# Optics for Engineers
# EECE5646 — Fall 2009
# Final Exam with Solutions

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10 December 2009

## 1 Polarization

Here we consider light scattered from a biological specimen consisting of small isotropic scatterers that reflect $R_{bkg} = 0.3$ of the incident light without changing the state of polarization, and some aligned fibrils that reflect according to the Jones matrix

$$\mathcal{M}_{fib} = \begin{pmatrix} \rho_x & 0 \\ 0 & \rho_y \end{pmatrix}, \tag{1.0.1}$$

where

$$R_x = |\rho_x|^2 = 0.4 \qquad R_y = |\rho_y|^2 = 0.2. \tag{1.0.2}$$

This matrix is multiplied by $A$, a complex scalar random variable and added to $\mathcal{M}_{bkg}$, so that there is a random phase (and possibly amplitude) relationship between the two types of scatterers. Let us assume that

$$\langle A \rangle = 0 \qquad \left\langle |A|^2 \right\rangle = 1. \tag{1.0.3}$$

Most students incorporated the A terms into a single matrix rather than dealing with the two matrices symbolically. That is not a problem, but seems a bit harder than what I did.

## 1.1 Matrix Multiplication

Now we are going to work with well–defined input fields, so the only random variables are in the matrix of the sample. If

$$\mathbf{E}_{out} = \left(\mathcal{M}_{bkg} + A\mathcal{M}_{fib}\right)\mathbf{E}_{in}, \tag{1.1.1}$$

write the expressions for the output field, $\mathbf{E}_{out}$ and its adjoint, $\mathbf{E}_{out}^{\dagger}$. Then write the general expression for $\mathbf{E}_{out}\mathbf{E}_{out}^{\dagger}$.

$$\mathbf{E}_{out} = \mathcal{M}\mathbf{E}_{in}, \tag{1.1.2}$$

and

$$\mathbf{E}_{out}^{\dagger} = \mathbf{E}_{in}^{\dagger}\mathcal{M}^{\dagger}. \tag{1.1.3}$$

$$\begin{gathered}
\mathbf{E}_{out}\mathbf{E}_{out}^{\dagger}\mathcal{M}\mathbf{E}_{in}\mathbf{E}_{in}^{\dagger}\mathcal{M}^{\dagger} = \\
\mathcal{M}_{bkg}^{\dagger}\mathbf{E}_{in}\mathbf{E}_{in}^{\dagger}\mathcal{M}_{bkg} + \\
A\mathcal{M}_{bkg}^{\dagger}\mathbf{E}_{in}\mathbf{E}_{in}^{\dagger}\mathcal{M}_{fib} + \\
A^{\dagger}\mathcal{M}_{fib}^{\dagger}\mathbf{E}_{in}\mathbf{E}_{in}^{\dagger}\mathcal{M}_{bkg} + \\
A^{\dagger}A\mathcal{M}_{fib}^{\dagger}\mathbf{E}_{in}\mathbf{E}_{in}^{\dagger}\mathcal{M}_{fib}
\end{gathered} \tag{1.1.4}$$

## 1.2 Expectation Values

Now make use of the statistical description of $A$, and write an equation for the coherency matrix of the output, $\left\langle \mathbf{E}_{out}\mathbf{E}_{out}^{\dagger} \right\rangle$.

$$\langle A \rangle = 0 \qquad \left\langle |A|^2 \right\rangle = 1, \tag{1.2.1}$$

so

$$\begin{gathered}
\left\langle \mathbf{E}_{out}\mathbf{E}_{out}^{\dagger} \right\rangle \mathcal{M}\mathbf{E}_{in}\mathbf{E}_{in}^{\dagger}\mathcal{M}^{\dagger} = \\
\mathcal{M}_{bkg}^{\dagger}\mathbf{E}_{in}\mathbf{E}_{in}^{\dagger}\mathcal{M}_{bkg} + \\
\mathcal{M}_{fib}^{\dagger}\mathbf{E}_{in}\mathbf{E}_{in}^{\dagger}\mathcal{M}_{fib}
\end{gathered} \tag{1.2.2}$$

## 1.3 Polarimetry System

Design a system that will allow the input polarization to be linear at −45°, 0°, 45°, or 90°, and to measure the same states of the output polarization. Assume you have access to perfect polarizers and waveplates.

