EECE5646 Midterm Exam

with Solutions

Prof. Charles A. DiMarzio
Department of Electrical and Computer Engineering
Northeastern University

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1 Imaging a Sphere

A glass sphere sits on a glass slide as shown in Figure 1.1. The sphere diameter is 100 micrometers, and the glass has an index of refraction of 1.5, and the background medium is oil with an index of refraction of 1.47.

1.1 Optical Path Length

Compute the optical path length as a function of x, for light transmitted through the sphere, at $y = 0$ where the sphere is centered at the origin of coordinates. Subtract the nominal value of the optical path length for just the glass slide and the oil. You many neglect refraction at the surface, because the indices of refraction are closely matched.

The thickness of the sphere at any point where $y = 0$ is

$$z = 2\sqrt{r^2 - x^2} \tag{1.1}$$

The difference in optical path length caused by the sphere is

$$\Delta(OPL) = 2\sqrt{r^2 - x^2}\delta n \tag{1.2}$$

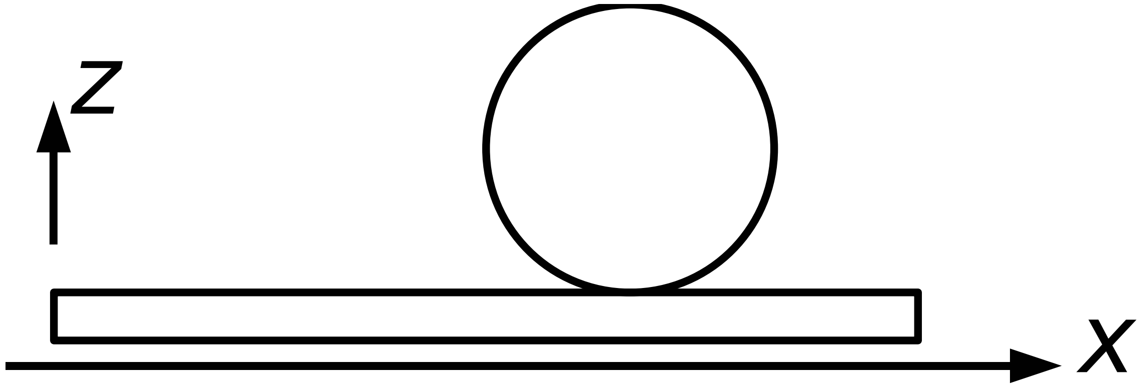

Figure 1.1: SPHERE IMAGING. A glass sphere sits on a glass microscope slide. Both the sphere and the slide have an index of refraction of 1.5. The background medium has an index of refraction of 1.47.

The maximum value is

$$\Delta(OPL)_{x=0} = 2r\delta n$$

$$\Delta(OPL)_{x=0} = 100\ \mu\text{m} \times (1.50 - 1.47) = 3\ \mu\text{m}. \tag{1.3}$$

I also gave credit for any solution that did the optical path length correctly.

1.2 Geometric Optics Image

Now assume that the surfaces of the glass are slightly rough, so that they scatter some light allowing us to see them. How deep does the bottom of the sphere appear as viewed by a microscope through the sphere from the top? Use the surface of the microscope slide as a reference point.

Use the equation for refraction at a spherical surface,

$$\frac{n}{s} + \frac{n'}{s'} = \frac{n' - n}{R} \tag{1.4}$$

$$s = 100\ \mu\text{m} \qquad R = -50\ \mu\text{m} \tag{1.5}$$

$$\frac{1.47}{s'} = \frac{1.47 - 1.50}{-50\ \mu\text{m}} - \frac{1.50}{100\ \mu\text{m}} \tag{1.6}$$

$$s' = -102.\ \mu\text{m}. \tag{1.7}$$

The image is 102 micrometers below the top, or 2 micrometers below the actual bottom.

2 Matrix Optics

2.1 Matrix between Focal Planes

Write the matrix from the front focal plane to the back focal plane of a lens of focal length, f, $\mathcal{M}_{FF'}$.

$$\mathcal{M}_{FF'} = \begin{pmatrix} 1 & f \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1/f & 1 \end{pmatrix} \begin{pmatrix} 1 & f \\ 0 & 1 \end{pmatrix} \tag{2.1}$$

$$\mathcal{M}_{FF'} = \begin{pmatrix} 1 & f \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & f \\ -1/f & 0 \end{pmatrix} \tag{2.2}$$

$$\mathcal{M}_{FF'} = \begin{pmatrix} 0 & f \\ -1/f & 0 \end{pmatrix} \tag{2.3}$$

Angle, α' in the back focal plane is only dependent on position, x in the front focal plane, and position, x', in the back focal plane is only dependent on angle, α in the front.

Assume collimated light with rays tilted by an angle, θ from the horizontal. Where is the image, as a function of θ and f?

