Homework Set 1 Solutions

For each problem please turn in a complete answer. In particular you should include any narrative required to understand your method and comment on your results where appropriate, label and place captions on any figures you include, and in general think of your homework as a short report. The intent is not to have you do a lot of writing for the sake of writing, and you can certainly assume that we know what the problem statement was, but your homework should be professionally done and we should not have to guess at how to interpret what you hand in.

The homework contains some challenges. If things are too challenging, you need to figure that out and ask about it ahead of time. So what we all know is the standard undergraduate approach of waiting until the night before to start the homework may be even more dangerous here than in most undergraduate classes. Start early and leave the procrastinating to us professionals.

Problem 1: Granite Chief

This isn’t really a subsurface problem, but it tests your understanding of imaging systems, resolution, and pixelation.\footnote{My spell checker rejects this word, suggesting “pixilation.” The Oxford English Dictionary definition for “pixilated” is, “Chiefly \textit{U.S. regional}. Slightly crazed; bewildered, confused; fey, whimsical; (also) intoxicated.” The OED does also list “pixelated” with my intended meaning, “Of an image: captured, reproduced, or displayed as pixels, usually with a grainy or low-resolution result”} Suppose that I wish to image a mountain 9,050 feet high at a distance of 2 miles. I want the top of the picture to be at the peak, and the bottom to be at the Valley floor, which is at 6,200 feet. The camera is an old CCD consisting of 640 by 480 pixels, spaced 10 micrometers apart. This is not a particularly good problem for hyperspectral imaging in the visible spectrum, because the mountain in question is almost completely white at this time of year (To be sure that this is correct, I have inspected the area personally.).
(a) What is the angular field of view? In other words, what angle does the target subtend (from the top of the mountain to the Valley floor, as viewed from the camera, located in the Valley?)

Solution:

The height from top to bottom is

\[ x = 9050 \text{ ft} - 6200 \text{ ft} = 2850 \text{ ft} = 877 \text{ m}. \]  

(1)

The object distance is

\[ s = 2 \text{ miles} = 3200 \text{ m}. \]  

(2)

The angle is approximately

\[ \alpha = \frac{x}{s} = 0.27 \text{ radians}. \]  

(3)

Strictly, we should take the arctangent, but because the angle is small, and we don’t need a very precise answer, it is sufficient to approximate \( \alpha = \tan \alpha \).

End of Solution

(b) What magnification is required?

Solution:

I want the image of the mountain to cover 480 pixels at 10 \( \mu \text{m} \) each, so

\[ x' = 480 \times 10 \mu \text{m} = 4.8 \times 10^{-3} \text{ m}. \]  

(4)

Therefore, I want the magnification to be

\[ m = -\frac{x'}{x} = -5.5 \times 10^{-6}. \]  

(5)

I used the minus sign because I know the image is going to be inverted.

End of Solution

(c) What is the size of the 10–micrometer pixel projected onto the side of the mountain?

Solution:

\[ x_{\text{pixel}} = \frac{x'_{\text{pixel}}}{m} = \frac{10 \mu \text{m}}{-5.5 \times 10^{-6}} = -1.8 \text{ m}. \]  

(6)

The pixel is about 1.8 m square.

End of Solution
(d) What focal length lens is needed?

**Solution:**

The image distance is

\[ s' = -\frac{x'}{x} s = 0.0176 \text{ m}, \tag{7} \]

and the focal length is given by

\[ \frac{1}{f} = \frac{1}{s} + \frac{1}{s'} = 0.0176 \text{ m}. \tag{8} \]

Note that the object distance, \( s \), is so long that it can be approximated as infinity, and we could have easily said \( f \approx s' \). If anything, we want to choose the focal length a little small, so that the mountain won’t overfill the camera, so I’d pick something a little smaller than \( f = 17 \text{ mm} \).

**End of Solution**

(e) If the camera has a lens diameter equal to one tenth of the focal length, what is the resolution? Does the image of a point underfill or overfill a pixel?

**Solution:**

The resolution (in angle) is

\[ 2.44 \frac{\lambda}{D} = 2.44 \times 10 \frac{\lambda}{f} = 6.9 \times 10^{-4} \text{ Radians}. \tag{9} \]

We can compare to the pixel by multiplying by \( f \) to get the size on the camera or \( s \) to get the size on the object. In the former case, it’s easy;

\[ 2.44 \times 10 \frac{\lambda}{f} \times f = 2.44 \times 10 \lambda = 1.2 \times 10^{-5} \text{ m}. \tag{10} \]

The spot is about 12 \( \mu \text{m} \) in diameter, so it slightly overfills the pixel.

**End of Solution**

**Problem 2: San Francisco Fog**

Solve Problem 2.2.10 in the text.

**Solution:**

The book solution is first.
EXERCISE 2.2-10

Transmission through Fog. Consider a large patch of fog sufficiently thick that, seen from the air, it is not possible to see the ground. Suppose that the optical properties of this fog bank are: \( \mu_s = 10^{-6} \text{ m}^{-1}, \ g = 0.8, \) and \( \mu_a = 7 \times 10^{-4} \text{ m}^{-1}. \)

(a) Compute the diffuse reflectance \( \mathcal{R}. \)
(b) Now suppose that embedded in the fog bank is a denser fog layer of height \( d = 100 \text{ m}, \) having the same properties, except that \( \mu_a' = 10^{-5} \text{ m}^{-1}, \) beginning at a depth \( d_1 \) and ending at a depth \( d_1 + d. \) Determine the reflectance and plot it as a function of \( d_1. \)

(a) \( \mu_s = 10^{-6} \text{ m}^{-1}, \ g = 0.8, \ \mu_a = 7 \times 10^{-4} \text{ m}^{-1} \)

Reduced scattering coefficient \( \mu_s' = (1 - g)\mu_s = 2 \times 10^{-7} \text{ m}^{-1}. \)

Since \( \mu_s' \ll \mu_a \), we can use the approximate formula (2.2-40) with \( d = \infty \) and \( \mu_a \) replaced by \( \mu_s' \) to obtain

\[
\mathcal{R} = \frac{\mu_s'}{2(\mu_s' + \mu_a)} = 1.43 \times 10^{-4}
\]

(b) Here also \( \mu_s' \ll \mu_a \) in all layers, we apply (2.2-40)

\[
\mathcal{R} = \frac{1}{2} \mu_a \left[ 1 - \exp(-2\mu_d) \right] + \mathcal{R}_0 \exp(-2\mu_d).
\]

at point A and then at point B.

