DEPARTMENT OF ELECTRICAL AND COMPUTER ENGINEERING

NORTHEASTERN UNIVERSITY

ECEG105 OPTICS FOR ENGINEERS Fall 2007

Homework Set 4: Diffraction

Problem 13 — Gaussian Beams.

A helium–neon laser (633 nm) produces a collimated output beam with a diameter of 0.7 mm.

a. If the laser is 20 cm. long, what are the radii of curvature of the two mirrors?

The output being collimated dictates that the wavefronts are plane there, which means that the front mirror is a plane mirror. For the rear mirror;

$$b = \frac{\pi d_0^2}{4\lambda} = \frac{\pi(7\times10^{-4}m)^2}{4\times10^{-9}m} = 0.61m$$

$$\rho = z + \frac{b^2}{z} = 0.2m + \frac{(0.61m)^2}{0.2} = 2.1m$$

b. How big is the beam at the back mirror?

$$d = d_0\sqrt{1+\left(\frac{z}{b}\right)^2} = 0.7mm\sqrt{1+\left(\frac{20}{61}\right)^2} = 0.74mm$$

c. I now wish to launch the beam into a fiber whose core is best matched to a Gaussian beam of 5 micrometer diameter. Figure out how to do this, using one or more lenses in a reasonable space.

There are a number of good solutions. The easiest is to start from the waist of the beam entering the fiber. Find where the beam is 0.7 mm in diameter.

$$d = d_0\sqrt{1+\left(\frac{z}{b}\right)^2}$$

$$\left(\frac{z}{b}\right) = \sqrt{\left(\frac{d}{d_0}\right)^2 - 1}$$

$$z = \pm b\sqrt{\left(\frac{d}{d_0}\right)^2 - 1} = \pm\frac{\pi d_0^2}{4\lambda}\sqrt{\left(\frac{d}{d_0}\right)^2 - 1}.$$

With all distances in micrometers,

$$z = \pm\frac{\pi 5^2}{4\times0.633}\sqrt{\left(\frac{700}{5}\right)^2 - 1} \approx \pm\frac{\pi 5^2}{4\times0.633}\frac{700}{5} = \pm4.4E3.$$

We take the negative square root because we are to the left of the waist, so the waist is at -4.4 mm. We place a lens at this location, and put the laser immediately in front of it. The focal length of the lens must be chosen so that the out radius of curvature is ρ given by

$$\rho = z + \frac{b^2}{z}$$

with

$$b = \left(\frac{\pi d_0^2}{4\lambda}\right) = \left(\frac{\pi 5^2}{4 \times 0.633}\right) = 31 \mu m$$

$$\rho = -4.4mm + \frac{0.031mm}{-4.4mm} \approx -4.4mm$$

Thus the lens must have a focal length of 4.4 mm.

Problem 14 — Resolution.

A car approaches along a long straight road. Initially it is so far away that the two headlights appear as one.

a. Using the Rayleigh criterion for resolution, at what distance do we begin to see that they are two separate lights? Use reasonable numbers for the spacing of the headlights and the diameter of the pupil of the eye.

$$d = 1.22 \frac{\lambda}{D} z,$$

where d is the spacing of the ights, and D is the aperture diameter of the pupil.

$$z = \frac{Dd}{1.22\lambda} = \frac{1.5m0.008m}{1.22500 \times 10^{-9}m} = 1.97 \times 10^4 m,$$

or about 20 km.

b. Now, consider the fact that we have two eyes, separated by a distance of a few centimeters. Does this change your answer? Why or why not?

The headlights image separately on each retina, so the eyes have no mechanism for comparing two different wavefronts across the two eyes. In order to improve the resolution, the signals received by the two eyes would have to be processed coherently in some way. Once the retinas have detected the signals this is no longer possible.

Problem 15 — Koehler Illumination .

A common way of illuminating an object for a microscope is to focus the filament of a tungsten lamp on the pupil of the system (which, if the lenses are modeled as thin, means focussing it in the plane of the objective lens). The image plane will then contain the two–dimensional Fourier transform of the filament. The idea, as I have mentioned in class, is that the illumination is thus very uniform. Start with the following code:

```
\% Koehler Illumination
\%
\% By Chuck DiMarzio, Northeastern University, April 2002
\%
```

```
\% coherent source
\%
[x,y]=meshgrid(1:256,1:256);
coh=zeros(size(x));
coh(60:69,50:200)=1;
coh(80:89,50:200)=1;
coh(100:109,50:200)=1;
coh(120:129,50:200)=1;
coh(140:149,50:200)=1;
coh(160:169,50:200)=1;
coh(180:189,50:200)=1;
subplot(2,2,1);imagesc(coh);colormap(flipud(bone));colorbar;
title('Coherent Source');
```

a. Now, take the 2–D Fourier transform and image the magnitude of the result. A couple of hints are in order here. Try "help fft2," and "help fftshift." If you haven't used subplots, try "help subplot," to see how to display all the answers in one figure.

b. The result above is not very uniform, but the prediction was that it would be. See if you can figure out why it doesn't work. You have two more subplots to use. On the third one, show the corrected source, and ...

c. on the fourth one show the correct image (Fourier transform).