The image will be at the back focal plane always, because the incoming rays are parallel. The height of the image will be

$$x' = f\theta \tag{2.4}$$

2.2 Microscope

Now put two of these matrices together, with focal lengths f_1 and f_2. This means that the lenses are separated so that the back focal plane of the first is at the front focal plane of the second. What is the resulting matrix? What does it mean?

$$\mathcal{M}_{F1F2'} = \begin{pmatrix} 0 & f_2 \\ -1/f_2 & 0 \end{pmatrix} \begin{pmatrix} 0 & f_1 \\ -1/f_1 & 0 \end{pmatrix} \tag{2.5}$$

$$\mathcal{M}_{F1F2'} = \begin{pmatrix} -f_2/f_1 & 0 \\ 0 & -f_1/f_2 \end{pmatrix} \tag{2.6}$$

Position in the final plane is only related to position in the first plane. This means that these are conjugate planes. If the first contains an object, the last will contain the image. The transverse magnification is

$$m = -f_2/f_1. \tag{2.7}$$

Unlike a general pair of conjugate planes, for these planes, angle at the final one is only related to angle at the first, and the angular magnification is

$$m_\alpha = 1/m \tag{2.8}$$

Suppose that the second lens has a focal length of 200 mm and the first lens works with it to produce an overall magnification of -40, and a camera with 1000 pixels, each being a square 10 micrometers on a side in each direction is placed at the image location.

What is the focal length of the first lens?

$$m = -f_2/f_1. \tag{2.9}$$

$$-40 = -200 \text{ mm}/f_1. \tag{2.10}$$

$$f_1 = 5 \text{ mm}. \tag{2.11}$$

How large are the pixels on the object?

The pixels on the object are $1/m$ times their actual size, or 0.25 micrometers.

What is the field of view on the object?

The field of view is 1000 times the size of a pixel, or 0.25 millimeters.

What is the diameter of the lenses to achieve a numerical aperture of 0.75?

The diameter of the pupil must be

$$D = 2f_1 \tan\theta \tag{2.12}$$

where

$$NA = \sin\theta \tag{2.13}$$

so

$$D = 2f_1 \frac{\sin\theta}{\cos\theta} = 2f_1 \frac{NA}{\sqrt{1 - NA^2}} = 11.3 \text{ mm} \tag{2.14}$$

Both lenses must have at least this diameter.

3 Scanning

Now consider the system in Problem 2, but use a linear array detector that is one pixel by 1000. We are going to obtain the pixels of an image in the y direction from these 1000 pixels. In the other dimension, we want to use a scanning mirror to scan across the object.

Where do we place the scanning mirror?

The purpose of the mirror is to scan across the object. We must place the mirror where the change in angle that it causes will translate to a change in position. In other words, we must place it at the back focal point of the objective, f_1, or the point we would call F_1'. Note that this is also the front focal point, F_2, of the second lens. This is the pupil of the microscope, and we know that we want to place the scanner in a real pupil.

How large must it be to achieve the same numerical aperture?

In the direction along the axis of the mirror, it must be the same diameter we calculated above, 11.3 mm. In the other direction, it must be larger by at least the secant of the angle of incidence. At 45 degrees, it must be larger by $\sqrt{2}$, or 16 millimeters.

How far must it move to obtain the same field of view in the x direction that we have in the y direction?

The distance moved on the object is going to be $f\theta$ where θ is the angular deviation of the beam. Thus

$$x = f_1\theta \tag{3.1}$$

$$0.25 \text{ mm} = 5 \text{ mm}\theta \tag{3.2}$$

$$\theta = 0.05 \text{ radians.} \tag{3.3}$$

The mirror needs to move only half this angle. If the mirror moves from 45 degrees to 46, the deviation of the beam goes from 90 to 92 degrees. Therefore, the mirror must move through about 25 milliradians, or about 1.4 degrees.

4 Spherical Aberrations

In this problem, we are going to explore the use of multiple lenses to reduce aberration.

4.1 Approach

Explain why the longitudinal aberration terms,

$$L_s = \frac{1}{s'(h)} - \frac{1}{s'(0)} \tag{4.1}$$

add if we place two lenses close together.

This equation describes a change in the optical power of the lens with height. The equation,

$$L_s = \frac{x_1^2}{8f^3}\frac{1}{n\left(n-1\right)}\left(\frac{n+2}{n-1}q^2 + 3\left(n+1\right)pq + \left(3n+2\right)\left(n-1\right)p^2 + \frac{n^3}{n-1}\right) \tag{4.2}$$

depends on h in the same way regardless of choices of n, p, and q. If the two lenses are close together their optical powers add. Therefore, the errors in their optical powers for a given value of h also add.

4.2 Simple Lens

We have an object 20 cm in front of a lens (assume thin lenses), and want a real image 30 cm behind it. The diameter of the lens is 2cm. What is the f–number of this lens?

$$\frac{1}{f} = \frac{1}{s} + \frac{1}{s'} \tag{4.3}$$

$$\frac{1}{f} = \frac{1}{20\text{ cm}} + \frac{1}{30\text{ cm}}. \tag{4.4}$$

$$f = 12\text{ cm}. \tag{4.5}$$

$$F = f/d = 12/2 = 6 \tag{4.6}$$

The lens is f/6.