Many solutions are possible. The point is to generate each of the four different input polarizations, and measure each of the four different outputs. Polarizers can be used with some loss, or waveplates can be used in various ways.

## 1.4 Coherency Matrix

Write equations for the components of the coherency matrix of the output, with an input vector having arbitrary linear polarization with unit power;

$$\mathbf{E}_{in} = \begin{pmatrix} \cos\zeta \\ \sin\zeta \end{pmatrix}. \tag{1.4.1}$$

$$\mathbf{E}_{in}\mathbf{E}_{in}^{\dagger} = \begin{pmatrix} \cos\zeta \\ \sin\zeta \end{pmatrix} \begin{pmatrix} \cos\zeta & \sin\zeta \end{pmatrix} = \begin{pmatrix} \cos^2\zeta & \cos\zeta\sin\zeta \\ \cos\zeta\sin\zeta & \sin^2\zeta \end{pmatrix} \tag{1.4.2}$$

Using Equation 1.2.2,

$$\left\langle \mathbf{E}_{out}\mathbf{E}_{out}^{\dagger} \right\rangle \mathcal{M}\mathbf{E}_{in}\mathbf{E}_{in}^{\dagger}\mathcal{M}^{\dagger} =$$
$$R_{bkg}\begin{pmatrix} \cos^2\zeta & \cos\zeta\sin\zeta \\ \cos\zeta\sin\zeta & \sin^2\zeta \end{pmatrix} +$$
$$\begin{pmatrix} \rho_x & 0 \\ 0 & \rho_y \end{pmatrix} \begin{pmatrix} \cos^2\zeta & \cos\zeta\sin\zeta \\ \cos\zeta\sin\zeta & \sin^2\zeta \end{pmatrix} \begin{pmatrix} \rho_x & 0 \\ 0 & \rho_y \end{pmatrix} \tag{1.4.3}$$

Note that we do not need to take the conjugates of the $\rho$ terms, because we assumed at the beginning that they were real.

$$\begin{pmatrix} \rho_x & 0 \\ 0 & \rho_y \end{pmatrix} \begin{pmatrix} \cos^2\zeta & \cos\zeta\sin\zeta \\ \cos\zeta\sin\zeta & \sin^2\zeta \end{pmatrix} \begin{pmatrix} \rho_x & 0 \\ 0 & \rho_y \end{pmatrix} = \begin{pmatrix} \cos^2\zeta\rho_x\rho_x & \cos\zeta\sin\zeta\rho_x\rho_y \\ \cos\zeta\sin\zeta\rho_x\rho_y & \sin^2\zeta\rho_y\rho_y \end{pmatrix} \tag{1.4.4}$$

Combining all these,

$$a = R_{bkg}\cos^2\zeta + R_x\cos^2\zeta \tag{1.4.5}$$

$$b = \left(R_{bkg} + \sqrt{R_xR_y}\right)\cos\zeta\sin\zeta \tag{1.4.6}$$

$$c = (R_{bkg} + R_y)\sin^2\zeta \tag{1.4.7}$$

## 1.5 Stokes Parameters

Plot the Stokes parameters and the degree of polarization as a function of $\zeta$ from 0 to 90°.

Equation 1.165 in the book contained an error. The third term should be $b + b^*$ rather than $b - b^*$. I will correct it in the future. I did not penalize anyone for using the wrong equation on this part.

$$I = a + c \tag{1.5.1}$$

$$M = a - c \tag{1.5.2}$$

$$C = b + b = 2\Re b \tag{1.5.3}$$

$$S = \frac{b - b}{j} = 2\Im b \tag{1.5.4}$$

$$V = \frac{\sqrt{M^2 + C^2 + S^2}}{I} \tag{1.5.5}$$

Plots in Figure 1.5.1

## 1.6 Discussion

Discuss the realism of this model. What can be done to make it better?

