Optics for Engineers

Chapter 6

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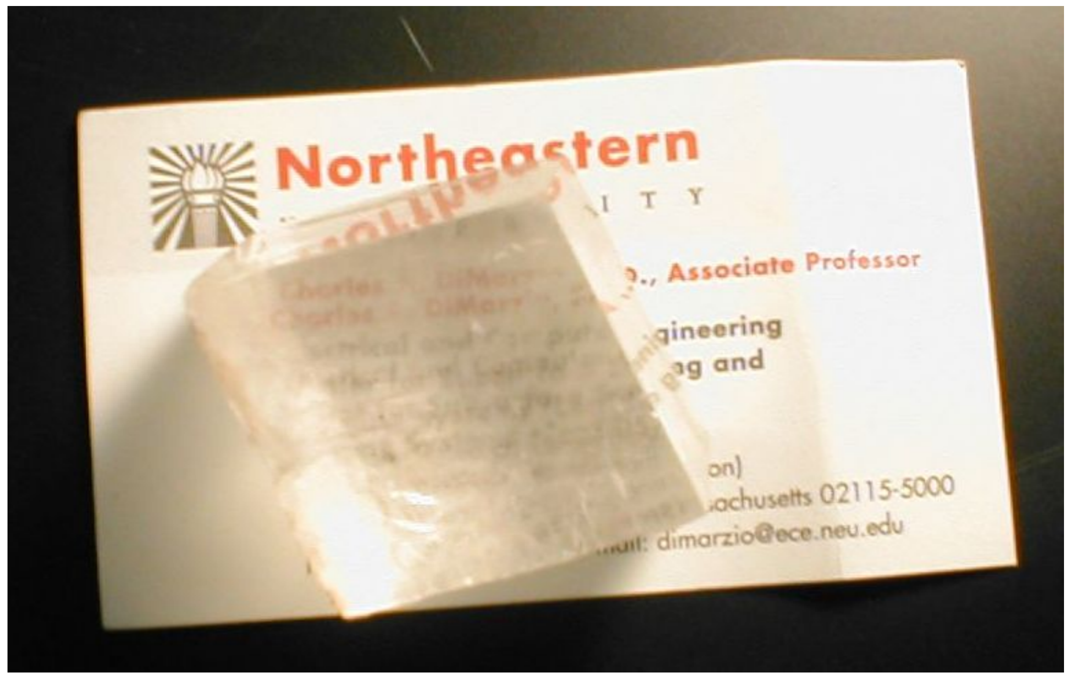
Overview of Polarized Light

- Fundamentals
- Devices
  (What They Do)
- Physics
  (How They Do It)
- Interfaces
- Jones Matrices
  (Bookkeeping)
- Coherency Matrices
  (Partial Polarization)
- Mueller Matrices
  (More Bookkeeping)



Transverse Waves

- From Ch. 1

$$\mathbf{k}\times\mathbf{E}=-\omega\mathbf{B} \qquad (1)\ \mathbf{B}\perp\mathbf{k} \qquad (2)\ \mathbf{B}\perp\mathbf{E}$$

$$\mathbf{k}\times\mathbf{H}=\omega\mathbf{D} \qquad (1)\ \mathbf{D}\perp\mathbf{k} \qquad (1)\ \mathbf{D}\perp\mathbf{H}$$

$$\mathbf{E}\times\mathbf{B}=\mathbf{S} \qquad (2)\ \mathbf{S}\perp\mathbf{E} \qquad (2)\ \mathbf{S}\perp\mathbf{B}$$

- Conclusions
  - $\mathbf{H}$, $\mathbf{D}$, $\mathbf{k}$ mutually perpendicular (from 1)
  - $\mathbf{E}$, $\mathbf{B}$ $\mathbf{S}$ mutually perpendicular (from 2)
  - $\mathbf{H}\parallel\mathbf{B}$ at Optical Wavelengths
  - $\mathbf{D}\parallel\mathbf{E}$, $\mathbf{k}\parallel\mathbf{S}$ Not Required
  - Only Two Numbers Specify Field for Known $\mathbf{k}$

Linear Polarization

- Vertical and Horizontal Basis

$$\mathbf{E} = \left[E_v\hat{v} + E_h\hat{h}\right] e^{j(\omega t - kz)}$$

- x, y Basis

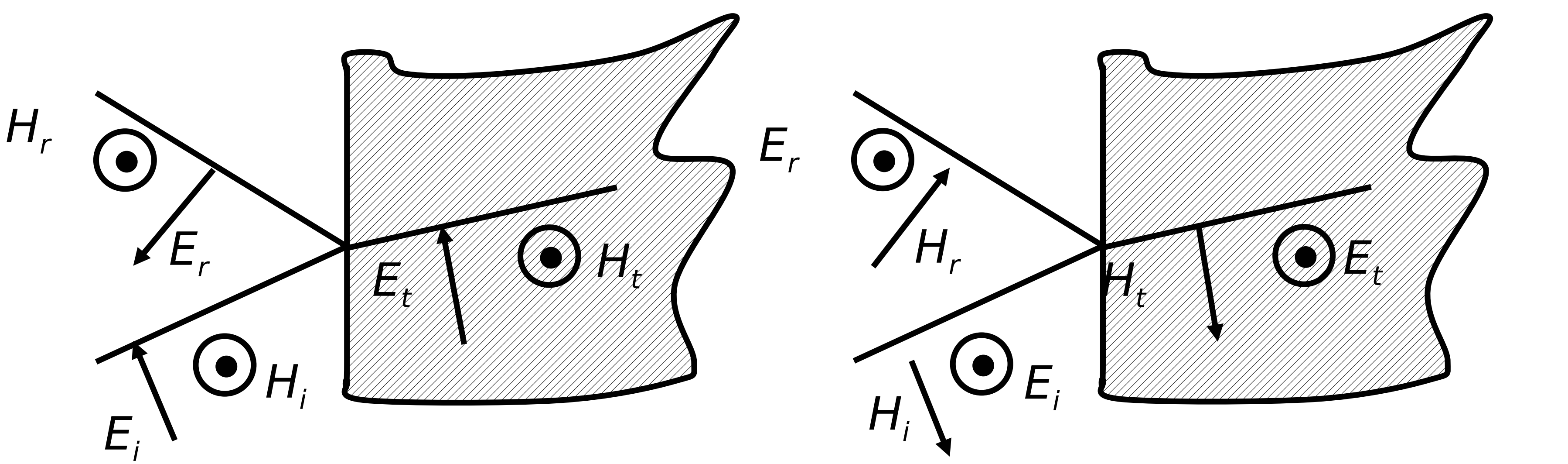
$$\mathbf{E} = \left[E_x\hat{x} + E_y\hat{y}\right] e^{j(\omega t - kz)}$$

$$\mathbf{H} = \left[-\frac{E_y}{Z}\hat{x} + \frac{E_x}{Z}\hat{y}\right] e^{j(\omega t - kz)}$$

S,P Basis at an Interface

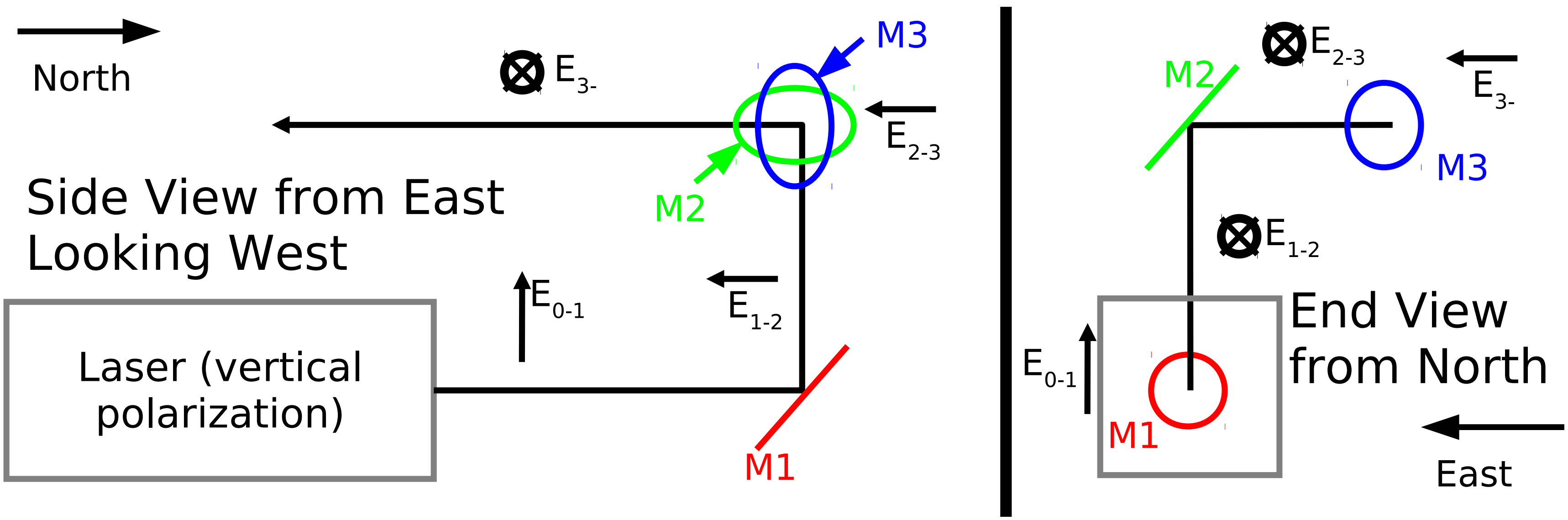
- P Means $\mathbf{E}$ Parallel to Plane of Incidence (More Later)
- S Means $\mathbf{E}$ Perpendicular (Senkrecht) to Plane of Incidence

$$\mathbf{E} = [E_s\hat{s} + E_p\hat{p}]\, e^{j(\omega t - kz)}$$



P Polarization (TM) S Polarization (TE)

Polarization Labels



Location	$\mathbf{k}$ Direction	$\mathbf{E}$ Direction	Label
Before $M1$	North	Up	Vertical
At $M1$			P
After $M1$	Up	South	
At $M2$			S
After $M2$	West	South	Horizontal
At $M3$			P
After $M3$	South	East	Horizontal

Circular Polarization

- Right–Hand Circular

$$\mathbf{E}_r = \frac{E_0}{\sqrt{2}}\left[\hat{x} + j\hat{y}\right] e^{j(\omega t - kz)}$$

$$(\mathbf{E})_{TD} = \frac{E_0}{\sqrt{2}}\Re\Big[\hat{x}\left(e^{j(\omega t)} + e^{-j(\omega t)}\right) j\hat{y}\left(e^{j(\omega t)} + e^{j(\omega t)}\right)\Big]$$

$$= \hat{x}\ E_0\sqrt{2}\cos\omega t + \hat{y}\ E_0\sqrt{2}\sin\omega t$$

  - Viewed from Source, $\mathbf{E}$ Rotates Like Right–Hand Screw

- Left–Hand Circular

$$\mathbf{E}_\ell = \left[E_0\hat{x} - jE_0\hat{y}\right] e^{j(\omega t - kz)}.$$

Superposition

- General Superposition

$$\mathbf{E} = A_r \frac{1}{\sqrt{2}} \hat{\mathbf{r}} + A_\ell \frac{1}{\sqrt{2}} \hat{\ell} \qquad \text{Circular Basis}$$

$$\mathbf{E} = A_P \frac{1}{\sqrt{2}} \hat{\mathbf{p}} + A_S \frac{1}{\sqrt{2}} \hat{\mathbf{s}} \qquad \text{P,S Basis}$$

- Example: X Polarization in Circular Basis

$$\frac{1}{\sqrt{2}} \hat{\mathbf{r}} + \frac{1}{\sqrt{2}} \hat{\ell} = E_x \hat{x}$$

- Q: What is $E_y \hat{y}$ in a Circular Basis?

Random Polarization

- Random or Unpolarized Light
  - Most Natural Light Is at Least Partially Random...
  - But it Is Harder to Describe

$$\langle E_x \rangle = \langle E_y \rangle = 0 \qquad \langle E_x E_x^* \rangle = \left\langle E_y E_y^* \right\rangle = \frac{S}{2} Z \qquad \left\langle E_x E_y^* \right\rangle = 0$$

- More on this Later

Polarizing Devices

- Ideal Polarizers
  Pass or Block
- Others Transform

- Linear Polarizer
  - *e.g.* Pass x, Block y
  - Characterization
    - Direction
      (x,y, other)
    - Insertion Loss
      (Pass Direction)
    - Extinction
      (Block Direction)

- The Waveplate
  (Retarder)
  - Change Relative Phase
  - Characterization
    - Axis Direction
    - Phase Difference
    - Insertion Loss
- The Rotator
  (Circular Retarder)
  - Rotate Linear Pol.
  - Phase Change E_r *vs.* E_ℓ
  - Characterization
    - Rotation Angle
      or Phase Shift
    - Insertion Loss

Linear Polarizer

- Input Polarization Example (θ Direction)

$$\mathbf{E}_{in} = E_x\hat{x} + E_y\hat{y} = E_o\left[\cos\left(\theta\right)\hat{x} + \sin\left(\theta\right)\hat{y}\right]$$

- Perfect x Polarizer

$$\mathbf{E}_{out} = 1 \times E_x\hat{x} + 0 \times E_y\hat{y} = E_o\cos\left(\theta\right)\hat{x}$$

- Irradiance

$$|\mathbf{E}_{in}|^2 = E_o^2 \qquad\qquad |\mathbf{E}_{out}|^2 = E_o^2\cos^2\theta$$

- Transmission (Malus Law for This Case)

$$\boxed{T = \frac{|\mathbf{E}_{out}|^2}{|\mathbf{E}_{in}|^2} \qquad\qquad T = \cos^2\theta}$$

Polarizers in "Real Life"

- General Equation

$$\mathbf{E}_{out} = \tau_x \times E_x\hat{x} + \tau_y \times E_y\hat{y} \qquad \tau_x \approx 1 \qquad \tau_y \approx 0$$

- Insertion Loss

$$1 - |\tau_x|^2 \qquad \text{or in dB,} \quad 10\log_{10}|\tau_x|^2$$

- Extinction

$$|\tau_y|^2 \qquad \text{or in dB,} \quad 10\log_{10}|\tau_y|^2$$

- Extinction Ratio

$$|\tau_x|^2 / |\tau_y|^2$$

  - Good Extinction $\approx 10^{-5}$ or 45dB

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Eigenvalues and Eigenvectors

- Polarizing devices are “easy to explain” for two states of polarization.
- For each of those states (eigenvectors) the output is the product of the input and a scalar...

$$E_{x:out} = \tau_x E_x \qquad E_{y:out} = \tau_x E_y.$$

- The scalar is the eigenvalue.
- Polarization makes eigenvalues and eigenvectors more easily understandable!
- See section on Jones Matrices for more details.