There are different degrees of correctness possible in part b, which will affect the results somewhat. You only need to develop a model which is "correct enough," to show the important features.

```
% m10353h.m Koehler Illumination
%
% By Chuck DiMarzio, Northeastern University, April 2002
%
% coherent source
%
[x,y]=meshgrid(1:256,1:256);
coh=zeros(size(x));
coh(60:69,50:200)=1;
coh(80:89,50:200)=1;
coh(100:109,50:200)=1;
coh(120:129,50:200)=1;
coh(140:149,50:200)=1;
coh(160:169,50:200)=1;
coh(180:189,50:200)=1;
subplot(2,2,1);imagesc(coh);colormap(flipud(bone));colorbar;
title('Coherent Source');
imgcoh=abs(fftshift(fft2(coh))).^2;
subplot(2,2,2);imagesc(imgcoh);colormap(flipud(bone));colorbar;
```

```
title('Coherent Image');
%
% incoherent source
inc=coh.*exp(i*2*pi*rand(size(coh)));
subplot(2,2,3);imagesc(real(inc));colormap(flipud(bone));colorbar;
title('Incoherent Source');
imginc=abs(fftshift(fft2(inc))).^2;
subplot(2,2,4);imagesc(imginc);colormap(flipud(bone));colorbar;
title('Incoherent Image');
```

Results are shown in Figure 14 The essential point here is this: Each point in the pupil plane illuminates the whole image plane. If the light is coherent, these contributions interfere in just the right places to produce the Fourier transform of the pupil function. In the case of incoherent light, the interference effects are random and tend to cancel, producing more–or–less uniform illumination. The more points contributing, the smaller will be the fluctuations.

Note that in the picture, I used "colormap(flipud(bone))" to produce something that would print well in black and white.

Problem 16 — Sectioning with a Confocal Microscope.

Consider a microscope with a digital camera, in which a particular pixel has a sensitivity function given by,

$$E_r = \frac{1}{w_r^2}$$

for

$$-\frac{w_r}{2} < x < \frac{w_r}{2} \quad \text{and} \quad -\frac{w_r}{2} < y < \frac{w_r}{2}$$

and 0 elsewhere, where

$$w_r = w_{r0}\sqrt{\left(\frac{z - z_{r0}}{b_r}\right)^2 + 1}.$$

The parameters in this equation are shown in Figure 15.

Note that a realistic function would be more complicated, but this one illustrates the concepts with sufficient clarity and does not involve messy integrals. Let's be specific about the dimensions:

$$w_{r0} = 5 \text{ micrometers}$$

$$b_r = 25 \text{ micrometers}$$

$$z_{r0} = 4 \text{ millimeters}$$

The goal of this problem is to look at the "sectioning" capability of the confocal microscope. We will look at two different z locations, one near the focus, and one not, and see what component of the total signal comes from each. For comparison, we will begin by doing the same calculations for a conventional bright–field microscope.

(a) Plot the sensitivity as a function of z for a suitable region around $z = z_{r0}$.

Next suppose that the transmitter (source) irradiance (power per unit area) obeys

$$E_t = \frac{P}{w_t^2}$$

where P is the source power. Here w_t is defined in a way similar to w_r above, except that the values may be different.

First consider brightfield imaging, where the transmitter parameters are

$$z_{t0} = 4 \text{ millimeters} \qquad w_{t0} = 1 \text{ millimeter} \qquad b_t = 1 \text{ meter}$$

Note that this means that, to a good approximation, $w_t = w_{t0}$ over the region of interest. Assume furthermore that the source power is one milliwatt.

The region of interest consists of particles which scatter light so that the received signal from one of them is

$$S_1 = E_t \sigma E_r.$$

The area, σ is called the scattering cross–section of the particle. We are interested in relative changes in signal, so we have left out a large number of multiplying parameters.

Now suppose the scatterers are evenly distributed with a density N_v per unit volume.

$$N_v = 10^{18}/m^3 \qquad \sigma = 10^{-12} m^2$$

We need to sum the signal over all the scatterers. We don't know exactly the location of each particle, and even if we did, this would be a tedious process. Fortunately, for large numbers of particles, we can approximate it to a good degree of accuracy by multiplying by the density and integrating. If the range of z is small, we simply need to integrate over the area and multiply by dz. Do this for **(b)** a layer dz thick, centered at $z = z_{r0}$ and again for **(c)** one where z is 50 micrometers beyond the focus, z_{r0}. Let $dz = 1$ micrometer.

Now, let's look at the confocal situation. For this case, set the transmitter parameters equal to those of the receiver. Once again, compute **(d)** the signal from a layer at the focus, $z = z_r 0$, **(e)** a layer 50 micrometers out of focus.