The major problem here is that the model assumes coherence across all the points and all the fibers. A better approach would be to use the incoherent summing approach with a term like $A$ within each category of scatterer.

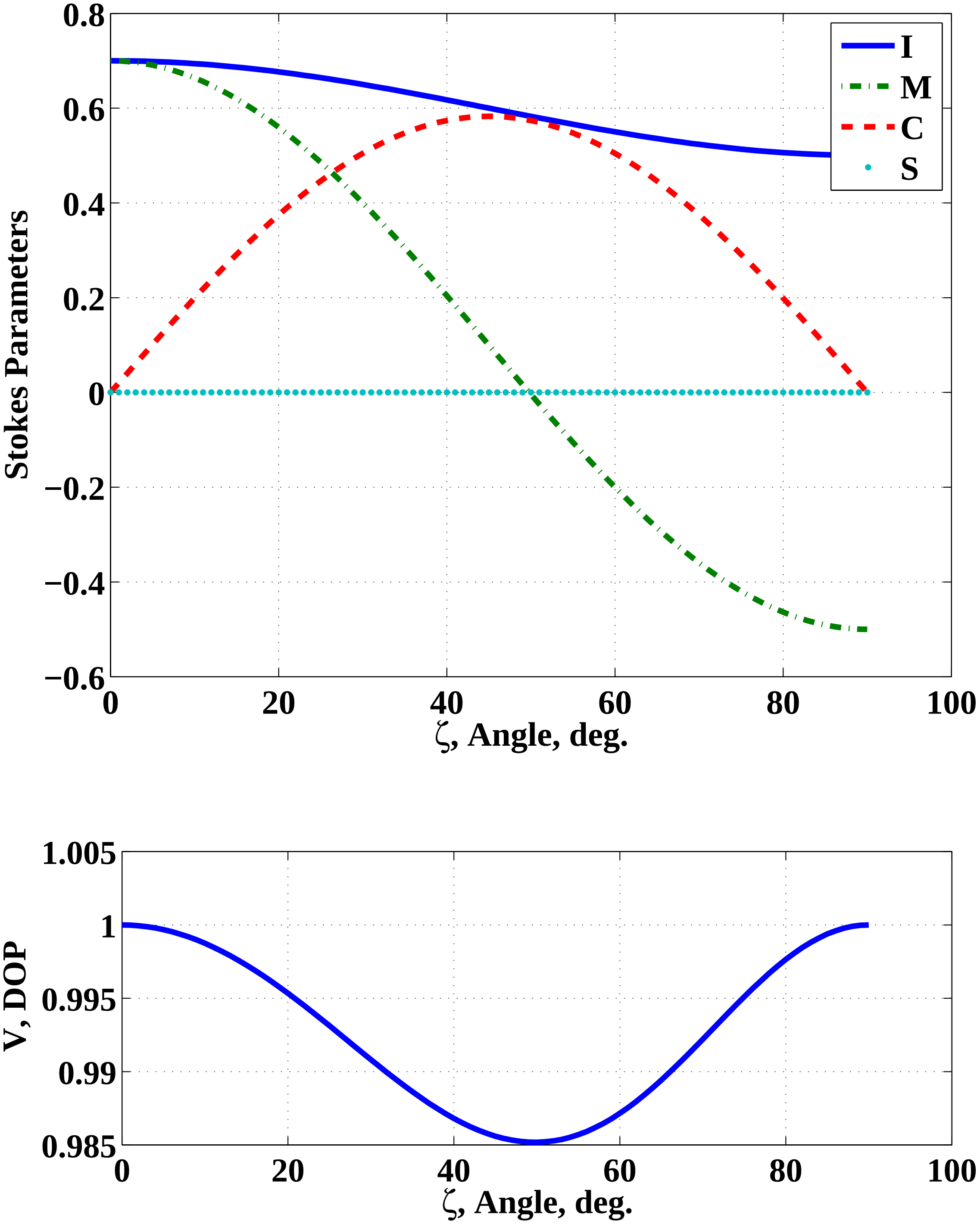

Figure 1.5.1: Stokes Parameters (top) and Degree of polarization (bottom)

