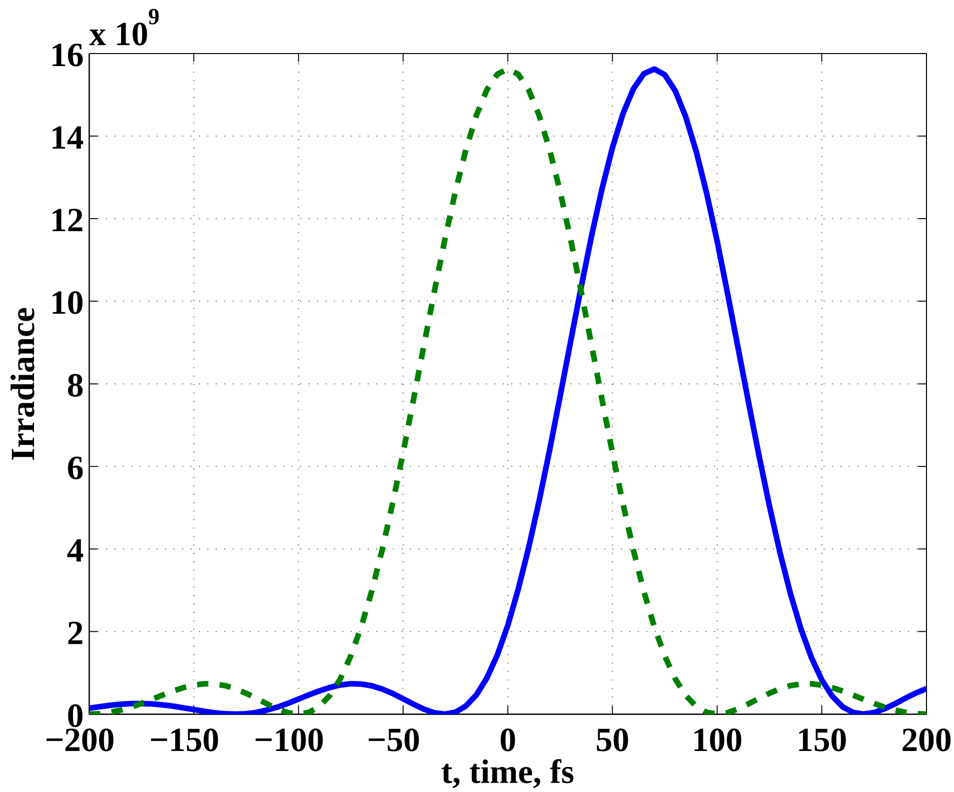
or almost a wavelength of variation across the spectrum. However, most of this is associated with the linear phase variation with frequency, and thus leads to a time delay but not an increase in pulse width. Specifically, we can write the phase as

$$\phi_m = \left(n + \frac{dn}{df} \times m \times FSR \right) \times (f + m \times FSR) \,. \tag{29}$$

Expanding

$$\phi_m = nf + nm \times FSR + \frac{dn}{df} \times m \times FSR \times f + \frac{dn}{df} \times m \times FSR \times m \times FSR. \tag{30}$$

Substituting this into the sum the maximum value (at $t = 0$) is not significantly degraded. The pulse does shift in time, of course.



```
% m11700fx3a.m sum of terms for chirp problem.
lambda0=730e-9;dt=100e-15;fsr=80e6;
df=1/dt;
dlambda=lambda0^2*df/2.99e8
ell=2e-3;
constant;
f0=c/lambda0;
nmodes=df/80e6
nmax=floor(nmodes/2);
dndlambda=(1.3285-1.3315)/210e-9;
dndf=dndlambda*dlambda/df;

fmodes=fsr*[-nmax:nmax];
lambdamodes=c./(c/lambda0+fmodes);
t=[-200e-15:5e-15:200e-15];
for nt=1:length(t);
  termstl=exp(1i*2*pi*fmodes*t(nt));
  terms=exp(1i*(2*pi*fmodes*t(nt)...
               +2*pi*ell*dndf*f0*fmodes/c...
               +2*pi*dndf*ell*fmodes.^2/c...
       ));
  fieldtl=sum(termstl);
  field=sum(terms);
  irradtl(nt)=abs(fieldtl).^2;
  irrad(nt)=abs(field).^2;
end;

figure:plot(t*1e15,irrad,'-',t*1e15,irradtl,'--');grid on;
xlabel('t, time, fs');
ylabel('Irradiance');
print -depsc 11700fx3a.eps
```

4 Radiometry

In this problem, we determine the signals we expect to see with three different cameras, viewing a terrestrial scene. We consider one camera in each of the three important bands: visible light from 400 to 800 nm, the mid–infrared band from 3 to 5 μm and the far–infrared band from 8 to 12 μm. We will consider imaging both with scattered sunlight and thermal emission.