The Wave Plate

- Input Polarization Example (θ Direction Again)

$$\mathbf{E}_{in} = E_x\hat{x} + E_y\hat{y} = E_o\left[\cos\left(\theta\right)\hat{x} + \sin\left(\theta\right)\hat{y}\right]$$

- Half–Wave Plate Eigenvalues

$$\tau_x = 1 \qquad \tau_y = -1$$

$$\mathbf{E}_{hwp} = E_o\left[\cos\left(\theta\right)\hat{x} - \sin\left(\theta\right)\hat{y}\right] \qquad \angle\mathbf{E}_{out} = -\theta$$

- Quarter–Wave Plate Eigenvalues

$$\tau_x = 1 \qquad \tau_y = j,$$

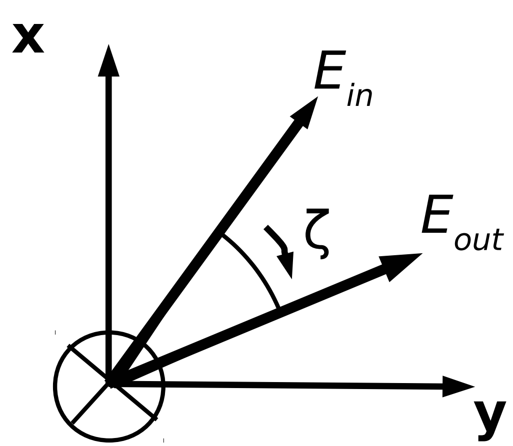
$$\mathbf{E}_{qwp} = E_o\left[\cos\left(\theta\right)\hat{x} + j\sin\left(\theta\right)\hat{y}\right]$$

- Circular Polarization at $\theta = 45^\circ$ (Q: Left or Right?)
- Other Waveplates Later

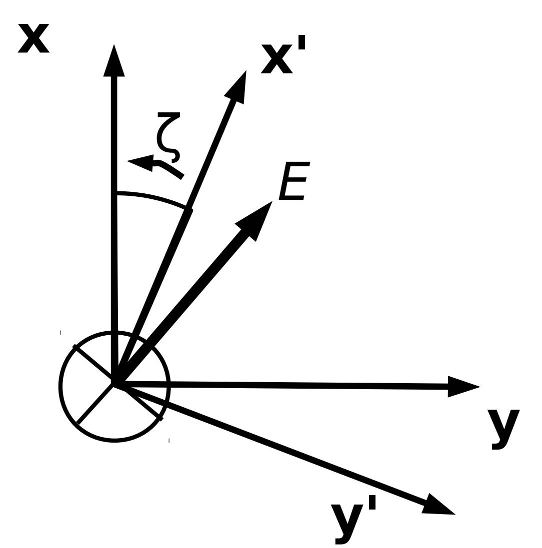
Rotator

- General Equation

$$\begin{pmatrix} E_{x:out} \\ E_{y:out} \end{pmatrix} = \begin{pmatrix} \cos\zeta_r & -\sin\zeta_r \\ \sin\zeta_r & \cos\zeta_r \end{pmatrix} \begin{pmatrix} E_{x:in} \\ E_{y:in} \end{pmatrix}$$



Polarization Rotator



Rotation of Coordinates (Later)

Interaction with Materials

- Field $\mathbf{E} = E_0\hat{x}e^{j\omega t}$
- Force $-e\mathbf{E}$
- Acceleration

$$\frac{d^2x}{dt^2} = -\mathbf{E}e/m - \kappa_x x$$

- Differential Equation

$$\frac{d^2x}{dt^2} - \frac{m}{\kappa_x e}x = \mathbf{E}$$

- Polarization

$$\mathbf{P}(t) = -n_v e x(t)\hat{x}$$

$$\mathbf{P} = \epsilon_0\chi\mathbf{E}$$

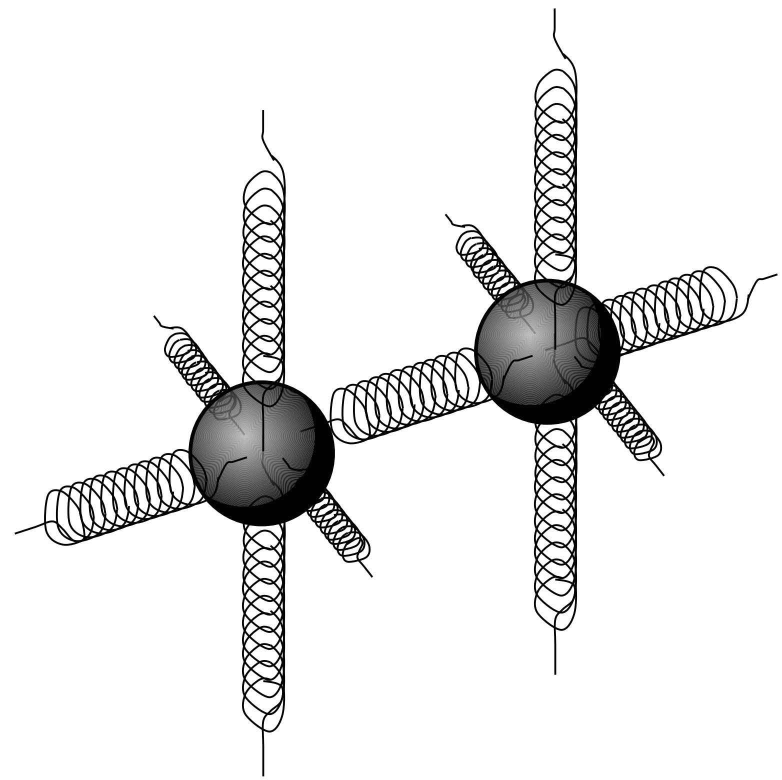
- Displacement

$$\mathbf{D} = \epsilon_0\mathbf{E}+\mathbf{P} = \epsilon\mathbf{E} = \epsilon_0\left(1+\chi\right)\mathbf{E}$$

- Anisotropic "Springs" (Tensor χ)

$$\mathbf{D} = \epsilon_0\mathbf{E}+\mathbf{P} =$$

$$\varepsilon\mathbf{E} = \epsilon_0\left(1+\chi\right)\mathbf{E}$$



Dielectric Tensor

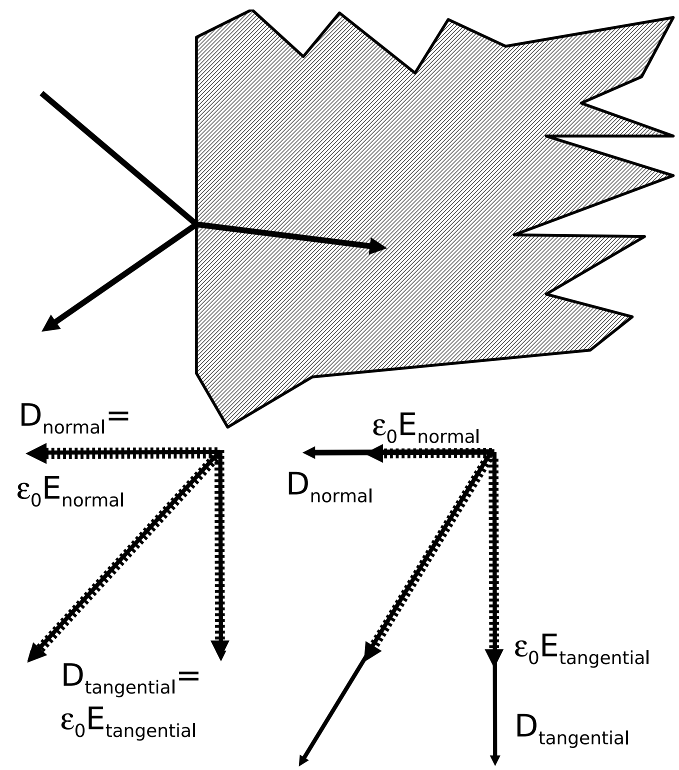
- General and Diagonal (Related by Coordinate Transform)

$$\mathcal{E} = \begin{pmatrix} \epsilon_{xx} & \epsilon_{xy} & \epsilon_{xz} \\ \epsilon_{yx} & \epsilon_{yy} & \epsilon_{yz} \\ \epsilon_{zx} & \epsilon_{zy} & \epsilon_{zz} \end{pmatrix} \qquad \mathcal{E} = \begin{pmatrix} \epsilon_{xx} & 0 & 0 \\ 0 & \epsilon_{yy} & 0 \\ 0 & 0 & \epsilon_{zz} \end{pmatrix}$$

- No Coupling Between Orthogonal States
  - In Isotropic Media, $\mathbf{D} \parallel \mathbf{E}$;
  - In Anisotropic Media for Polarization $\parallel$ Principal Axes
- Coupling Between Orthogonal States (All Other Cases)
  - Resolve Input into Two Components
  - Solve
  - Add Results

Light at an Interface: Boundary Conditions

- Boundary Conditions
  - See Chapter 1
  - Apply to S and P
- Relate E and H
  - Incident
  - Reflected and Transmitted
  - Maxwell's Equations
  - Next Page



$$
\begin{aligned}
\nabla \cdot \mathbf{D} = \rho = 0 \quad & \rightarrow \quad \Delta D_{normal} = 0, & (1)\\
\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad & \rightarrow \quad \Delta E_{tangential} = 0, & (2)\\
\nabla \cdot \mathbf{B} = 0 \quad & \rightarrow \quad \Delta B_{normal} = 0 & (3)\\
\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} = \frac{\partial D}{\partial t} \quad & \rightarrow \quad \Delta H_{tangential} = 0 & (4)
\end{aligned}
$$

Light at an Interface: Field Relationships (S Pol)

$$\mathbf{E}_i = E_i\hat{x}e^{-jkn_1(\sin\theta_i y+\cos\theta_i z)}$$

Incident Wave

$$\mathbf{H}_i = \frac{E_i}{Z_0/n_1}\left(\sin\theta_i\hat{z} - \cos\theta_i\hat{y}\right)e^{-jkn_1(\sin\theta_i y+\cos\theta_i z)}$$

$$\mathbf{E}_r = E_r\hat{x}e^{-jkn_1(\sin\theta_r y-\cos\theta_r z)}$$

Reflected Wave

$$\mathbf{H}_r = \frac{E_r}{Z_0/n_1}\left(\sin\theta_r\hat{z} + \cos\theta_r\hat{y}\right)e^{-jkn_1(\sin\theta_r y-\cos\theta_r z)}$$

$$\mathbf{E}_t = E_t\hat{x}e^{-jkn_2(\sin\theta_t y+\cos\theta_t z)}$$

Transmitted Wave

$$\mathbf{H}_i = \frac{E_t}{Z_0/n_2}\left(\sin\theta_t\hat{z} - \cos\theta_t\hat{y}\right)e^{-jkn_2(\sin\theta_t y+\cos\theta_t z)}$$

Snell's Law Again: Where the Light Goes

- Boundary Conditions Must Apply at All y (Along Boundary)
- Pick Electric Fields (or Magnetic)
- Exponents Cannot Vary with y (Sine terms equal on prev. page)

$$kn_1 \sin\theta_i = kn_1 \sin\theta_r = kn_2 \sin\theta_t$$

- Reflection Angle

$$\theta_i = \theta_r$$

- Transmission Angle (Snell's Law)

$$n_1 \sin\theta_i = n_2 \sin\theta_t$$

Fresnel Coefficients: How Much Light Goes Each Way (1)

- Analogy to Transmission Lines

- Magnetic Field Boundary Condtions (S Polarization Only)

$$\frac{E_r}{Z_0/n_1}\cos\theta_i - \frac{E_i}{Z_0/n_1}\cos\theta_i = \frac{E_t}{Z_0/n_2}\cos\theta_t$$

  – Eliminate θ_t (Other Approaches Possible)

$$\cos\theta_t = \sqrt{1-\sin^2\theta_i\left(\frac{n_1}{n_2}\right)^2} \qquad E_i - E_r = E_t\frac{\sqrt{\left(\frac{n_2}{n_1}\right)^2 - \sin^2\theta_i}}{\cos\theta_i}$$

- Electric Field Boundary Conditions (S Polarization Only)

$$E_i + E_r = E_t$$

Fresnel Coefficients: How Much Light Goes Each Way (2)

- Boundary Condtions (Previous Page)

$$E_i - E_r = E_t \frac{\sqrt{\left(\frac{n_2}{n_1}\right)^2 - \sin^2\theta_i}}{\cos\theta_i} \qquad\qquad E_i + E_r = E_t$$

- Difference Divided by Sum

$$\boxed{\rho_s = \frac{E_r}{E_i} = \frac{\cos\theta_i - \sqrt{\left(\frac{n_2}{n_1}\right)^2 - \sin^2\theta_i}}{\cos\theta_i + \sqrt{\left(\frac{n_2}{n_1}\right)^2 - \sin^2\theta_i}} \qquad \tau_s = 1 + \rho_s}$$

- Note that Fields are Not Conserved ($\tau + \rho \neq 1$)

Fresnel Coefficents Summarized

- S Polarization: E_y, H_x, H_z (Just Derived)

$$\rho_s = \frac{E_r}{E_i} = \frac{\cos\theta_i - \sqrt{\left(\frac{n_2}{n_1}\right)^2 - \sin^2\theta_i}}{\cos\theta_i + \sqrt{\left(\frac{n_2}{n_1}\right)^2 - \sin^2\theta_i}} \qquad \tau_s = 1 + \rho_s$$

- P Polarization: H_y, E_x, E_z (Trust Me)

$$\rho_p = \frac{\sqrt{\left(\frac{n_2}{n_1}\right)^2 - \sin^2\theta_i} - \left(\frac{n_2}{n_1}\right)^2\cos\theta_i}{\sqrt{\left(\frac{n_2}{n_1}\right)^2 - \sin^2\theta_i} - \left(\frac{n_2}{n_1}\right)^2\cos\theta_i} \qquad \tau_p = (1 + \rho_p)\frac{n_1}{n_2}$$

Air To Glass

$$|\rho_p| = |\rho_s|$$

at $\theta = 0^\circ$

Q: Why Abs?

$$|\rho_p| = |\rho_s| = 1$$

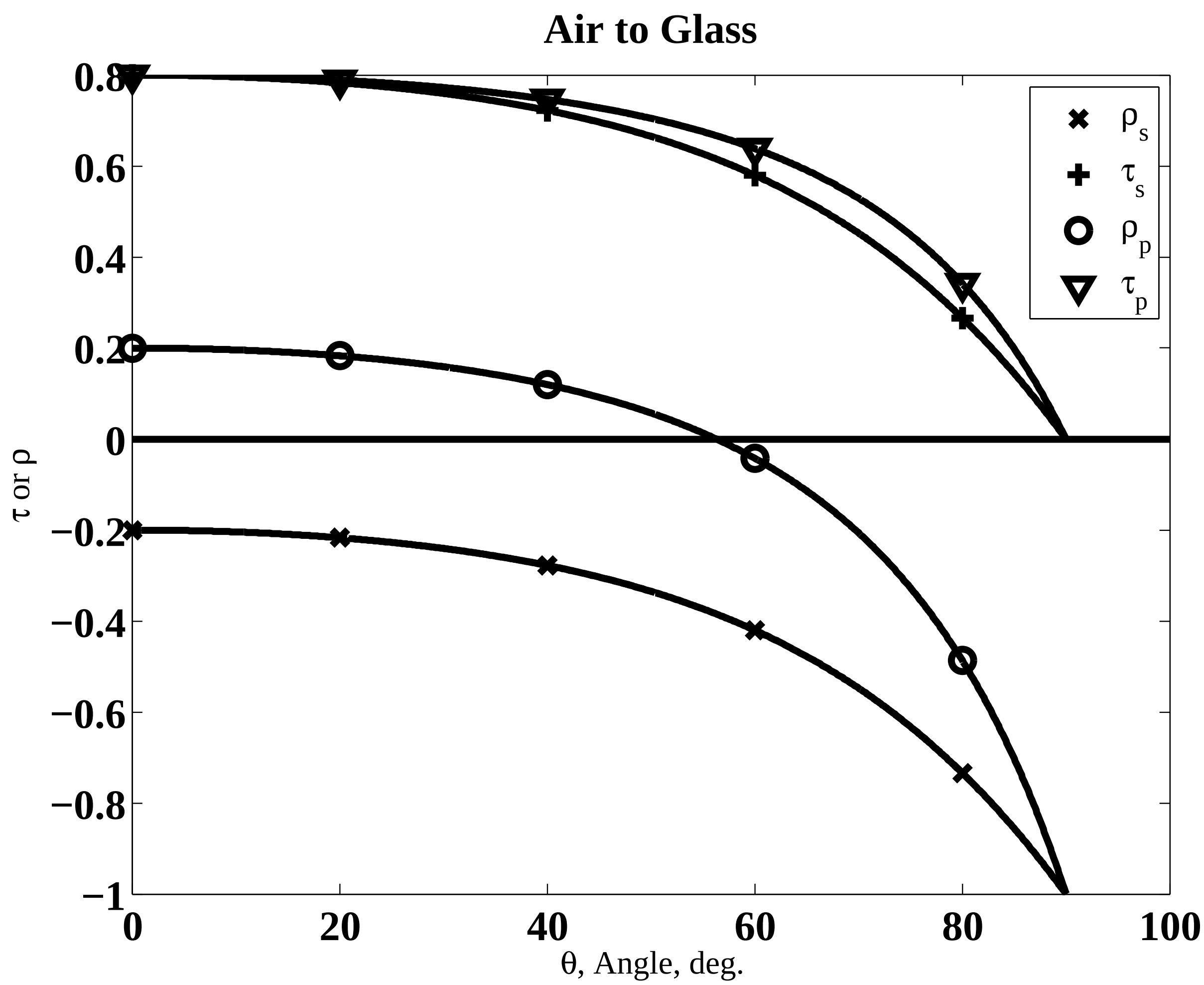
at $\theta = 90^\circ$

and

$$\rho_p = 0$$

at $\theta \approx 56^\circ$

(Brewster's Angle)