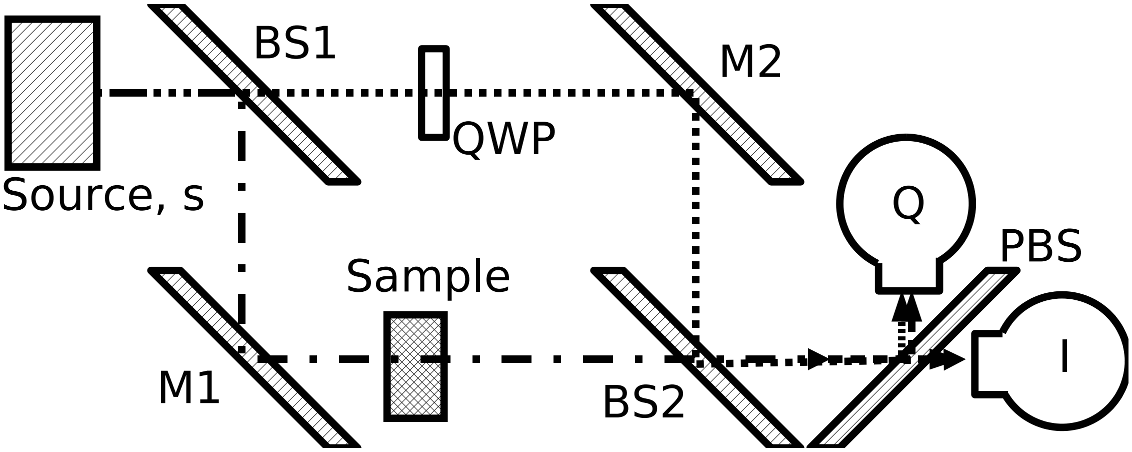

Figure 2.0.1: Optical Quadrature Microscope.

# 2 Interference

Here we consider an alternative to the Optical Quadrature Microscope discussed in class and in the homework. Instead of using circular polarization in the reference and linear polarization in the signal, as shown in Figure 2.0.1, we add a quarter–wave plate in the signal arm and use one of the two orthogonal states of circular polarization in the reference and one in the signal.

## 2.1 Analysis

Show that the polarizers select the in–phase and quadrature signals in the same way as the Optical Quadrature Microscope.

Take the x and y components of the two orthogonal states of polarization

$$\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\left[\begin{pmatrix}1 & 0\\ 0 & i\end{pmatrix}+\tau_s\begin{pmatrix}1 & 0\\ 0 & i\end{pmatrix}\right]=[1+\tau_s]\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} \tag{2.1.1}$$

$$\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\left[\begin{pmatrix}1 & 0\\ 0 & i\end{pmatrix}+\tau_s\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\right]=[i-i\tau_s]\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix} \tag{2.1.2}$$

The x output is

$$|1+\tau_s|^2=1+|\tau_s|^2+\tau_s+\tau_s^*=1+|\tau_s|^2+2\Re\tau_s \tag{2.1.3}$$

and the y output is

$$|i-i\tau_s|^2=1+|\tau_s|^2+i\tau_s-i\tau_s^*=1+|\tau_s|^2+2\Im\tau_s \tag{2.1.4}$$

The “mixing” erms are the ones we want.

Depending on how you chose your definitions of circular polarization, you may have different results, but they should differ in phase by 90 degrees, as these do.

## 2.2 Polarimetric Imaging

Modify the design so that multiple measurements can be made to determine the Jones matrix of the sample.

There are many solutions. Use polarizers and waveplates to deliver and detect the desired polarization. My choice would to start with linear polarization from the laser and rotate a half–wave plate from zero to 45 degrees to acheive x and y polaizations respectivly. Then use a polarizer, after the sample to select x or y. Follow that with a quarter–wave plate at 45 degrees to the polarizer (rotate the pair together) to produce circular polarization. The rotation of this polarizer will introduce a 90–degree phase shift which can be removed computationally.