Hint: Calculation of the photon count in each band will require integration. One good approach is to calculate the spectral photon count, N_λ in photons per unit wavelength at each end of the band and then do a two–point trapezoidal integration using

$$N_{band} \approx \frac{N_\lambda(\lambda_1) + N_\lambda(\lambda_2)}{2} \times (\lambda_2 - \lambda_1)\,. \tag{31}$$

4.1 The Cameras

Here we relate the spectral photon count at the camera (in photons per unit wavelength) to the target spectral radiance. In the remaining sections, we will use numbers appropriate to the different wavelengths and targets. For simplicity and comparison, let's assume that all cameras are the same, with square pixels, 20 μm on a side and an f/2 lens. Of course the camera technologies would be quite different, using different detector materials, different lens materials, and more.

Assume that the target has a spectral radiance, $L_\lambda(\lambda)$. Write an equation for the spectral flux on a pixel in Watts per unit area.

Now, assume that the integration time is (1/30) sec, and write an equation for the spectral photon count, $N_\lambda(\lambda)$, in photons per unit wavelength.

$$\Omega \approx \frac{\pi D_{pupil}^2}{4f^2} = \frac{\pi}{16} \approx 0.2. \tag{32}$$

$$\begin{aligned}\Phi_\lambda(\lambda) = L_\lambda(\lambda)\,A\Omega &\approx L_\lambda(\lambda) \times \left(20 \times 10^{-6}\ \mathrm{m}\right) \times 0.2 \\ &= L_\lambda(\lambda) \times 8 \times 10^{-11}\ \mathrm{m}^2 \end{aligned} \tag{33}$$

The spectral photon count is

$$\begin{aligned}N_\lambda(\lambda) = \frac{\Phi_\lambda(\lambda)\,\tau}{h\nu} = \frac{\Phi_\lambda(\lambda)\,\tau\lambda}{hc} &= \frac{L_\lambda(\lambda)\,\tau\lambda}{hc} \times 8 \times 10^{-11}\ \mathrm{m}^2 \\ &\approx L_\lambda(\lambda) \times \lambda \times 1.3 \times 10^{13}\ \mathrm{m}^2/\mathrm{W}. \end{aligned} \tag{34}$$

4.2 Thermal Emission

Now consider the thermal emission of the target at 300 Kelvins. Compute the number of photons detected in a pixel in each of the three wavelength bands.

$$L_\lambda(\lambda) = M_\lambda(\lambda)/\pi \tag{35}$$

where $M_\lambda(\lambda)$ is given by the Planck Equation. We convert to photons per micrometer of wavelength using Equation 34 and then integrate using Equation 31.

4.3 Scattered Sunlight

Repeat the process for scattered sunlight. Assume that the sun is a 5000 K black–body source with 1000 W/m^2 irradiance at the surface of the target. Assume that the surface of the target reflects ten percent of the light in a Lambertian pattern.

Compute the spectral fraction,

$$f_\lambda(\lambda) = \frac{M_\lambda(\lambda)}{\sigma T^4}, \tag{36}$$

where σ is the Stefan–Boltzmann constant.

The solar irradiance on the surface is

$$E_\lambda(\lambda) = 1000\ \mathrm{W/m^2} f_\lambda(\lambda). \tag{37}$$

The radiant exitance scattered is ten percent of this level, and it is emitted in a Lambertian pattern, so

$$L_\lambda(\lambda) = 1000\ \mathrm{W/m^2} \times \frac{0.1}{\pi} \times f_\lambda(\lambda). \tag{38}$$