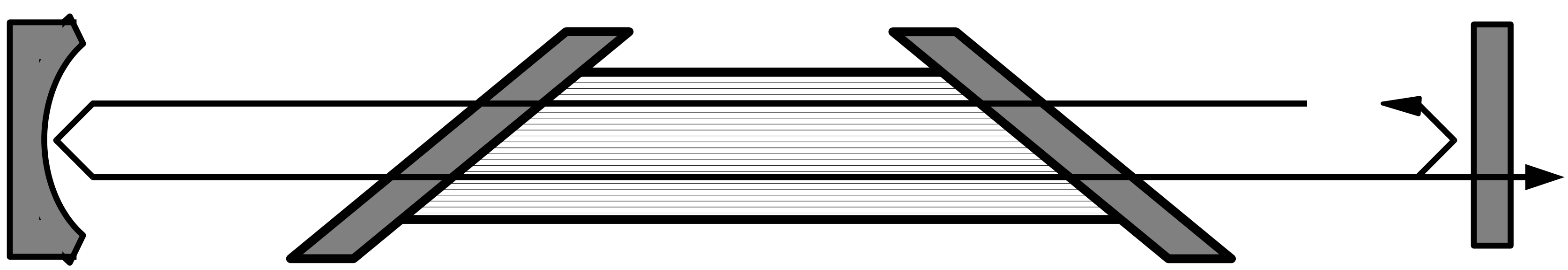


Brewster's Angle

- $\rho_p = 0$ Means No Reflection
- 100% Transmission (Different from $\tau_p = 1$ Q: Why?

$$\tan\theta_B = \frac{n_2}{n_1}$$

- Application: Windows in Laser (Polarized Laser)



- Q: What is the Direction of Polarization?

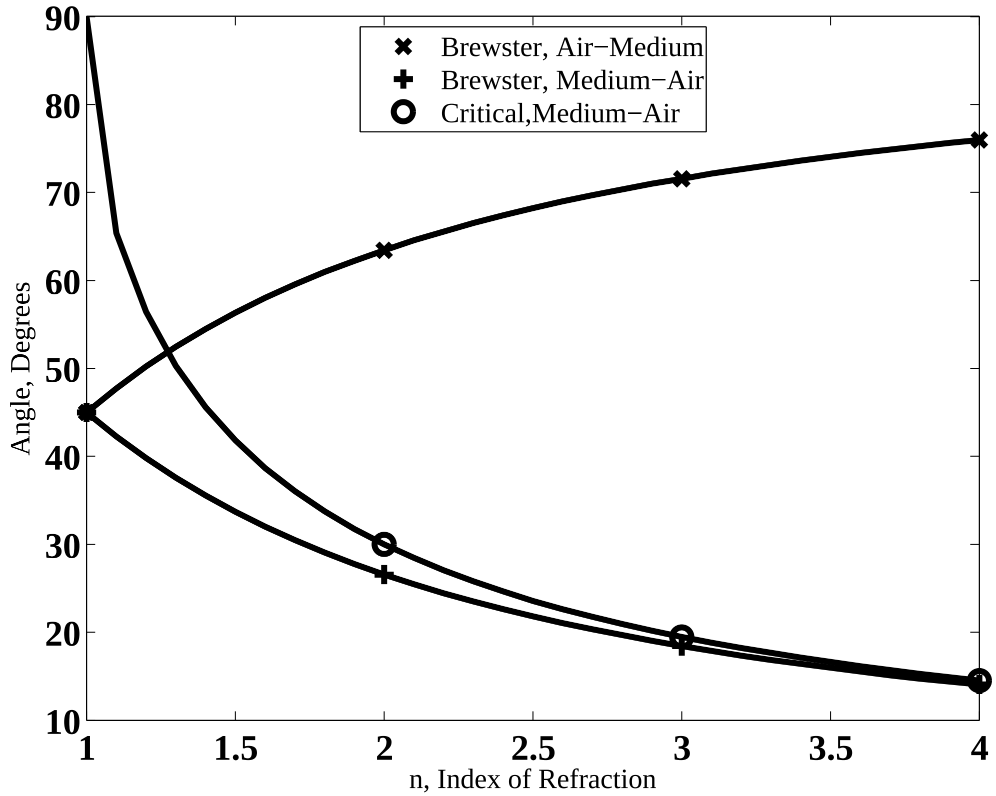
Critical Angle

- Critical Angle ($n_1 > n_2$)

$$\sin\theta_C = \frac{n_2}{n_1}$$

- Brewster's Angle

$$\tan\theta_B = \frac{n_2}{n_1}$$



Irradiance and Power

- Irradiance

$$I = \frac{|\mathbf{E}|^2}{Z}, \qquad I = \frac{dP}{dA'} = \frac{dP}{\cos\theta dA}$$

- Reflection

$$\boxed{\frac{I_r}{I_i} = R = \rho\rho^*}$$

- Transmision

$$\frac{I_t}{I_i} = T = \tau\tau^* \frac{Z_1}{Z_2}\frac{\cos\theta_t}{\cos\theta_i} = \tau\tau^* \frac{n_2}{n_1}\frac{\sqrt{\left(\frac{n_2}{n_1}\right)^2 - \sin^2\theta_i}}{\cos\theta_i}$$

- Conservation

$$T + R = 1$$

Transmission Calculations

In–Practice

- It's easy to make a mistake in calculating τ and T
  - Different materials for input and output
  - Different angles for input and output
  - Different relationship to ρ for S and P
- It's easier to calculate R and then get T from conservation

$$T = 1 - R$$

- If you know the phase (or don't care), $|\tau| = \sqrt{T}$.

Fresnel Reflection at Normal Incidence

- Reflection

$$R(0) = \left|\frac{(n_2/n_1) - 1}{(n_2/n_1) + 1}\right|^2$$

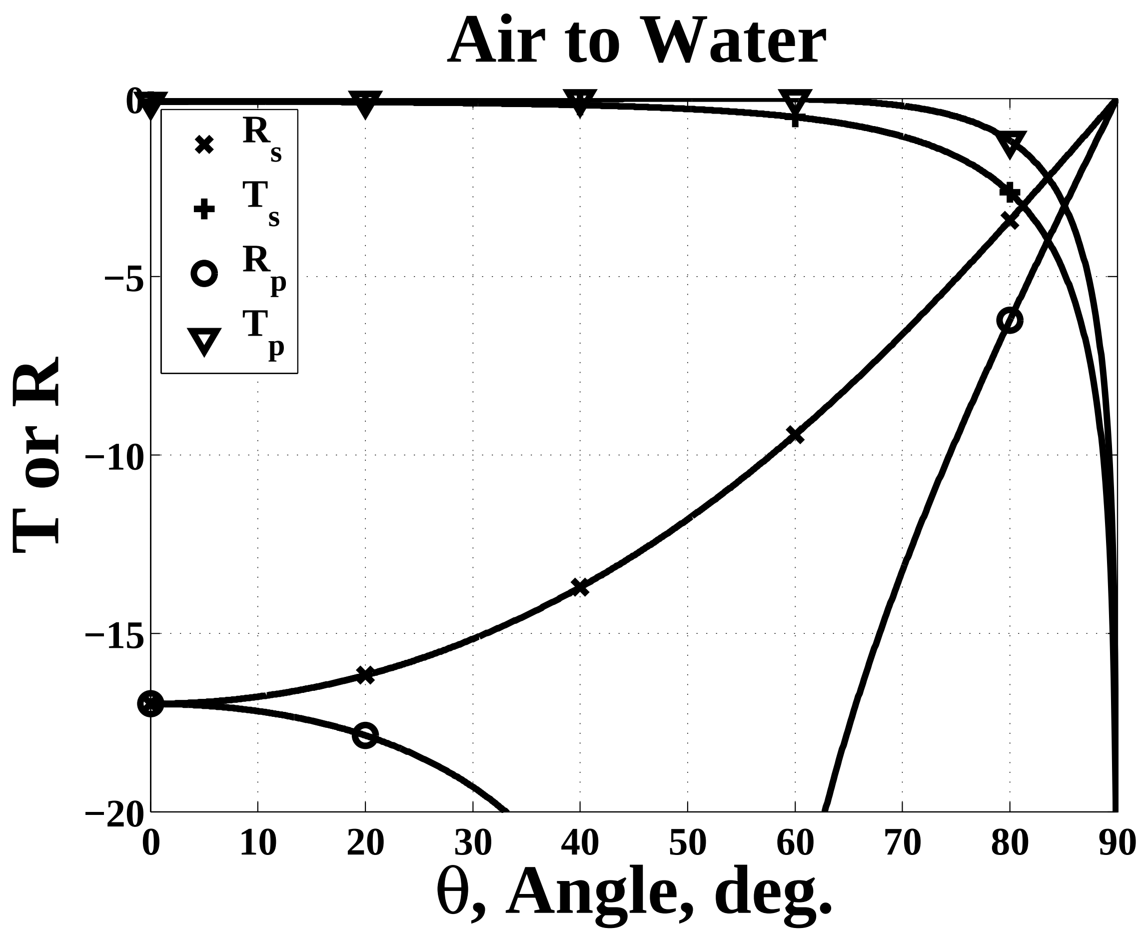
- Special Case (Air to Medium)

$$R(0) = \left|\frac{n - 1}{n + 1}\right|^2$$

- Examples

Air–Water:	$n = 1.33$	$R(0) = 0.02$
Air–Glass:	$n = 1.5$	$R(0) = 0.04$
Air–Germanium (IR):	$n = 4$	$R(0) = 0.36$

Air to Water (dB)



Air–Water:

$$R(0) = 0.02$$

Generally:

$$R_s(0) = R_p(0)$$

$$R\left(90^{\circ}\right) = 1$$

Elsewhere

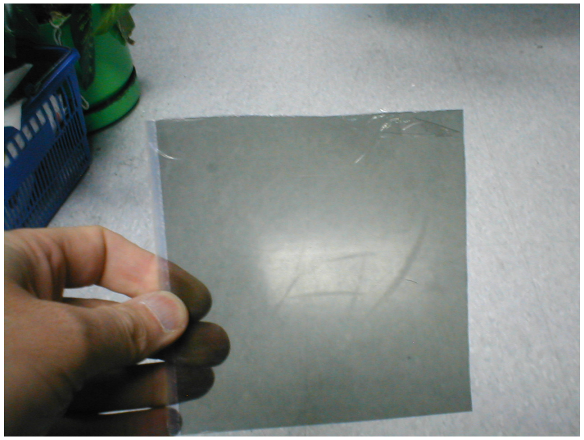
$$R_s(0) > R_p(0)$$

$$R_p\left(\theta_b\right) = 0$$

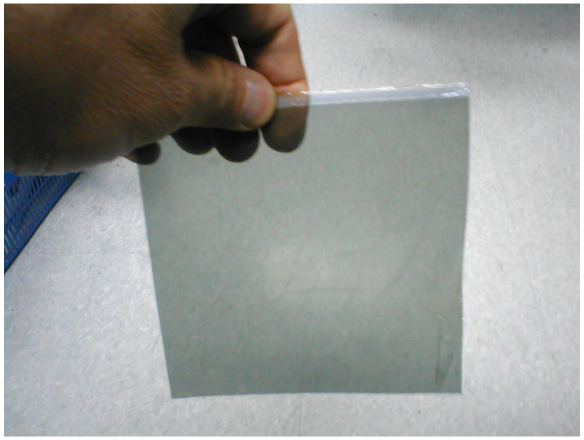
$R\left(\theta\right)$ for n to $n' = R\left(\theta'\right)$ for n' to n