# 3 Diffraction Grating

Here we consider a diffraction grating used as a rear mirror in an Argon Ion laser. The laser operates on several gain lines, two of which are of interest here, centered at 514 nm and 488 nm. On each of these gain lines, there are multiple longitudinal cavity modes under the gain line with sufficient gain to exceed threshold. The cavity length is 0.3 meters.

Suppose that I wish to install the grating at an angle of 20 degrees with respect to the laser beam path for operation at 488 nm, on the first order.

## 3.1 Grating Spacing

What is the line spacing required for the grating?

This is the Littrow configuration in the notes:

$$n\lambda = 2d\sin\theta$$

$$n = 1 \qquad \lambda = 448nm \qquad \theta = 20^\circ$$

$$d = 0.713\mu m$$

## 3.2 Angle for Green Line

If I now wish to work on the 514–nm line, what is the new angle?

Change the wavelength in the above equation, keeping $d$ the same

$$\lambda = 514nm \qquad \theta = 21.12^\circ$$

## 3.3 Nominal Diffraction Pattern

Now if the grating is properly designed for 488 nm, then the phase change across each grating opening will be zero, and there will be a jump of $2\pi$ in phase between steps, and thus can be neglected. Suppose that the laser Gaussian beam diameter is 500 micrometers. Assume that there is a band of 10% of the grating spacing between facets, over which no transmission occurs. Using an FFT routine, plot the far–field pattern of the diffracted beam.

```
% m12420dg.m solution to diffraction grating problem
% November 2009,
% by Chuck DiMarzio, Northeastern University
```

```
% December 2003
lambdades=0.488; % Design wavelength in microns
lambdaop=0.514; % Operating wavelength in microns
thetades=20; % Design angle, in degrees
xperiod=lambdades/2/sind(thetades) % d, Period of grating in microns
thetaop=asind(lambdaop/2/xperiod) % Angle in degrees for operating wavelength
span=12500; % units are microns; total span
xstep=xperiod/20;
w=250; % Gaussian beam width in microns
xcenter=span/2;
dutycycle=0.9; % duty cycle of the grating
x=[0:xstep:span]; % Set up an array for x in microns
pix=length(x);
if( isinteger(pix/2));
x=[x,span+xstep]; % Make sure it's even because I want to find the
% middle
end;
pix=length(x);
gbeam=exp(-((x-xcenter)/w).2̂); % w
grating=((x/xperiod-floor(x/xperiod))¡dutycycle);
pupilfield=grating.*gbeam;
diffractionfield=fftshift(fft(fftshift(pupilfield)));
angldes=[1:length(x)]*lambdades/x(length(x)); % Angle axis at design
% wavelength
angldes=angldes-angldes(pix/2);
bestvalue=abs(diffractionfield(pix/2+1))2̂ % Save and print the peak value
```

## 3.4 Pattern Change with Tilt

When the grating is tuned to the appropriate angle for 514 nm, then the phase step from one grating facet to the next is still $2\pi$, but there is an extra tilt to each facet, equal to the difference between the two angles. This results in a loss in performance. Plot the far–field pattern under this condition.

```
tilt=(21.13-20)/180*pi;
const=2*pi*i*tilt/lambdaop;
```

```
newpupilfield=pupilfield.*exp(const*((x/xperiod-floor(x/xperiod))));
newdiffractionfield=fftshift(fft(fftshift(newpupilfield)));
newvalue=abs(newdiffractionfield(pix/2+1))2̂ % Print the reduced peak value
anglop=[1:length(x)]*lambdaop/x(length(x)); % angle axis at operational
% wavelength anglop=anglop-anglop(pix/2);
figure;plot(angldes,abs(diffractionfield),'-',...
anglop,abs(newdiffractionfield),'–');
xlabel('Angle, Rad');ylabel('Diffraction Pattern');grid on;
% Nominal width of pattern
wd=lambdaop/w/pi;
elk=axis;
axis([-3*wd,3*wd,0,elk(4)]);
```

Results are in Figure 3.4.1 Note that the new one is lower than the old by a bit less than 1%.