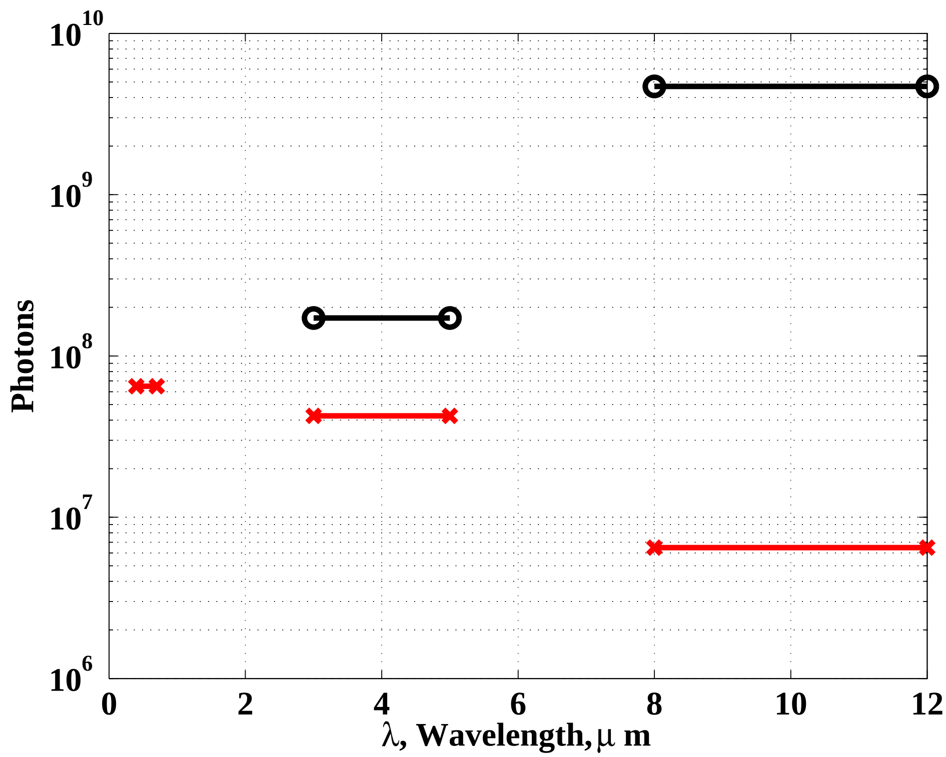
From here we follow the same procedure as in the previous section.

4.4 Summary

Present the results in some reasonable way, and discuss them. Consider which situations produce usable signals, which bands are more appropriate to thermal or reflected–light imaging, and which would work at night.

Results are shown in the figure below, except for the visible emission, which is about 10^{-15} photons. This means that we'd need to increase the imaging time to about a million years to obtain a mean number of one photon. The rest of the values are all sufficient to produce good signals.

In the visible spectrum, we can image with reflected sunlight, but not thermal emission, as expected. At night, we would need active illumination. In the far–infrared, thermal emission is a couple of orders of magnitude stronger than reflected sunlight. A camera designed for this band would likely not see the reflected–light images. In the mid–infrared, both modes produce nearly equal signals. Both infrared modes will work at night, but the mid–infrared one might look different because of the absence of reflected sunlight.




```
% m11700fx4.m Solution to camera problem for second exam

constant % sets constants
T_over_hc=(1/30)/h/c
A_Omega_T_over_hc= (20e-6)^2*.2*(1/30)/h/c

% Thermal radiance at 300K
lambda=[0.4,0.7,3,5,8,12]; % wavelength in microns
mlambda=bbspec(300,lambda); % Spectral radiant exitance W/m^2/micron

llambda=mlambda/pi
nlambda=llambda.*(lambda*1e-6)*1.3e13; % photons per micron of wavelenth
% above lambda must be in microns because it comes from lambda/hc and
% hc is in joule-meters. In hindsight, should have done this whole
% thing in MKS units.

pvis300=sum(nlambda([1:2]))/2*(lambda(2)-lambda(1))
pmid300=sum(nlambda([3:4]))/2*(lambda(4)-lambda(3))
pfar300=sum(nlambda([5:6]))/2*(lambda(6)-lambda(5))

% Reflected sunlight

mlambdasun=bbspec(5000,lambda);
flambdasun=mlambdasun/(stefan_boltzmann*5000^4); % Spectral Fraction
```

```
llambdarefl=1000*flambdasun*0.1/pi % 1000W/m^2 times reflectivity
nlambdarefl=llambdarefl.*(lambda*1e-6)*1.3e13; % photons
                                              % per micron of
                                              % wavelegth

pvisrefl=sum(nlambdarefl([1:2]))/2*(lambda(2)-lambda(1))
pmidrefl=sum(nlambdarefl([3:4]))/2*(lambda(4)-lambda(3))
pfarrefl=sum(nlambdarefl([5:6]))/2*(lambda(6)-lambda(5))

figure;semilogy(lambda([3:4]),pmid300*[1,1],'-ko',...
lambda([5:6]),pfar300*[1,1],'-ko',...
lambda([1:2]),pvisrefl*[1,1],'-rx',...
                lambda([3:4]),pmidrefl*[1,1],'-rx',...
lambda([5:6]),pfarrefl*[1,1],'-rx');grid on;
xlabel('\lambda, Wavelength, \mu m');
ylabel('Photons');

print -depsc 11700fx4a.eps
```