Polished–Floor Reflection

No Polarizer



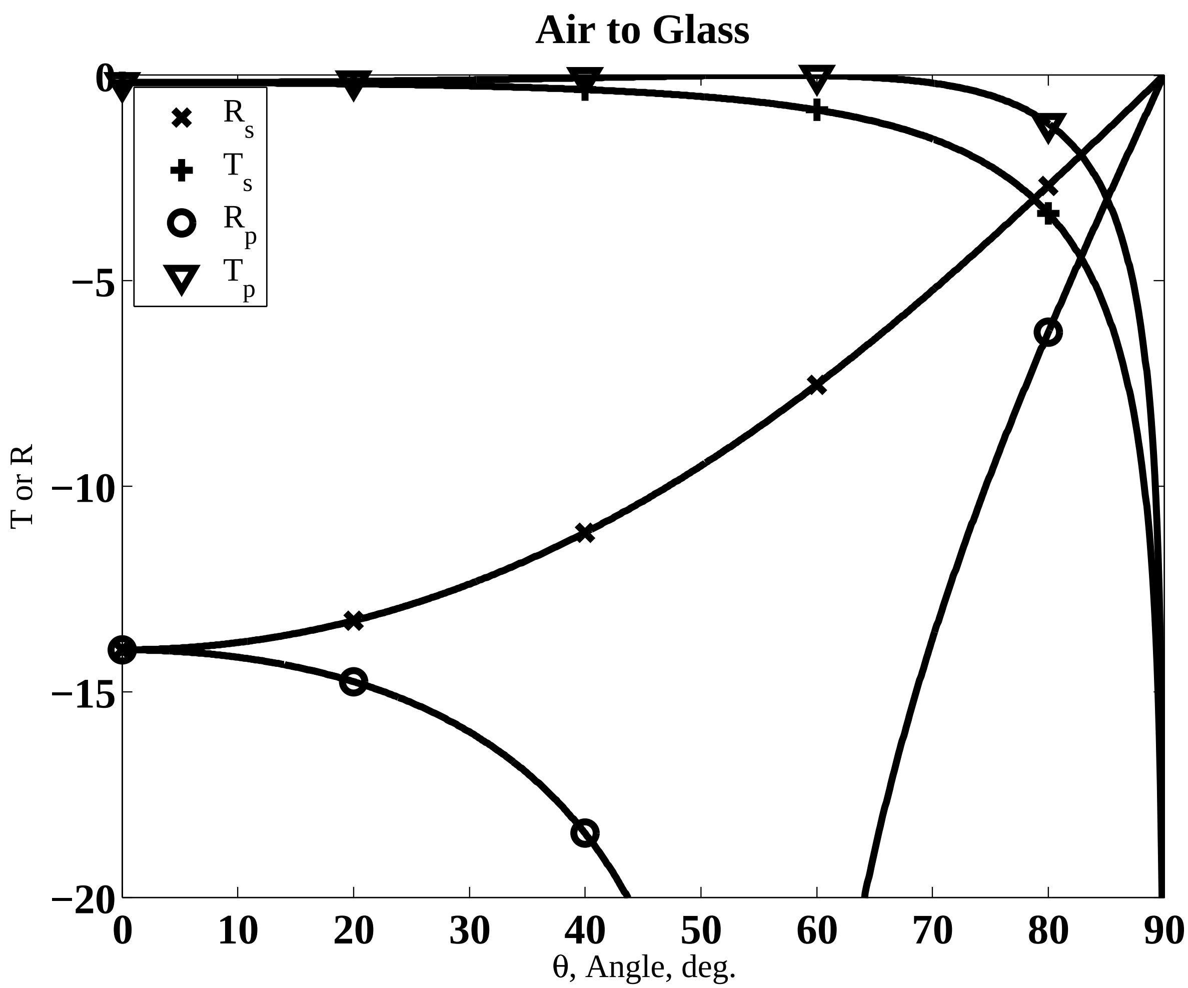
Horizontal Polarizer



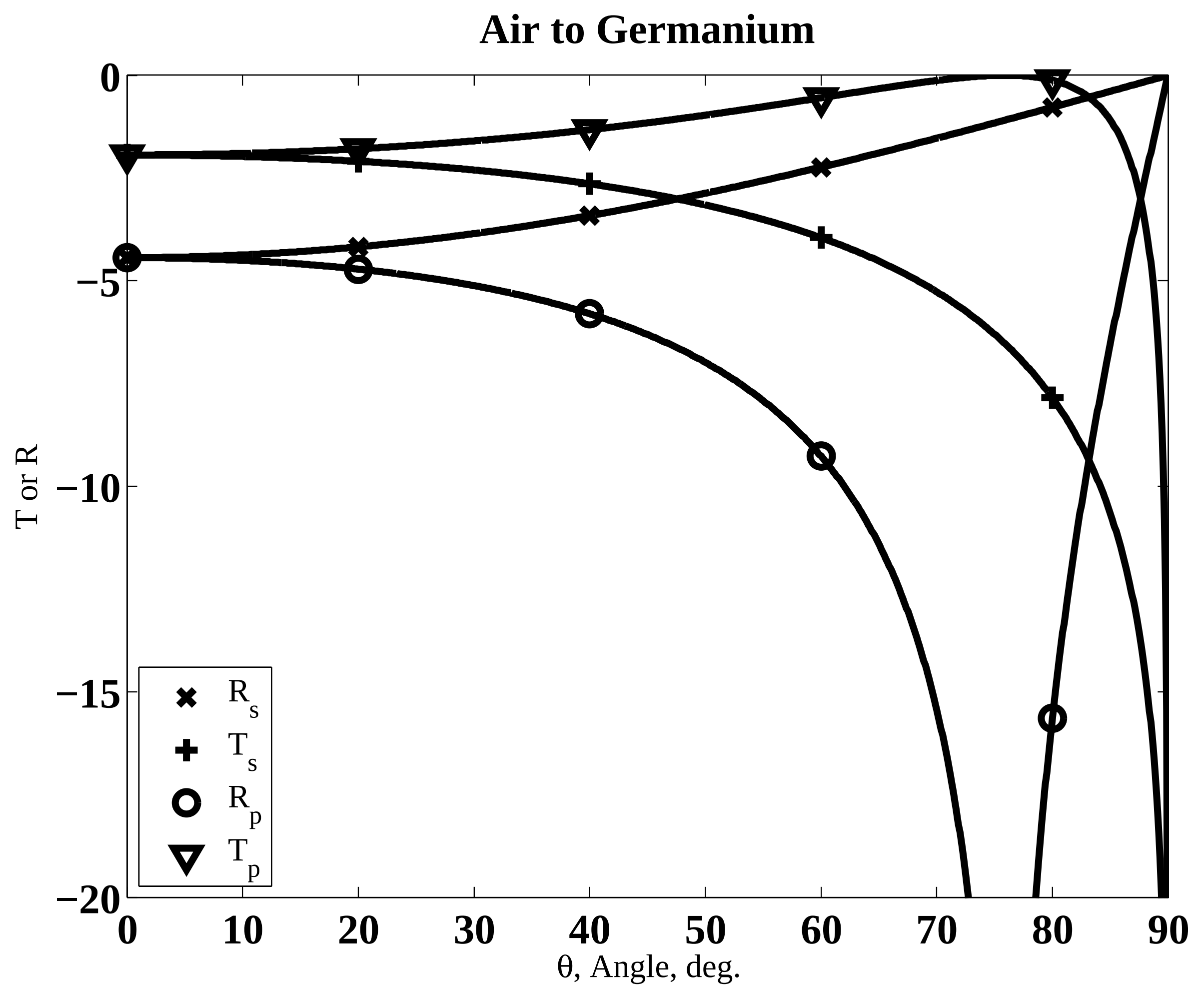
Vertical Polarizer

Q: Which is Which?

Air to Glass (dB)



Air to Germanium (dB)



Total Internal Reflection

- Fresnel Equations

$$\rho_s = \frac{E_r}{E_i} = \frac{\cos\theta_i - \sqrt{\left(\frac{n_2}{n_1}\right)^2 - \sin^2\theta_i}}{\cos\theta_i + \sqrt{\left(\frac{n_2}{n_1}\right)^2 - \sin^2\theta_i}} \qquad \rho_p = \frac{\sqrt{\left(\frac{n_2}{n_1}\right)^2 - \sin^2\theta_i} - \left(\frac{n_2}{n_1}\right)^2 \cos\theta_i}{\sqrt{\left(\frac{n_2}{n_1}\right)^2 - \sin^2\theta_i} - \left(\frac{n_2}{n_1}\right)^2 \cos\theta_i}$$

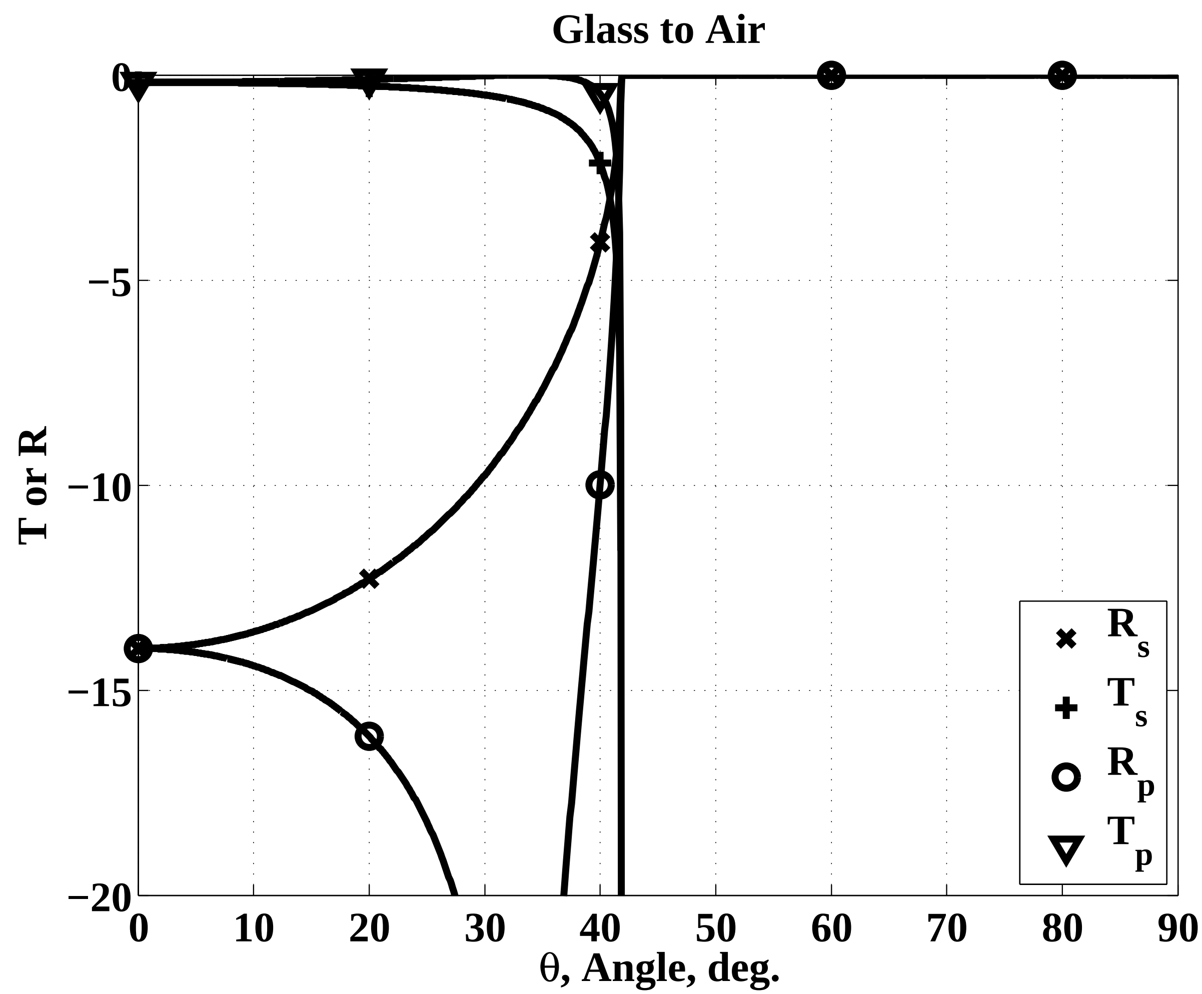
- Beyond the Critical Angle ($\sin\theta > \frac{n_2}{n_1} \rightarrow \arg\sqrt{\bullet} < 0$)

$$|\text{Numerator}| = |\text{Denominator}| \qquad \text{(Both Polarizations)}$$

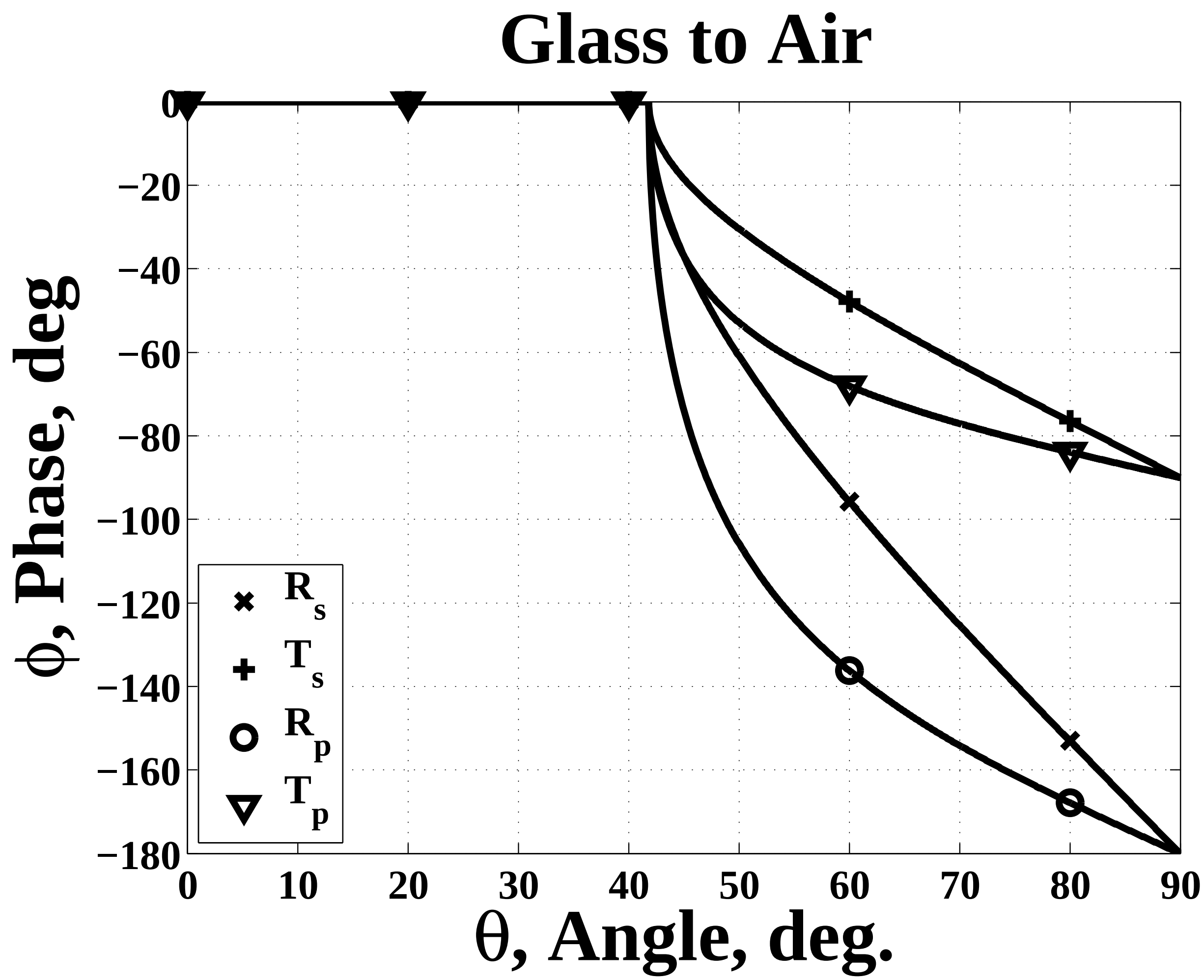
$$R = 1 \qquad T = 0 \qquad |\rho| = 1 \qquad \rho = e^{j\phi}$$

$$\tan\phi_s = -2\frac{\sqrt{\sin^2\theta_i - \left(\frac{n_2}{n_1}\right)^2}}{\cos\theta_i} \qquad \tan\phi_p = 2\frac{\left(\frac{n_2}{n_1}\right)^2 \cos\theta_i}{\sqrt{\sin^2\theta_i - \left(\frac{n_2}{n_1}\right)^2}}$$

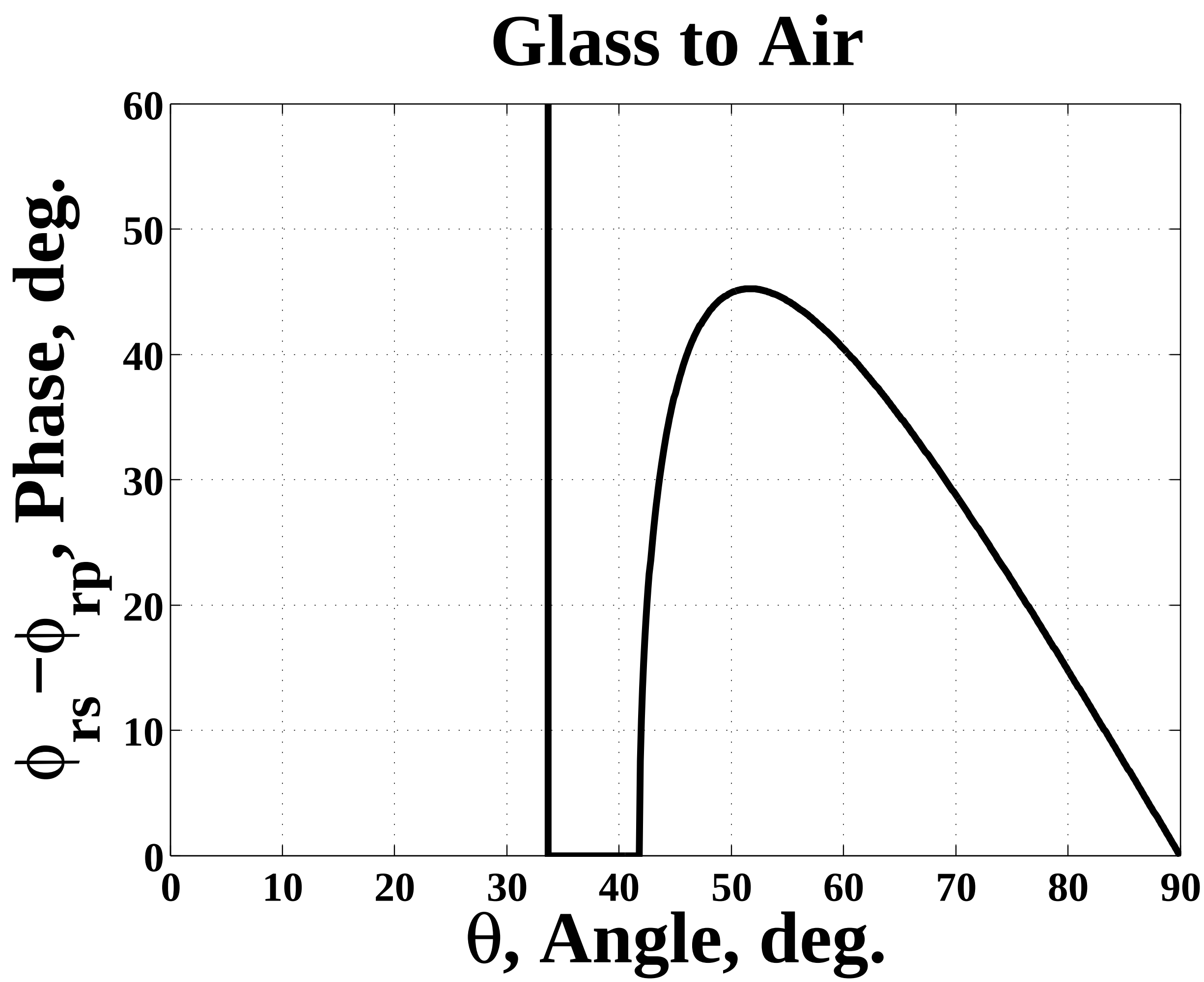
Glass to Air (dB)