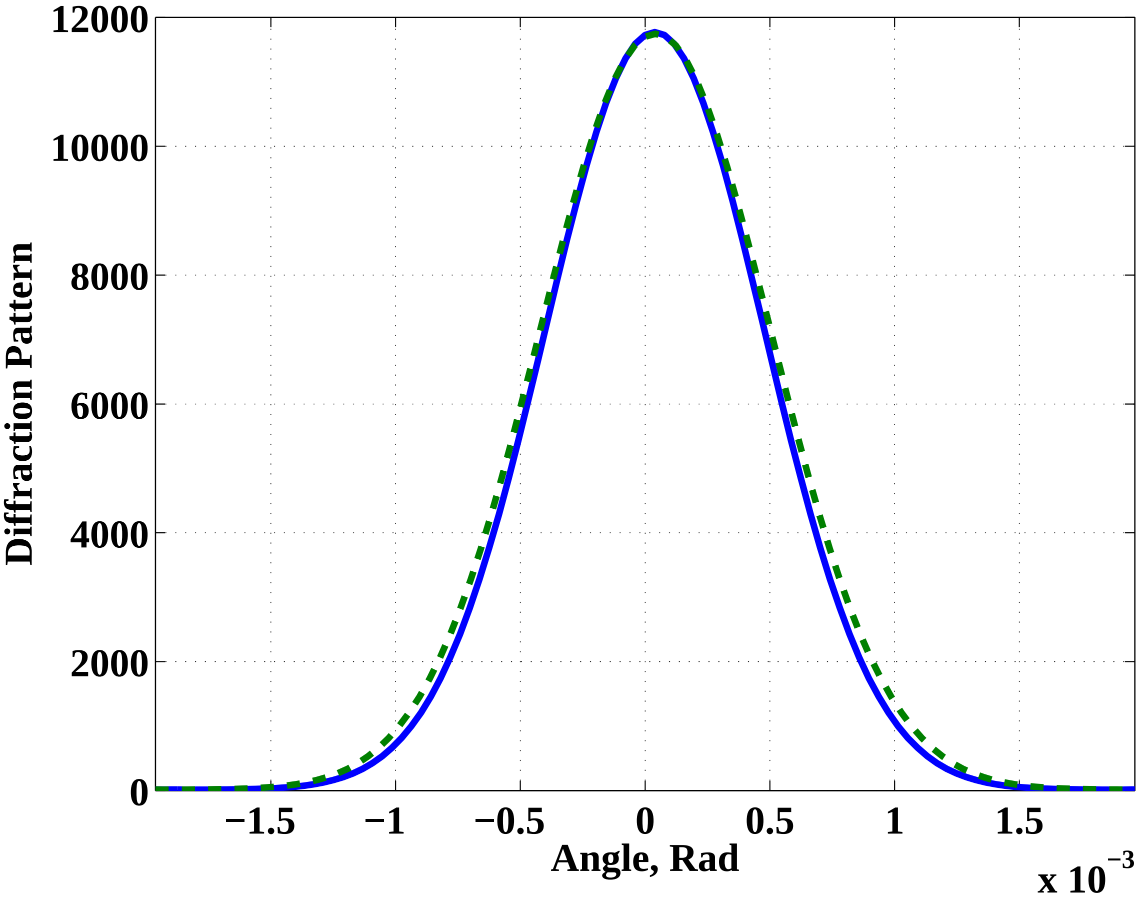

Figure 3.4.1: Grating Patterns. The original pattern (solid line) peaks at $1.386 \times 10^8$ and the second one peaks at $1.379 \times 10^8$. The second pattern, being a bit lower, is also wider, but not significantly.

# 4 Compact Disc Player

Here we look at a simple model of a compact disc player. A metal surface has a depression of a certain depth, and a diameter of two micrometers. It is covered with a plastic layer with index of refraction equal to 1.5. A laser beam twice the diameter of this "pit" is incident normally. The beam can be assumed to be a uniform plane wave over the "pit" and, in fact, over the surrounding area, just to make things easy. The laser has a free–space wavelength of 750 nm. Now some of the light will be reflected from the flat surface surrounding the "pit," and some will be reflected from the pit. Both reflected beams will diffract as they leave the surface. For light from the "pit" the phase will be shifted because of the depth of the pit, and this light will interfere with light which is reflected from the surroundings.