Now, continuing on to get the parameters for the rest of the problem,

$$p = \frac{s'-s}{s'+s} = \frac{30-20}{30+20} = 0.2 \tag{4.7}$$

The optimal q is

$$q_{min} = -2\frac{n^2-1}{n+2}p = -0.1429 \tag{4.8}$$

For each part, use the appropriate q with Equation 4.2 and

$$ds = s'^2 L_s \qquad dx = ds \times h/s' \tag{4.9}$$

Compute the transverse spherical aberration, δx, for

- A symmetric biconvex lens,

  $q = 0$, $dx = 0.0199$ cm (close to best)

- The simple lens optimized for minimum spherical aberration,

  $q = -0.1429$, $dx = 0195$ cm (best)

- A plano–convex lens, curved side toward the image,

  $q = -1$, $dx = 0344$ cm (bad)

- A plano–convex lens, curved side toward the object,

  $q = 1$, $dx = 0459$ cm (worse)

4.3 Compound Lens

Now let's try a compound lens. We know a plano–convex lens is a pretty good choice to collimate a point source, and used backward, it is good to focus light from a collimated source. Let's put two plano–convex lenses together. Let the first one have a focal length equal to the object distance and the second have one equal to the image distance. Orient the lenses correctly and put them close to each other. Compute the resulting transverse spherical aberration, δx.

Use

$$\begin{aligned} p_1 &= 1 \qquad q_1 = -1 \\ p_2 &= -1 \qquad q_2 = 1 \end{aligned} \tag{4.10}$$

Add the two Ln_s contributions, and use Equations 4.9.

$$dx = 0.0057 \text{ cm} \tag{4.11}$$

which is more than 3 times better than the best single lens.

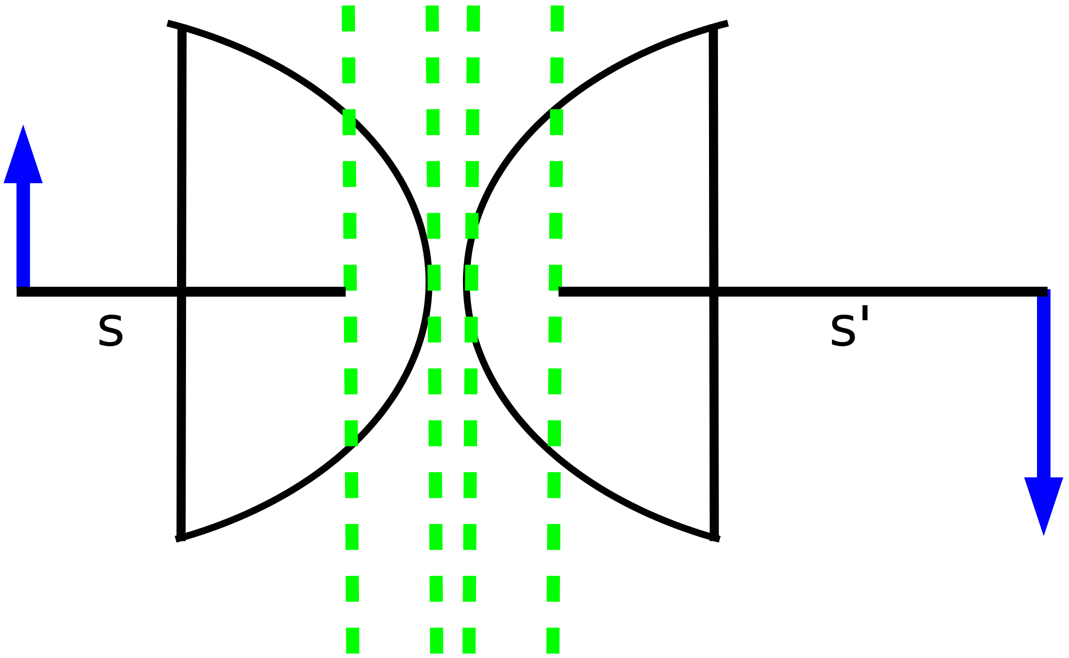

Figure 4.1: COMPOUND LENS. The two lenses together reduce the aberrations by more than a factor of 3 in comparison to the best single lens with spherical surfaces.

4.4 Configuration

Assume that the two lenses in the compound–lens problem are both 1 cm thick. Draw the setup and show the distances at which you would place the object and image planes to begin your alignment.

The drawing is in Figure 4.1, which is not to scale. The distance, s is 20 cm and s' is 30 cm. Distances are measured from the appropriate principal planes, which are 2/3 of the thickness into their respective lenses, or $0.667uncm$. Therefore, the object is to the left of the first vertex by

$$20 \text{ cm} - 0.667 \text{ cm} = 19.33 \text{ cm} \tag{4.12}$$

and the image is to the right of the last vertex by

$$30 \text{ cm} - 0.667 \text{ cm} = 29.33 \text{ cm} \tag{4.13}$$