Glass to Air (Phase)



Glass to Air (Phase Difference)



See Fresnel Rhomb Later in this Chapter for Application

“Reflections” on Fresnel Coefficients

Take–Away Message

- Normal–incidence reflection goes up with index mismatch.

$$R_s(0) = R_p(0)$$

- Reflection increases with angle for S polarization.
- Reflection decreases to zero at Brewster’s angle and then increases for P polarization.
- Reflection is always greater for S than P
- Grazing–angle reflection is $R_s(90^\circ) = R_p(90^\circ) = 1$.
- Conservation: $R_s + T_s = 1$ and $R_p + T_p = 1$
- $R(\theta)$ at $n_1 \rightarrow n_2$ interface equals $R(\theta')$ for propagation in opposite direction.

Complex Index of Refraction

- Plane Wave

$$Ee^{j(\omega t - n\mathbf{k}\cdot\mathbf{r})}$$

- Complex Index of Refraction, $n = n_r - jn_i$

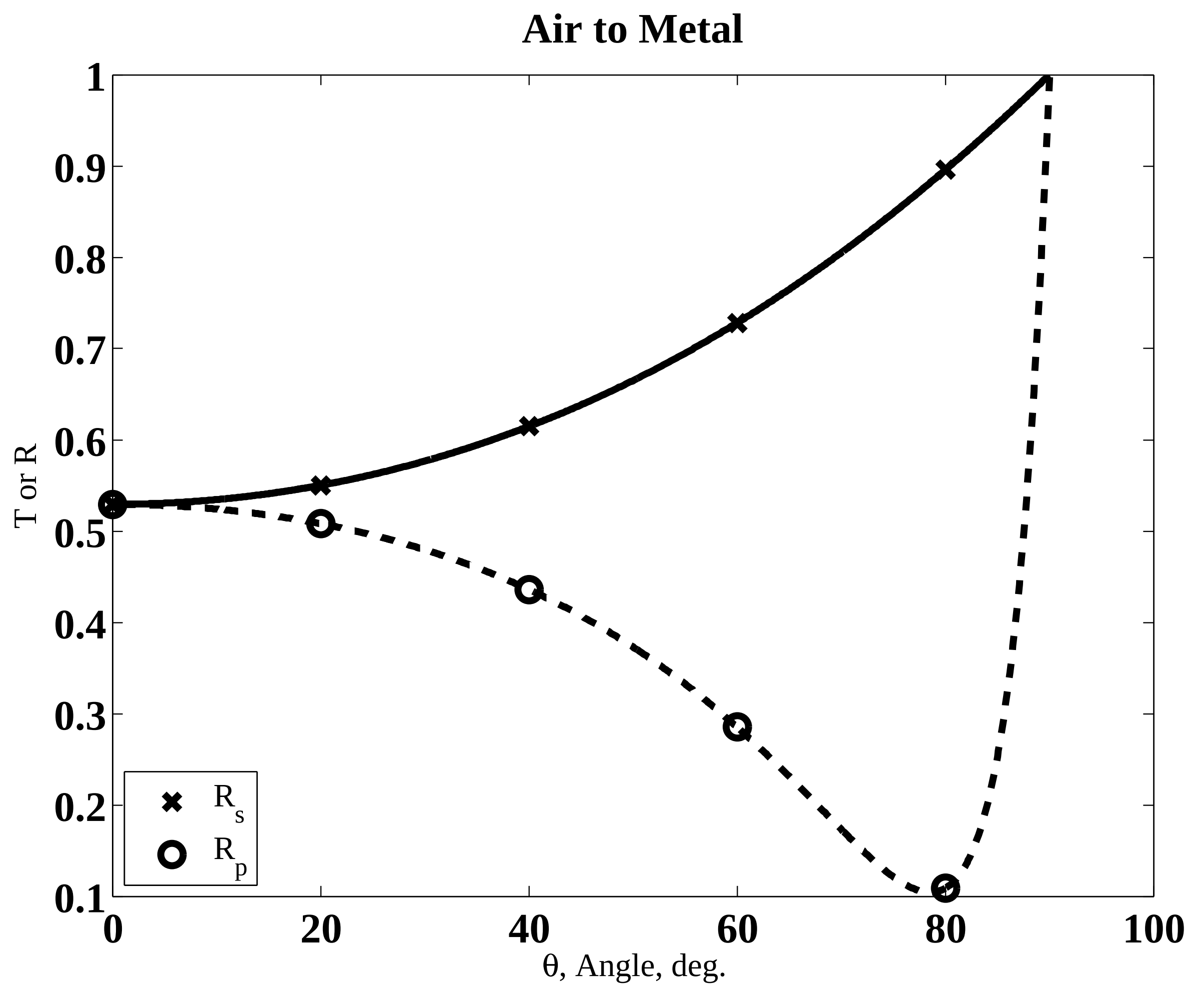
$$Ee^{j(\omega t - n_r\mathbf{k}\cdot\mathbf{r} + jn_i\mathbf{k}\cdot\mathbf{r})} = Ee^{j(\omega t - n_r\mathbf{k}\cdot\mathbf{r})}e^{-n_i\mathbf{k}\cdot\mathbf{r}}$$

  – Decaying Wave in the $\mathbf{k}$ direction

- Boundary Conditions at an Interface (Again)
  – Transverse $\mathbf{k}$ Conserved (Real and Imaginary)
  – Input $n_i k_{transverse} = 0$ Because n_i=0
  – Output $\Im\mathbf{k}$ Must Be in $\hat{z}$ Direction

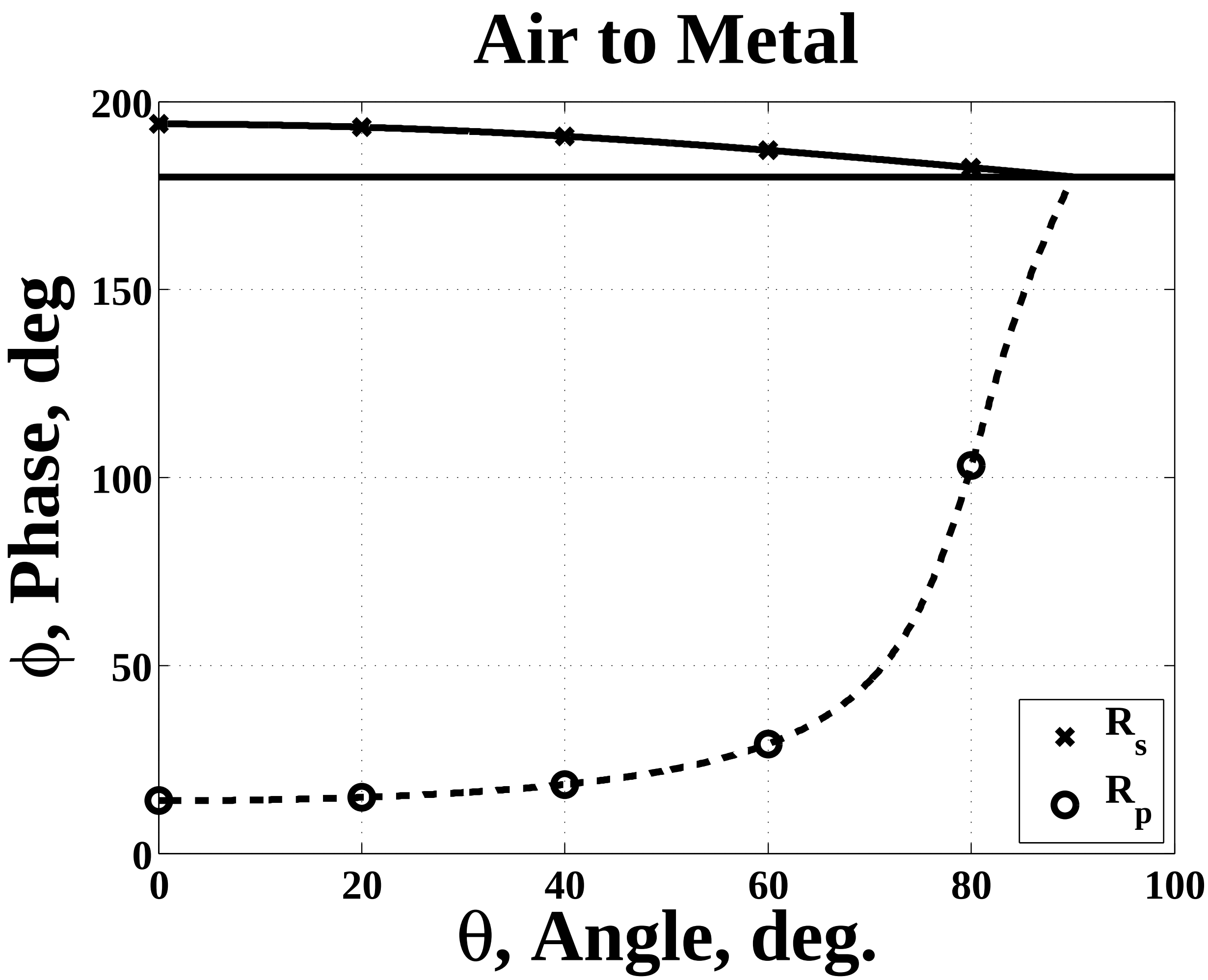
$$Ee^{j(\omega t - n_r\mathbf{k}\cdot\mathbf{r})}e^{-(n_i k)\hat{z}\cdot\mathbf{r}}$$

Air to Metal (Power on Linear Scale)



Note Pseudo–Brewster Angle (Vertical Axis Begins at 0.1)

Air to Metal (Phase)



Note Pseudo–Brewster Angle (Large Phase Change)

Devices for Polarization

- Polarizers Block One Polarization
  - Reflect it
  - Absorb it
- Waveplates Retard Phases of Linear Polarization
  - Birefringence
  - Total–Internal Reflection
- Rotators Retard Phases of Circular Polarization
  - Chiral Molecules (Reciprocal, to Be Defined Later)
  - Magneto–Optical Devices (Non–Reciprocal)

Brewster Plates

- At Brewster's Angle $T_p = 1$, $T_s < T_p$

$$\rho_s = \frac{\cos\theta_i - \left(\frac{n_2}{n_1}\right)^2 \cos\theta_i}{\cos\theta_i + \left(\frac{n_2}{n_1}\right)^2 \cos\theta_i} = \frac{1 - \left(\frac{n_2}{n_1}\right)^2}{1 + \left(\frac{n_2}{n_1}\right)^2} \qquad R_s = \rho_s \rho_s^* = \left(\frac{1 - \left(\frac{n_2}{n_1}\right)^2}{1 + \left(\frac{n_2}{n_1}\right)^2}\right)^2$$

- Transmission: $T_{pbp}^2 = 1$ (Neglecting Absorption)

$$T_{sbp}^2 = (1 - \rho_s \rho_s^*)^2 = \left[1 - \left(\frac{1 - \left(\frac{n_2}{n_1}\right)^2}{1 + \left(\frac{n_2}{n_1}\right)^2}\right)^2\right]^2 = \frac{16\left(\frac{n_2}{n_1}\right)^4}{\left[1 + \left(\frac{n_2}{n_1}\right)^2\right]^4}$$

Brewster Plates and Stacks

- Extinction Ratio (Absorption Cancels)

$$\frac{T_{pbp}}{T_{sbp}} = \frac{\left[1 + \left(\frac{n_2}{n_1}\right)^2\right]^4}{16\left(\frac{n_2}{n_1}\right)^4}$$

  - Glass in Air $R_s \approx 0.15$, Extinction ≈ 1.38 (Terrible)
  - Germanium In Air $R_s \approx 0.78$, Extinction ≈ 20.4

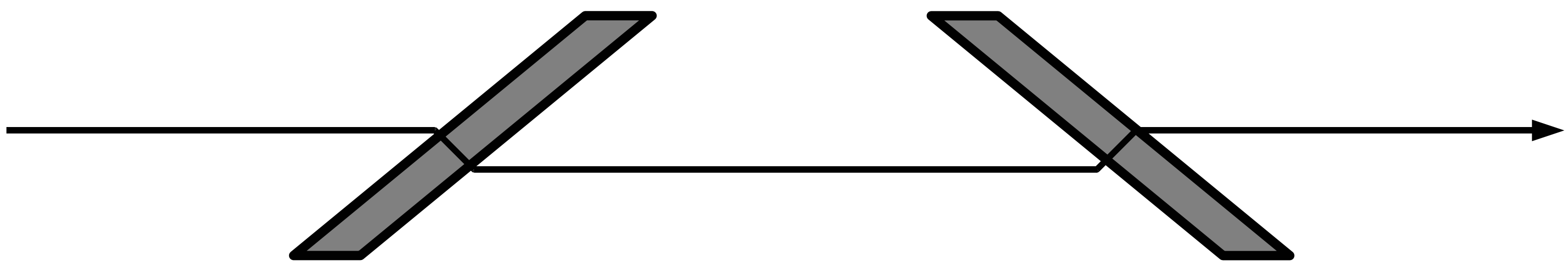
- Stack of m Plates

$$\left(T_{pbp}/T_{sbp}\right)^m$$

  - 10 Plates: 24.5 for Glass. 2 Plates: 400 for Germanium.
  - In Theory 10 Germanium plates gives 10^{13}.

Tent Polarizers

- Avoid Dogleg Problem
- Multiple Pairs: Tent–in–a–Tent
- Plate Size Proportional to $1/\tan\theta_B$ (Big?)