## 4.1 Interference

What is the depth of the pit to produce the best destructive interference on axis in the reflected beam?

The round trip of twice the pit depth must produce an optical path length difference of one half wave,

$$OPD = \frac{\lambda_0}{2} = 2nt, \tag{4.1.1}$$

so

$$t = \frac{\lambda_0}{4n}\frac{750nm}{4 \times 1.5} = 125nm.. \tag{4.1.2}$$

## 4.2 Diffraction Angle

What is the diffraction angle (full width between nulls) of the light reflected from the pit? Remember that the wavelength in the diffraction equation is that in the plastic, and not the free space wavelength.

The diffraction angle in the medium is

$$\alpha_p = 2.44\frac{\lambda_p}{D} = 2.44\frac{\lambda_0/n}{D} \tag{4.2.1}$$

$$\alpha_p = 2.44\frac{750 \times 10^{-9}m/1.5}{2 \times 10^{-6}m} = 0.61rad \tag{4.2.2}$$

## 4.3 Refraction

Here's a little bit of geometric optics. Show that the angle of the diffracted light is air is the same as it would be from a "pit" of the same diameter in air. Use the small–angle approximation here.

In air, the angle can be determined by Snell's law, so, within the small–angle approximation,

$$\alpha_p = n2.44\frac{\lambda_0/n}{D} = 2.44\frac{\lambda_0}{D} \tag{4.3.1}$$

$$\alpha_p = 2.44\frac{750 \times 10^{-9}m}{2 \times 10^{-6}m} = 0.91rad \tag{4.3.2}$$

## 4.4 Field from Pit

What is the on–axis scattered field of light from the pit? Remember the relationship between field and irradiance. Call the incident field $U_{inc}$.

The on–axis scattered irradiance is

$$I_{scat} = \frac{\pi}{4}\frac{PD^2}{\lambda^2 R^2}, \tag{4.4.1}$$

and the power is

$$P = I_{inc}\pi D^2/4, \tag{4.4.2}$$

so

$$I_{scat} = I_{inc}\frac{\pi^2}{16}\frac{D^4}{\lambda^2 R^2} I_{inc}\frac{\pi^2}{16}\frac{(2 \times 10^{-6}m)^4}{(750 \times 10^{-9}m)^2(5e-3m)^2} = 7.1 \times 10^{-7} I_{inc}, \tag{4.4.3}$$

and

$$U_{scat} = U_{inc}\frac{\pi}{4}\frac{(2 \times 10^{-6}m)^2}{(750 \times 10^{-9}m)(5 \times 10^{-3}m)} = 8.4 \times 10^{-4} U_{inc}, \tag{4.4.4}$$

## 4.5 Field from Surface

What is the on–axis scattered field of light from the surrounding surface?

For the larger beam, the field would be the field diffracted from a 4–micrometer circle, minus that diffracted by the pit.

$$U_{beam} = U_{inc}\frac{\pi}{4}\frac{(4 \times 10^{-6}m)^2}{(750 \times 10^{-9}m)(5 \times 10^{-3}m)} - 8.4E{-}4U_{inc} = 0.0033U_{inc}. \tag{4.5.1}$$

Now we must subtract the field from the pit.

$$U_{surface} = U_{beam} - U_{scat} = 0.0033U_{inc} - 8.4 \times 10^{-4}U_{inc} = 026 \times 10^{-3}U_{inc} \tag{4.5.2}$$

## 4.6 Fringe Visibility

What is the ratio of maximum to minimum irradiance?

The maximum occurs when these add, and the minimum when they subtract, so the contrast is

$$\frac{\text{Max}}{\text{Min}} = \left(\frac{Ubeam - Uscat}{Ubeam + Uscat}\right)^2 = \left(\frac{0.0026 - 0.00084}{0.0026 + 0.00084}\right)^2 = 0.26 \tag{4.6.1}$$

This was the answer I was expecting. If you computed the visibilty, that is

$$\frac{\text{Max-Min}}{\text{Max+Min}} = 0.58 \tag{4.6.2}$$