Applying this formula at point A, we use \( R_0 = 1.43 \times 10^{-4}, d = d_1 = 100, \mu_s' = 2 \times 10^{-6}, \mu_a = 7 \times 10^{-4} \), and obtain \( R = 1.45 \times 10^{-4}. \)

Applying this formula at point B, we use \( R_0 = 1.45 \times 10^{-4}, d = d_1, \mu_s' = 2 \times 10^{-7}, \mu_a = 7 \times 10^{-4} \), and obtain the following graph for the reflectance at point B as a function of \( d_1 \). As expected, as \( d_1 \) increases, \( R \) no longer depends on \( d_1 \) since the wave never reaches the boundary.

Now, here is my own solution.

First Layer:
\[
\int_0^{d_1} \frac{1}{2} \mu_s' e^{-2(\mu_s'+\mu_a)} dz = 
\]

\(11\)

\[
\frac{1}{2} \mu'_s \left(1 - e^{-2(\mu'_s+\mu_a)d_1}\right)
\]

\(12\)

**Second Layer:**

\[
e^{-2(\mu'_s+\mu_a)d_1} \times \int_{d_1}^{d_1+d_1} \frac{1}{2} \mu'_s e^{-2(\mu'_s+\mu_a)} dz =
\]

\(13\)

\[
e^{-2(\mu'_s+\mu_a)d_1} \times \frac{1}{2} \mu'_s \left(1 - e^{-2(\mu'_s+\mu_a)d_1}\right)
\]

\(14\)

**Third Layer:**

\[
e^{-2(\mu'_s+\mu_a)d_1} e^{-2(\mu'_s+\mu_a)d} \times \int_{d_1+d_1}^{\infty} \frac{1}{2} \mu'_s e^{-2(\mu'_s+\mu_a)} dz =
\]

\(15\)

\[
e^{-2(\mu'_s+\mu_a)d_1} e^{-2(\mu'_s+\mu_a)d} \times \frac{1}{2} \mu'_s \frac{1}{\mu'_s + \mu_a}_.
\]

\(16\)

**Matlab Code:**

%%% two2ten.m  Solution to Problem 2.2-10 in Saleh’s Subsurface sensing and imaging text.
% by Chuck DiMarzio
% Northeastern University
% March 2011
us0=1e-6;g0=0.8;ua0=7e-4;us0p=us0.*(1-g0);
d=100;d1=[0:10:4000];
us1=1e-5;g1=g0;ua1=ua0;us1p=us1.*(1-g1);
% R with no intermediate layer for part A.
% multiply by ones(size(d1)) to make it have the right dimensions for plot
R0=us0p/2/(us0p+ua0).*ones(size(d1));
% 3 terms for 3 layers.
term1=us0p/2/(us0p+ua0).*(1-exp(-2*(us0p+ua0).*d1));
term2=us1p/2/(us1p+ua1).*(1-exp(-2*(us1p+ua1).*d1))...
\*exp(-2*(u0p+u0).*d1);

\text{term3} = \frac{u0p}{2} / (u0p+u0) * \*exp(-2*(u0p+u0).*d1) * \*exp(-2*(u1p+u1).*d);

\text{R} = \text{term1} + \text{term2} + \text{term3};

figure; plot(d1, term1, 'b-.',
            d1, term2, 'g--',
            d1, term3, 'r:',
            d1, R, 'k-',
            d1, R0, 'c'); grid on;

xlabel('d_1, m'); ylabel('R');

legend('Layer 1', 'Layer 2', 'Layer 3', 'Total', 'Uniform');
Problem 3: Acoustic Propagation

Solve Problem 2.2-3 in the text.

Solution:

EXERCISE 2.2-3

Attenuation of an Acoustic Wave in Water. The attenuation coefficient of an ultrasonic wave in water at 1 MHz corresponds to a half-value thickness of 3.4 m. What is the maximum depth at which a target can be detected using a probe wave producing an intensity of 100 mW/cm² if the detector’s sensitivity is 1 mW/cm²? Assume that the transducer is immersed in water so that reflection is not present. Determine the maximum depth for subsurface imaging if the frequency is increased to 2 MHz.

For a medium with attenuation coefficient $\alpha$, the half-value thickness is $(\ln 2)/\alpha$.

Therefore, a half-value thickness of 3.4 m corresponds to $n = (\ln 2)/3.4 = 0.204$ m$^{-1}$.

An intensity of 100 mW/cm² is reduced to a value 1 mW/cm² in a distance $d$ such that

\[
\frac{1}{100} = \exp(-\alpha d),
\]

from which $\alpha d = \ln(100)$, or $d = \ln(100)/0.204 = 22.6$ m.

For water, $\alpha \propto \omega^a$, where $a = 2$. If the frequency is doubled, the attenuation coefficient is quadrupled, i.e., $\alpha = 0.815$ m$^{-1}$. In this case the maximum depth is reduced by a factor of 4, from 22.6 m to 4.65 m.

End of Solution