- Often Used for High Power
- More Practical for Infrared (Using Germanium)

Other Polarizers

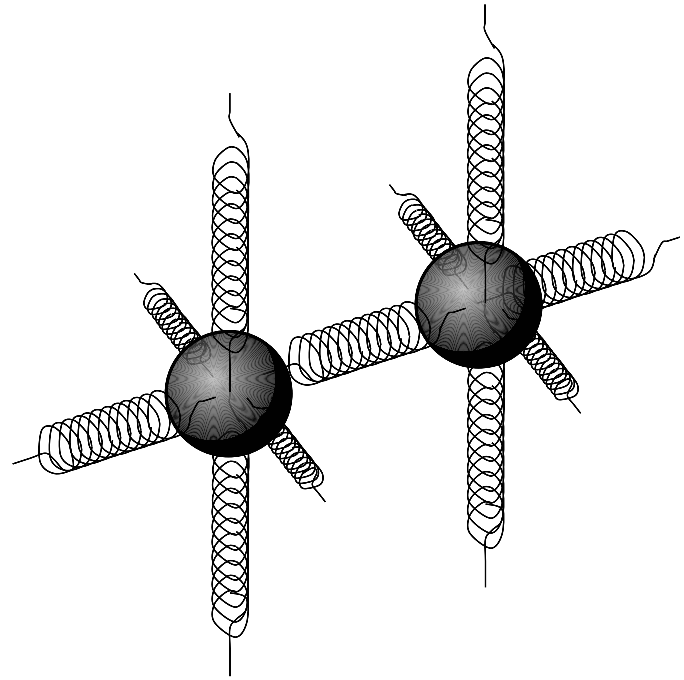
- Wire Grid
  - Conductive Cylinders
  - Pass $\perp$ Axes
  - Diffraction Issues (Ch. 8)
  - Low Power
  - Extinction to 300
- Glan–Thompson
  - Prism Polarizer
  - Based on Birefringence
  - Extinction Ratio to 10^5
  - Limited Power? (Adhesive)
- Beamspitting Cubes
  - Use Both Polarizations
  - Fair Performance
  - Moderate Power
- Polaroid H–Sheets
  - Polyvinyl Alcohol/Iodine
  - Similiar to Wire Grid
  - Specification
    * T% Total (unpol in)
    * HN–50 is Perfect
    * HN for "Neutral"
    * HR for Infrared
    * Good = *e.g.* HN–42
    * Normally Uncoated
  - Limited Passband
  - Limted Power
  - Large Size
  - Low Cost

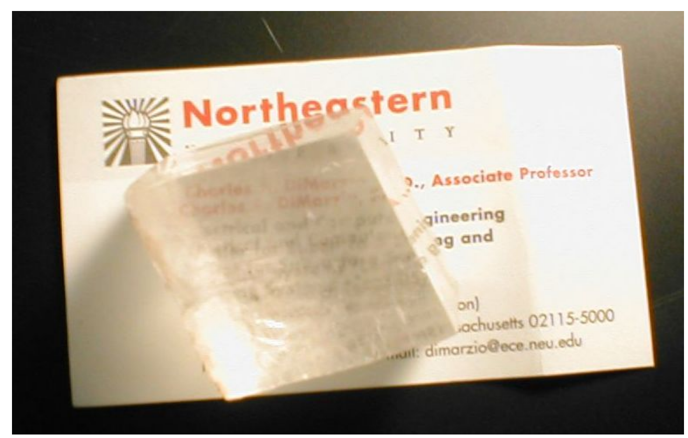
Birefringence

- Two Indices of Refraction
  - Different Ray Bending (Double Image)
  - Different Speeds
- Epsilon Tensor
  - 3–D Matrix
  - Can be Diagonalized
  - Two or Three Eigenvalues
    - ∗ Uniaxial

$$\mathcal{E} = \begin{pmatrix} \epsilon_{xx} & 0 & 0 \\ 0 & \epsilon_{yy} & 0 \\ 0 & 0 & \epsilon_{yy} \end{pmatrix}$$

      - · Ordinary Ray (y Polarized)
      - · Extraordinary Ray (x)
    - ∗ Biaxial (All 3 Different)





Wave in Birefringent Crystal

- Before the Crystal ($z < 0$)

$$\mathbf{E}_{in} = \left(E_{xi}\hat{x} + E_{yi}\hat{y}\right)e^{j(\omega t - kz)}$$

- In the Crystal ($0 < z < \ell$)

$$\mathbf{E} = \tau_1\left(E_{xi}\hat{x}e^{j(\omega t - kn_{xx}z)} + E_{yi}\hat{y}e^{j(\omega t - kn_{yy}z)}\right)$$

- At the End

$$\mathbf{E} = \tau_1\left(E_{xi}\hat{x}e^{j(\omega t - kn_{xx}\ell)} + E_{yi}\hat{y}e^{j(\omega t - kn_{yy}\ell)}\right)$$

- Beyond the Crystal ($\ell < z$)

$$\mathbf{E} = \tau_1\tau_2\left(E_{xi}\hat{x}e^{j[\omega t - kn_{xx}\ell - k(z-\ell)]} + E_{yi}\hat{y}e^{j[\omega t - kn_{yy}\ell - k(z-\ell)]}\right)$$

After the Birefringent Crystal

- From Previous Page

$$\mathbf{E}=\tau_1\tau_2\left(E_{xi}\widehat{x}e^{j[\omega t-kn_{xx}\ell-k(z-\ell)]}+E_{yi}\widehat{y}e^{j[\omega t-kn_{yy}\ell-k(z-\ell)]}\right)$$

- Regroup

$$\mathbf{E}=\tau_1\tau_2\left(E_{xi}e^{-jk(n_{xx}-1)\ell}\widehat{x}+E_{yi}e^{-jk(n_{yy}-1)\ell}\widehat{x}\right)e^{j(\omega t-kz)}\qquad(\ell<z)$$

- Simply

$$\mathbf{E}_{out}=\left(E_{xo}\widehat{x}+E_{yo}\widehat{y}\right)e^{j(\omega t-kz)}\qquad(\ell<z),$$

with

$$E_{xo}=\tau_1\tau_2E_{xi}e^{-jk(n_{xx}-1)\ell}\qquad E_{yo}=\tau_1\tau_2E_{yi}e^{-jk(n_{yy}-1)\ell}$$

- Phase Difference between E_{xo} and E_{yo}

Phases at Output of Birefringent Crystal

- Previous Equation

$$E_{xo} = \tau_1 \tau_2 E_{xi} e^{-jk(n_{xx}-1)\ell}$$

$$E_{yo} = \tau_1 \tau_2 E_{yi} e^{-jk(n_{yy}-1)\ell}$$

- Phase Difference

$$\delta\phi = k\ell \left(n_{yy} - n_{xx}\right)$$

- Half–Wave Plate

$$\delta\phi_{hwp} = \pi = k\ell \left(n_{yy} - n_{xx}\right)$$

$$OPD = n_{yy}\ell - n_{xx}\ell = \frac{\lambda}{2}$$

  – Reflects Polarization

- Quarter–Wave Plate

$$\delta\phi_{qwp} = \frac{\pi}{2} = k\ell \left(n_{yy} - n_{xx}\right)$$

$$n_{yy}\ell - n_{xx}\ell = \frac{\lambda}{4}.$$

  – Quartz at 589.3nm,

$$n_{yy} - n_{xx} = 1.5534 - 1.5443$$

  – Thickness 16.24μm

- 5/4–Wave Plate

$$\frac{d\delta\phi_{qwp}}{dT} = k\ell \frac{d\left(n_{yy} - n_{xx}\right)}{dT}$$

$$\frac{d\delta\phi_{5qwp}}{dT} = 5k\ell \frac{d\left(n_{yy} - n_{xx}\right)}{dT}$$

  – Watch Out for Dispersion

Retardation Dispersion

- Wavelength *vs.* OPL

$$\frac{d\delta\phi_{qwp}}{d\lambda} = \frac{2\pi}{\lambda^2}\ell\left(n_{yy} - n_{xx}\right) \qquad \frac{d\delta\phi_{5qwp}}{d\lambda} = 5\frac{2\pi}{\lambda^2}\ell\left(n_{yy} - n_{xx}\right)$$

  - Example:
    * Bandwidth of $\delta\lambda = 100nm$ at $\lambda = 800nm$
    * Phase Dispersion 6° for Zero Order 1/4–Wave Plate
    * 30° for 5/4–Wave Plate

- Birefringence Dispersion

$$\delta\phi\left(\lambda\right) = \frac{2\pi}{\lambda}\ell\left(n_{yy}\left(\lambda\right) - n_{xx}\left(\lambda\right)\right)$$

- Use One Against the Other to Make a Wide–Band QWP

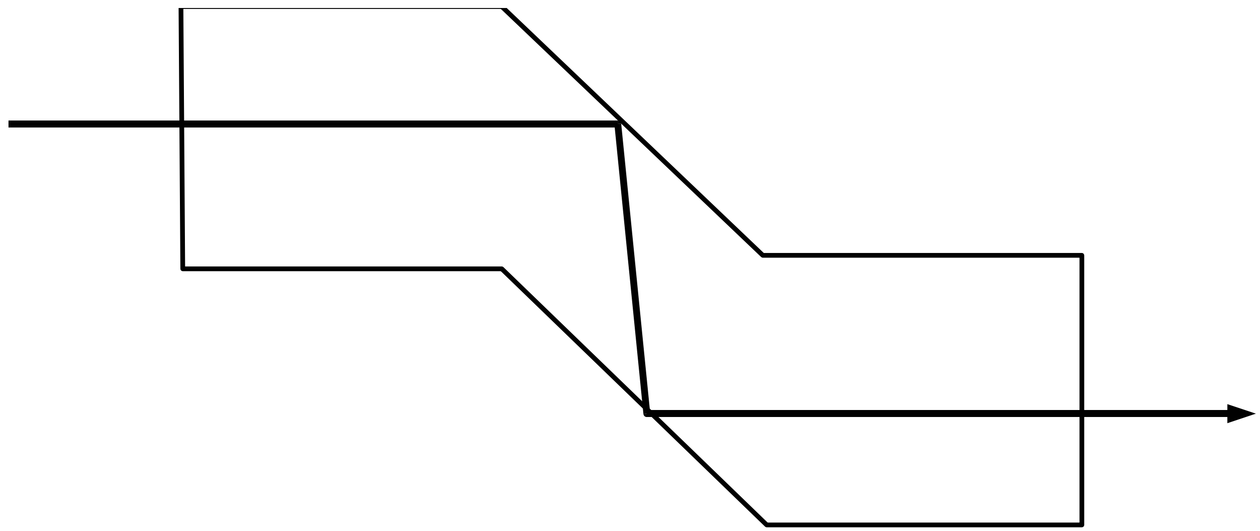
Electrically–Induced Birefringence

- Electric Field Alters Symmetry
- Birefringence Proportional to DC Voltage

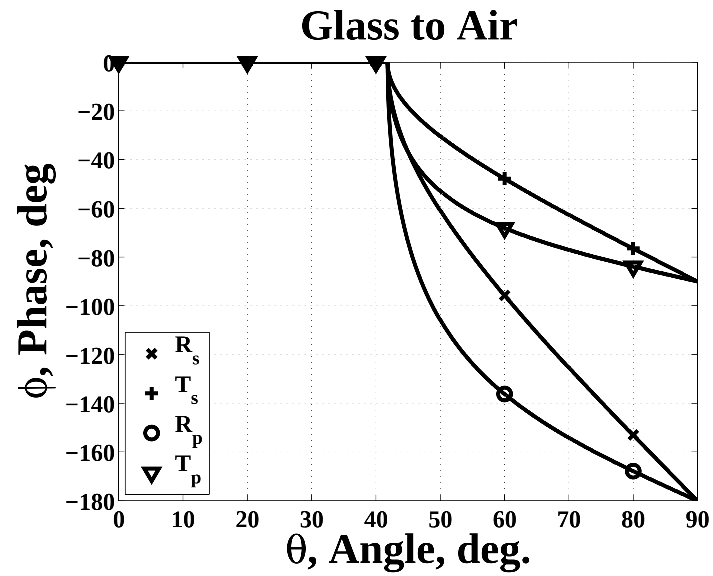
$$\delta\phi = \pi \frac{V}{V_\pi}$$

- Applications
  - Phase Modulation (Field Paralel to One Axis)
  - Frequency Modulation
    (Phase Modulation in Laser Cavity, Ch. 7)
  - Amplitude Modulation
    (Field at 45° with Crossed Polarizer at Output)

Fresnel Rhomb



- Phase Difference in TIR
  - See Fresnel Equations
  - Top Two Curves →
  - Two Reflections for 90°
- Less Dispersion in Birefringence
- Difficult Alignment (?)



Polarization Rotator

- Reciprocal Rotator (*e.g.* Sugar in Water)

$$\delta\zeta = \kappa C \ell$$

  - $\kappa =$ Specific Rotary Power
  - $C =$ Concentration
  - $\ell =$ Length
- Rotation in Either Direction
  - Left (Levulose) $\kappa > 0$
  - Right (Dextrose) $\kappa < 0$
- Same Sign for Reverse Propagation (*e.g.* Reflection)
  - Round–Trip Restores Original Polarization

- Non–Reciprocal Rotator (*e.g.* Faraday Rotator)
  - Underlying Physics (DC Magnetic Field)

$$\mathbf{a} = -\frac{e}{m}\mathbf{v} \times \mathbf{B}$$

  - Result ($v=$ Verdet Constant)

$$\delta\zeta = v\mathbf{B} \cdot \hat{z}\ell$$

- Reverse Propagation

$$\delta\zeta = v\mathbf{B} \cdot (-\hat{z})\ell$$

  - Round–Trip Doubles Rotation
  - Application: Faraday Isolator

Jones Vectors and Matrices

- Jones Vectors, $\mathbf{E}$
  - x and y Components for $\hat{z}$ Propagation
  - Alternative Basis Sets
- Jones Matrices, $\mathcal{J}$
  - Devices that Change Polarization
  - Transformations that Change Coordinates

$$\mathbf{E}_1 = \mathcal{J}\mathbf{E}_0$$

- Cascading Matrices (Right to Left)

$$\mathbf{E}_m = \mathcal{J}_m \mathcal{J}_{m-1} \ldots \mathcal{J}_2 \mathcal{J}_1 \mathbf{E}_0$$

Irradiance and Power

- Basic Equations

$$P = IA = \frac{\mathbf{E}^\dagger \mathbf{E}}{Z} A$$

  - $\mathbf{E}^\dagger$ is Hermitian Adjoint
  - Conjugate Transposed

$$\mathbf{E} = \begin{pmatrix} E_x \\ E_y \end{pmatrix}$$

$$\mathbf{E}^\dagger = \begin{pmatrix} E_x^* & E_y^* \end{pmatrix}$$

- Power

$$P = IA = \frac{E_x^* E_x + E_y^* E_y}{Z} A$$

- Common Approach
  - Assumes $Z_{out} = Z_{in}$
  - Input

$$\mathbf{E}_0^\dagger \mathbf{E}_0 = 1$$

  - Transmission

$$T = \mathbf{E}_{out}^\dagger \mathbf{E}_{out}$$

  - Output

$$I_{out} = T I_{in}$$

$$P_{out} = T P_{in}$$

  - Field Amplitudes Lost

Some Basic Jones Matrices: Polarizers

- Diagonal Matrices
  - Input

$$\mathbf{E}_0 = \begin{pmatrix} E_{x0} \\ E_{y0} \end{pmatrix}$$

  - Output

$$\mathbf{E}_1 = \begin{pmatrix} j_{11} & 0 \\ 0 & j_{22} \end{pmatrix} \begin{pmatrix} E_{x0} \\ E_{y0} \end{pmatrix}$$

$$E_{x1} = j_{11} E_{x0}$$

$$E_{y1} = j_{22} E_{y0}$$

  - No Cross–Coupling
  - E_x & E_y are Eigenvectors

- Perfect $\hat{x}$ Polarizer

$$\boxed{\mathcal{P}_x = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}$$

- Perfect $\hat{y}$ Polarizer

$$\mathcal{P}_y = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

  - Later: Arbitrary Polarizer
- Realistic $\hat{x}$ Polarizer

$$\boxed{\mathcal{P}_x = \begin{pmatrix} \tau_x & 0 \\ 0 & \tau_y \end{pmatrix}}$$