(f) Integrate the signal over z from zero to infinity. Hint; Integrate from $-\infty$ to ∞, because it is easier and the result will be almost the same. Estimate the fraction of the total signal coming from a layer at $z = z_{r0}$ if the thickness is $dz = b_r$.

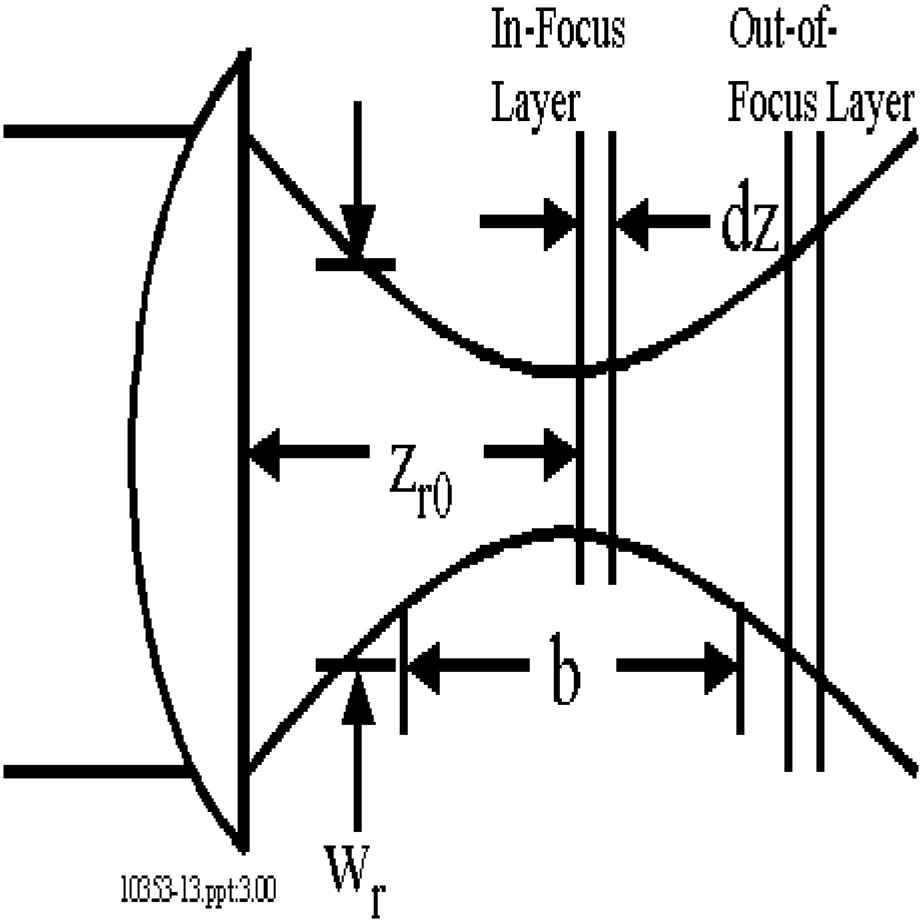

Figure 14: Results for problem 15

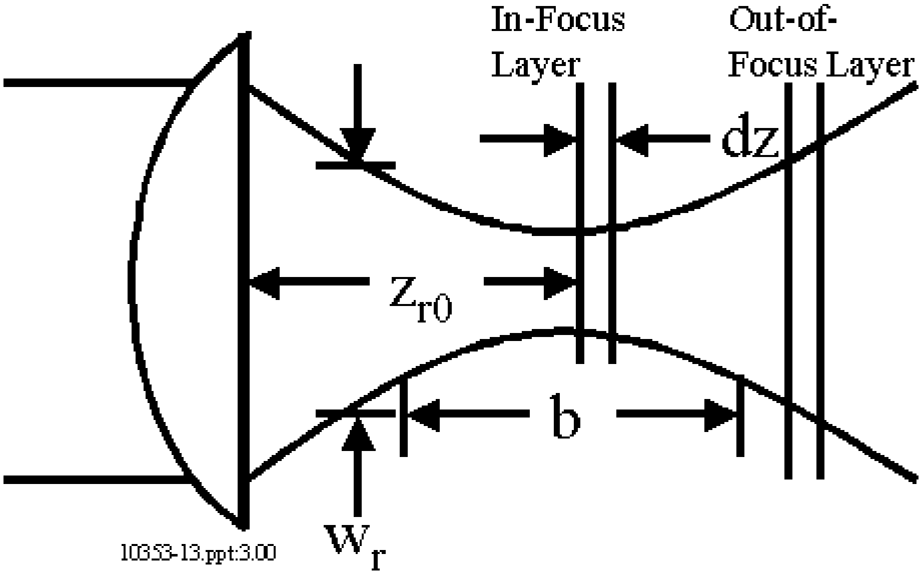

Figure 15: Layout for Problem 1. The drawing shows the receiver beam boundaries. A similar diagram could be drawn for the transmitter, possibly with different numerical values. See the problem text for definition of the labelled distances.