Realistic Polarizer Example (1)

- Insertion Loss (Fresnel Reflections ≈ 4% per Surface)

$$\tau_x = \sqrt{1 - 0.08}$$

- Extinction Ratio = 10,000

$$\tau_y = \tau_x / \sqrt{10,000}$$

- Input Polarization at Angle ζ

$$\mathbf{E}_0 = \begin{pmatrix} \cos\zeta \\ \sin\zeta \end{pmatrix} \qquad |\mathbf{E}_0|^2 = 1$$

- Output Field

$$\mathbf{E}_1 = \mathcal{P}_x \mathbf{E_0} = \begin{pmatrix} \tau_x & 0 \\ 0 & \tau_y \end{pmatrix} \begin{pmatrix} \cos\zeta \\ \sin\zeta \end{pmatrix}$$

- Output Field

$$\mathbf{E}_1 = \begin{pmatrix} \tau_x \cos\zeta \\ \tau_y \sin\zeta \end{pmatrix}$$

- Transmission

$$T = \mathbf{E}_1^\dagger \mathbf{E}_1$$

- Adjoint of Product

$$(\mathcal{A}\mathcal{B})^\dagger = \mathcal{B}^\dagger \mathcal{A}^\dagger.$$

- Output Power

$$\mathbf{E}_1^\dagger \mathbf{E}_1 = \mathbf{E}_0^\dagger \mathcal{P}_x^\dagger \mathcal{P}_x \mathbf{E}_0$$

Realistic Polarizer Example (2)

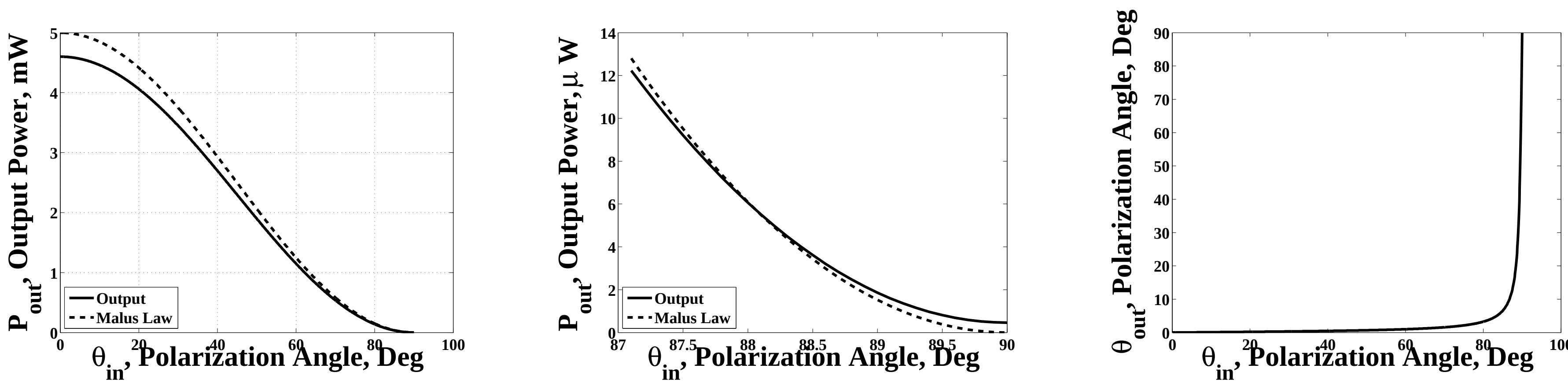
- Transmission (Remember ζ is Angle of Input Polarization)

$$T = \begin{pmatrix} \cos\zeta & \sin\zeta \end{pmatrix} \begin{pmatrix} \tau_x^* & 0 \\ 0 & \tau_y^* \end{pmatrix} \begin{pmatrix} \tau_x & 0 \\ 0 & \tau_y \end{pmatrix} \begin{pmatrix} \cos\zeta \\ \sin\zeta \end{pmatrix}$$

$$T = T_x \cos^2\zeta + T_y \sin^2\zeta$$

- Angle of Output Polarization

$$\tan\zeta_{out} = \frac{E_{yout}}{E_{xout}} = \frac{\tau_x \cos\zeta}{\tau_y \sin\zeta}$$



Jones Matrix for a Waveplate

- Phase Difference, ϕ

$$\boxed{\mathcal{W} = \begin{pmatrix} e^{j\phi/2} & 0 \\ 0 & e^{-j\phi/2} \end{pmatrix}}$$

- An Alternate Notation

$$\mathcal{W} = \begin{pmatrix} e^{j\phi} & 0 \\ 0 & 1 \end{pmatrix}$$

- Others Possible
  - Overall Phase Shift
  - Normally Present
  - Normally Not Important

- Quarter–Wave Plate

$$\mathcal{Q} = \begin{pmatrix} e^{-j\pi/4} & 0 \\ 0 & e^{j\pi/4} \end{pmatrix}$$

- or. . .

$$\boxed{\mathcal{Q} = \begin{pmatrix} 1 & 0 \\ 0 & j \end{pmatrix}}$$

- Half–Wave Plate

$$\boxed{\mathcal{H} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}$$

Coordinate Transforms

- So far, all matrices have been diagonal.
  - They are written in a way that they act on eigenvectors.
  - The diagonal elements are the eigenvalues.
- Otherwise...
  - we can resolve the input vector into components along the eigenvalues,
  - solve each problem independently,
  - and recombine to get the result.
- Equivalently...
  - we can use a transform matrix to find the eigenvector components of the input,
  - Multiply by the diagonal matrix,
  - and use the inverse transform back to original coordinates.

Rotator and Coordinate Rotation

- Rotator (Angle ζ)

$$\boxed{\mathcal{R}(\zeta) = \begin{pmatrix} \cos\zeta & -\sin\zeta \\ \sin\zeta & \cos\zeta \end{pmatrix}}$$

  - $\zeta > 0$ Rotates Right
  - $\hat{x}$ to $\hat{y}$
  - $\hat{y}$ to $-\hat{x}$
- Represents a Device
  - Actual Rotation of Polarization
  - *e.g.* Sugar/Water or Faraday Rotator

- Coordinate Rotation
  - for Devices at Arbitrary Angles
  - to Change Coordinates by Choice
- Old Coordinates $\hat{x}$,$\hat{y}$
- New Coordinates $\hat{x}'$,$\hat{y}'$ Rotated by $+\zeta$
- Mathematically Same as Rotating Vector by $-\zeta$

$$\mathcal{R}(-\zeta)\,\mathbf{E}_1 = \mathcal{R}^{\dagger}(\zeta)\,\mathbf{E}_1$$

Rotated Device Jones Matrix

- Rotate Coordinates of Input Vector to Eigenvectors of Device
  - Original Coordinates $\widehat{x}$,$\widehat{y}$
  - New Coordinates $\widehat{x}'$,$\widehat{y}'$

$$\mathbf{E}_1' = \mathcal{R}\left(-\zeta\right)\mathbf{E}_1 = \mathcal{R}^\dagger\left(\zeta\right)\mathbf{E}_1$$

- Operate with the Device in its Own Coordinates $\widehat{x}'$,$\widehat{y}'$

$$\mathbf{E}_2' = \mathcal{J}'\mathcal{R}^\dagger\left(\zeta\right)\mathbf{E}_1$$

- Rotate Back to Original Coordinates

$$\mathbf{E}_2 = \mathcal{R}\left(\zeta\right)\mathbf{E}_2' = \mathcal{R}\left(\zeta\right)\mathcal{J}'\mathcal{R}^\dagger\left(\zeta\right)\mathbf{E}_1$$

- Do it All at Once... $\mathbf{E}_2 = \mathcal{J}\mathbf{E}_1$ ... where

$$\boxed{\mathcal{J} = \mathcal{R}\left(\zeta\right)\mathcal{J}'\mathcal{R}^\dagger\left(\zeta\right)}$$

Coordinate Transform Example (Page 1)

- Input Field

$$\mathbf{E}_{in} = \begin{pmatrix} E_x \\ E_y \end{pmatrix},$$

- Polarizer ($\hat{x}$ Polarizer Rotated through ζ)

$$\mathcal{P}_\zeta = \mathcal{R}\left(\zeta\right)\mathcal{P}_x\mathcal{R}^\dagger\left(\zeta\right)$$

- Output Field

$$\mathbf{E}_{out} = \mathcal{P}_\zeta\mathbf{E}_{in} = \mathcal{R}\left(\zeta\right)\mathcal{P}_x\mathcal{R}^\dagger\left(\zeta\right)\begin{pmatrix} E_x \\ E_y \end{pmatrix}$$

- Option 1: Matrix–by–Matrix Multiplication

Coordinate Transform Example (Page 2)

- Option 2: New Matrix for Device

$$\mathcal{P}_\zeta = \begin{pmatrix} \cos\zeta & -\sin\zeta \\ \sin\zeta & \cos\zeta \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \cos\zeta & \sin\zeta \\ -\sin\zeta & \cos\zeta \end{pmatrix}$$

- Polarizer Matrix in $\hat{x}$,$\hat{y}$ Coordinates

$$\mathcal{P}_\zeta = \begin{pmatrix} \cos^2\zeta & -\cos\zeta\sin\zeta \\ -\cos\zeta\sin\zeta & \sin^2\zeta \end{pmatrix}$$

- Output (No Matter How We Do the Multiplication)

$$\begin{pmatrix} \cos^2\zeta & -\cos\zeta\sin\zeta \\ -\cos\zeta\sin\zeta & \sin^2\zeta \end{pmatrix} \begin{pmatrix} E_x \\ E_y \end{pmatrix} = \begin{pmatrix} E_x\cos^2\zeta - E_y\cos\zeta\sin\zeta \\ -E_x\cos\zeta\sin\zeta + E_y\sin^2\zeta \end{pmatrix}$$

Q: Compare to Malus' Law, Rotating Input or Rotating Device.

Coordinate Transform Example (Page 3)

- Output (Previous Page)

$$\begin{pmatrix} \cos^2\zeta & -\cos\zeta\sin\zeta \\ -\cos\zeta\sin\zeta & \sin^2\zeta \end{pmatrix} \begin{pmatrix} E_x \\ E_y \end{pmatrix} = \begin{pmatrix} E_x\cos^2\zeta - E_y\cos\zeta\sin\zeta \\ -E_x\cos\zeta\sin\zeta + E_y\sin^2\zeta \end{pmatrix}$$

- Input: $E_x = \cos\zeta_1$, $E_y = \sin\zeta_1$
- Use Trigonometric Identities
- Output Angle (Always ζ for Perfect Polarizer):

$$\tan\zeta_{out} = \frac{-E_x\cos\zeta\sin\zeta + E_y\sin^2\zeta}{E_x\cos^2\zeta - E_y\cos\zeta\sin\zeta} =$$

$$\frac{\sin\zeta}{\cos\zeta} \times \frac{-E_x\cos\zeta + E_y\sin\zeta}{E_x\cos\zeta - E_y\sin\zeta} = \frac{\sin\zeta}{\cos\zeta}$$

Rotated Device Couples Polarizations

- Polarizer Matrix in $\hat{x}'$,$\hat{y}'$ Coordinates

$$\mathcal{P}_{x'} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad \mathcal{P}_{x'}\mathbf{E}_{y'} = 0$$

- Polarizer Matrix in $\hat{x}$,$\hat{y}$ Coordinates

$$\mathcal{P}_{\zeta} = \begin{pmatrix} \cos^2\zeta & -\cos\zeta\sin\zeta \\ -\cos\zeta\sin\zeta & \sin^2\zeta \end{pmatrix}$$

- Any Device Matrix is Diagonal in its Eigenvector Coordinates

$$\mathcal{P}_{x'} = \begin{pmatrix} \tau_{x'} & 0 \\ 0 & \tau_{y'} \end{pmatrix}$$

Q: What is $\mathcal{P}_x$ for this $\mathcal{P}_{x'}$?

Three–Polarizer Thought Experiment

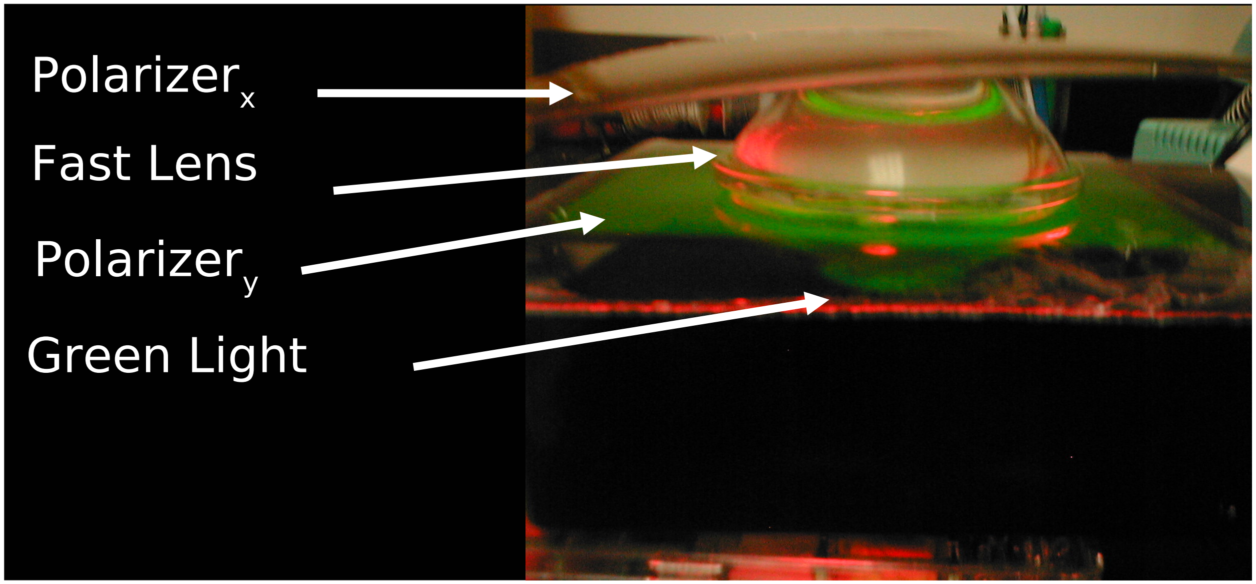
- Combined Matrix

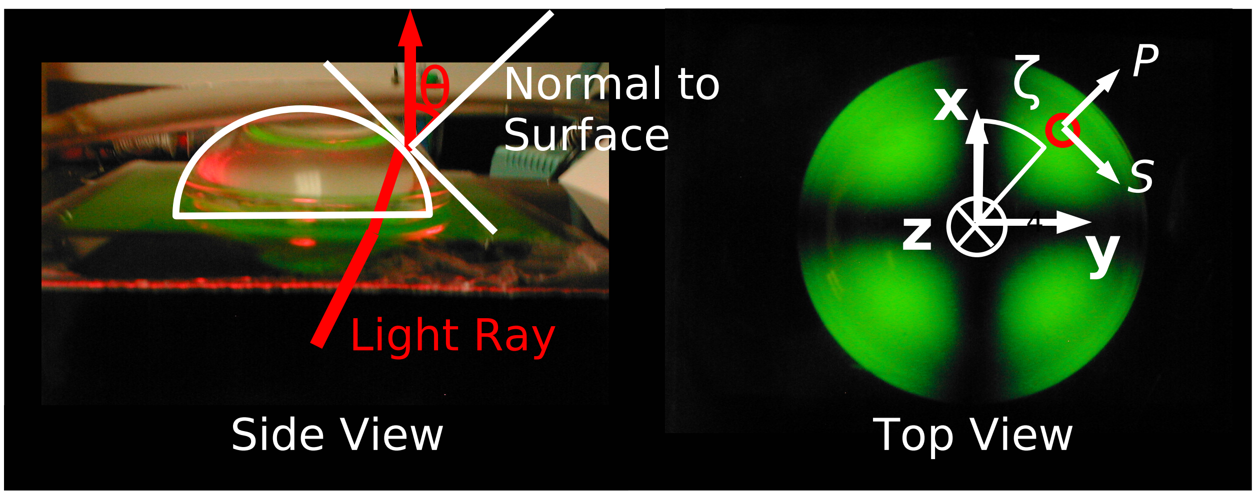
$$\mathcal{P}_x\mathcal{P}_\zeta\mathcal{P}_y = \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}\cos^2\zeta & -\cos\zeta\sin\zeta\\ -\cos\zeta\sin\zeta & \sin^2\zeta\end{pmatrix}\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}$$

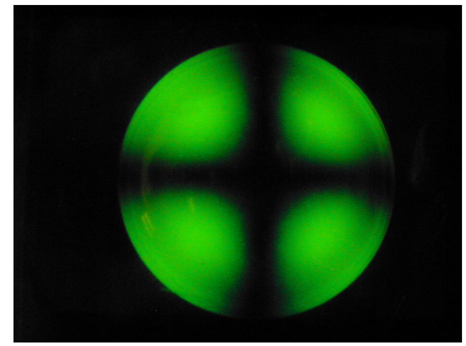
$$= \begin{pmatrix}0 & -\cos\zeta\sin\zeta\\ 0 & 0\end{pmatrix}$$

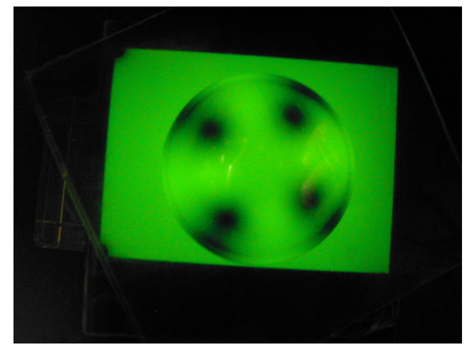
  - At Zero Degrees, $T = 0$
  - At 90 Degrees, $T = 0$
  - At 45 Degrees, . . .
    - $T = 0.25$ for $\hat{y}$ Input
    - $T = 0.125$ for Random Input (later)

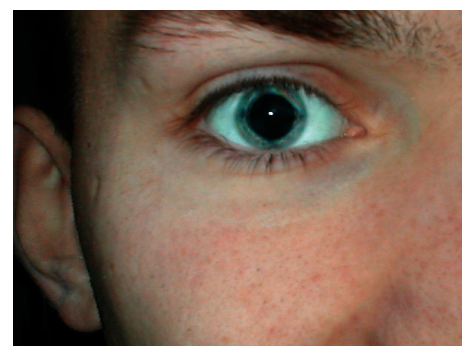
Coordinate Transforms Gone Wild: Maltese Cross











Maltese Cross Analysis (1)

- Solution With Fresnel Reflection & Coordinate Transforms
- Curved Lens Surface as a Polarizer
  - Fresnel Reflection with Varying Plane of Incidence
  - Natural Coordinate System (Eigenvectors): P,S

Fresnel Coef. Matrix $$\mathcal{F}'(\theta,0) = \begin{pmatrix} \tau_p(\theta) & 0 \\ 0 & \tau_s(\theta) \end{pmatrix}$$

- Working Coordinate System: $\hat{x}$,$\hat{y}$
  ($\zeta = 0$ when P is in $\hat{x}$ Direction)

$$\mathcal{F}(\theta,\zeta) = \mathcal{R}(\zeta)\,\mathcal{F}'(\theta,0)\,\mathcal{R}^{\dagger}(\zeta)$$

Maltese Cross Analysis (2)

- Assume Polarizers Are Perfect (Avoids dealing with partial polarization)

- Assume $\hat{x}$ Polarization out of First Polarizer

$$\mathbf{E}_{out} = \mathcal{P}_y \mathcal{R}(\zeta)\, \mathcal{F}'(\theta, 0)\, \mathcal{R}^\dagger(\zeta) \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

- Final Output

$$\mathbf{E}_{out} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \tau_p(\theta)\cos^2\zeta - \tau_s(\theta)\sin^2\zeta \\ \tau_p(\theta)\cos\zeta\sin\zeta - \tau_s(\theta)\cos\zeta\sin\zeta \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ \tau_p(\theta)\cos\zeta\sin\zeta - \tau_s(\theta)\cos\zeta\sin\zeta \end{pmatrix}$$

Maltese Cross Analysis (3)

- Output Field (Previous Page)

$$\mathbf{E}_{out} = \begin{pmatrix} 0 \\ \tau_p(\theta)\cos\zeta\sin\zeta - \tau_s(\theta)\cos\zeta\sin\zeta \end{pmatrix}$$

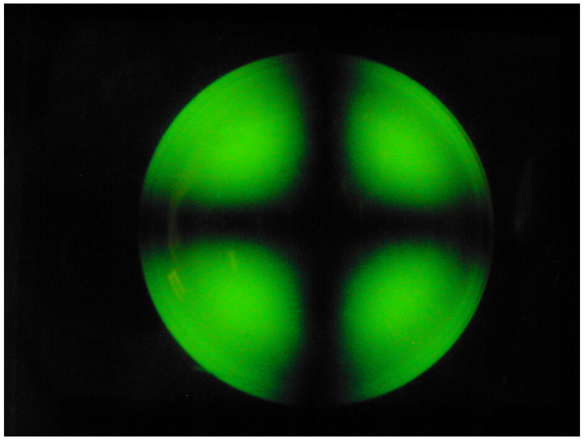
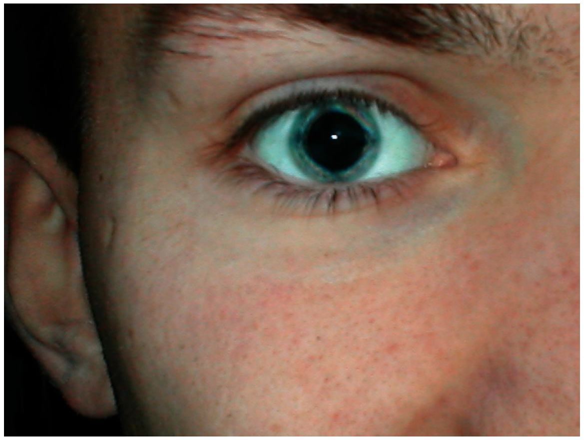
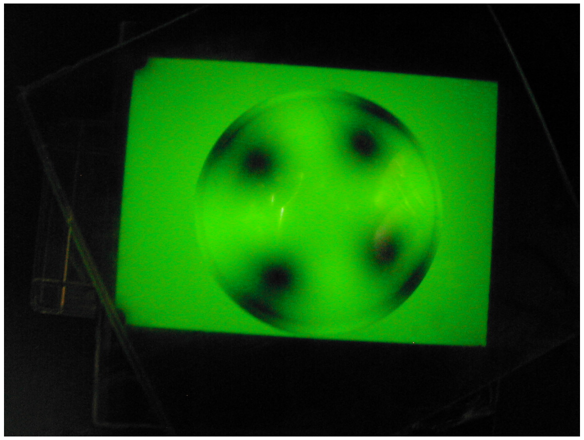
- For $\zeta = 0$ P Matches $\hat{x}$ ($\sin\zeta = 0$)
  - Input to Lens is P, Output is P (Eigenvector)
  - Output of Lens is Blocked by Final Polarizer; $\mathbf{E}_{out} = 0$
- For $\zeta = 90^\circ$ S Matches $\hat{x}$ ($\cos\zeta = 0$)
  - Input to Lens is S, Output is S (Eigenvector)
  - Output of Lens is Blocked by Final Polarizer; $\mathbf{E}_{out} = 0$
- Otherwise
  - Input to Lens is Superposition of P and S
  - P is Transmitted More than S
  - Output is Different Superpostion of P and S
  - Different Angle from Input; Not Completely Blocked

Maltese Cross Analysis (4)

- Output Field (Bottom of Page 2)

$$\mathbf{E}_{out}=\begin{pmatrix}0\\ \tau_p(\theta)\cos\zeta\sin\zeta-\tau_s(\theta)\cos\zeta\sin\zeta\end{pmatrix}$$

- At the Center (Normal Incidence)
  - Degenerate Eigenvalues; $\tau_p=\tau_s$
  - Zero Output

Q: What are the equations if polarizers are parallel (Right Picture)?

Another Transformation: Linear Basis to Circular

- QWP at 45°

$$\mathcal{Q}_{45} = \mathcal{R}_{45}\mathcal{Q}\mathcal{R}_{45}^{\dagger}$$

  – Simple Result

$$\mathcal{Q}_{45} = \mathcal{R}_{45}\mathcal{Q}\mathcal{R}_{45}^{\dagger} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & j \\ j & 1 \end{pmatrix}$$

- Converts
  – $\hat{x}$ to RHC
  – $\hat{y}$ to LHC
- Matrix for a Device: Physical Change of Polarization
- Coordinate Transform: Field Doesn't Change, Numbers Do

- Coordinate Transform

$$\mathbf{E} = \mathcal{Q}_{45}^{\dagger}\mathbf{E}'$$

$$\mathcal{J} = \mathcal{Q}_{45}\mathcal{J}'\mathcal{Q}_{45}^{\dagger}$$

- Example in $\hat{x}$,$\hat{y}$

$$\mathbf{E}_x = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

- in RHC/LHC

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & j \\ j & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ j \end{pmatrix}$$

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ j \end{pmatrix} = \frac{1}{\sqrt{2}}\hat{E}_r + \frac{j}{\sqrt{2}}\hat{E}_\ell$$

- Can Minimize Ambiguities in x, y or P,S

Matrix Properties: Unitary Matrices

- Transform Matrices Must not Change Power

$$\mathbf{E}_{out}^{\dagger}\mathbf{E}_{out} = \mathbf{E}_{in}^{\dagger}\mathcal{J}^{\dagger}\mathcal{J}\mathbf{E}_{in} = \mathbf{E}_{in}^{\dagger}\mathbf{E}_{in} \qquad \text{for all } \mathbf{E}_{in}$$

$$\mathcal{J}^{\dagger}\mathcal{J} = \mathcal{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

- Lossless Device (*e.g.* Perfect Waveplate or Rotator)

- Realistic Waveplate or Rotator
  - Unitary Matrix Multiplied by Scalar
  - Potential Simplification of Complicated Equations
  - Also Useful for "Single–Mode" Fiber

Matrix Properties: Eigenvectors

- Eigenvectors Are Natural Polarizations of the Device

$$\mathbf{E}_{out} = \text{Eigenvalue} \times \mathbf{E}_{in}$$

- Matrix is Diagonal in Coordinates Based on Eigenvectors

- Example: X Polarizer

$$\mathcal{P}_x = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad \text{Ideal}$$

$$\mathcal{P}_x = \begin{pmatrix} \tau_x & 0 \\ 0 & \tau_y \end{pmatrix} \qquad \text{Realistic}$$

Rotator Eigenvectors

- Matrix and Eigenvectors

$$\mathcal{R}(\zeta) = \begin{pmatrix} \cos\zeta & -\sin\zeta \\ \sin\zeta & \cos\zeta \end{pmatrix} \qquad \mathbf{E}_{RHC} = \begin{pmatrix} 1 \\ j \end{pmatrix} \qquad \mathbf{E}_{LHC} = \begin{pmatrix} 1 \\ -j \end{pmatrix}$$

- RHC Eigenvalue Solution

$$\mathcal{R}(\zeta)\,\mathbf{E}_{RHC} = \tau_{RHC}\mathbf{E}_{RHC}$$

$$\begin{pmatrix} \cos\zeta & -\sin\zeta \\ \sin\zeta & \cos\zeta \end{pmatrix} \begin{pmatrix} 1 \\ j \end{pmatrix} = \begin{pmatrix} \cos\zeta - j\sin\zeta \\ \sin\zeta + j\cos\zeta \end{pmatrix} = \begin{pmatrix} e^{-j\zeta} \\ je^{-j\zeta} \end{pmatrix} = e^{j\zeta} \begin{pmatrix} 1 \\ j \end{pmatrix}$$

- Eigenvalues

$$\tau_{rhc} = e^{j\zeta} \qquad \tau_{lhc} = e^{-j\zeta}$$

Circular Polarizer

- Configuration: QWP, Linear Polarizer at 45 Degrees, QWP
- Jones Matrix

$$\mathcal{J} = \mathcal{Q}_{90}\mathcal{P}_{45}\mathcal{Q} = \begin{pmatrix} j & 0 \\ 0 & 1 \end{pmatrix} \frac{1}{2} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & j \end{pmatrix} = \frac{1}{2} \begin{pmatrix} j & 1 \\ -1 & j \end{pmatrix}$$

- Eigenvectors

$$\mathbf{E}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix} = \mathbf{E}_{RHC} \qquad \mathbf{E}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix} = \mathbf{E}_{LHC}$$

- Eigenvalues

$$\tau_1 = 1 \qquad \tau_2 = 0$$

Non–Orthogonal Eigenvectors

- Polarizers at Zero and 10 Degrees

$$\mathcal{M} = \mathcal{R}_{10}\mathcal{P}\mathcal{R}_{10}^{\dagger}\mathcal{P} = \begin{pmatrix} \cos^2\zeta & 0 \\ \cos\zeta\sin\zeta & 0 \end{pmatrix}$$

- Eigenvectors

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \cos 10^\circ \\ \sin 10^\circ \end{pmatrix}$$

- Eigenvalues

$$0 \quad \text{and} \quad \cos 10^\circ$$

  - $\hat{y}$ Polarization Blocked by First Polarizer
  - $\hat{x}$ Polarization Passed by First Polarizer and $\cos 10^\circ$ Component Transmitted by Second
  - Output always at 10°

Jones Matrix Application: Amplitude Modulator

- Input $\hat{x}$ Polarized
- Electro–Optical Modulator

$$\mathbf{E}_{out} = \mathcal{P}_y \mathcal{R}_{45} \mathcal{M}\left(V\right) \mathcal{R}_{45}^{\dagger} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

- Electrically–Induced Birefringence

$$\mathcal{M}\left(V\right) =$$

$$\begin{pmatrix} e^{j\pi V/(2V_\pi)} & 0 \\ 0 & e^{-j\pi V/(2V_\pi)} \end{pmatrix}$$

- Output

$$\mathbf{E}_{out} = \mathcal{P}_y \mathcal{R}_{45} \mathcal{M}\left(V\right) \mathcal{R}_{45}^{\dagger} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

- At $V = 0$, $T = 0$:

$$\mathbf{E}_{out} = \mathcal{P}_y \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

- At $V = V_\pi$, $T = 1$:

$$\mathbf{E}_{out} = \mathcal{P}_y \begin{pmatrix} 0 & -1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

- At $V = V_\pi/2$, $T = 0.5$
  - $\Delta V \rightarrow$ Modulation
- Bias: $\mathcal{P}_y \mathcal{R}_{45} \mathcal{Q} \mathcal{M}\left(V\right) \mathcal{R}_{45}^{\dagger}$

$$\mathcal{Q}\mathcal{M}\left(V\right) = \begin{pmatrix} e^{j\frac{\delta\phi}{2}} & 0 \\ 0 & e^{-j\pi\frac{\delta\phi}{2}} \end{pmatrix}$$

$$\delta\phi = \pi V/\left(2V_\pi\right) + \pi/4$$

T/R Switch (Optical Circulator)

- Common Aperture
  - $T + R = 1$
  - Round–Trip

$$(1-R)\,F_{target}R$$

  - Optimize (Not Great)

$$d\left[(1-R)\,R\right]/dR = 0$$

$$R = \frac{1}{2} \qquad R(1-R) = \frac{1}{4}$$

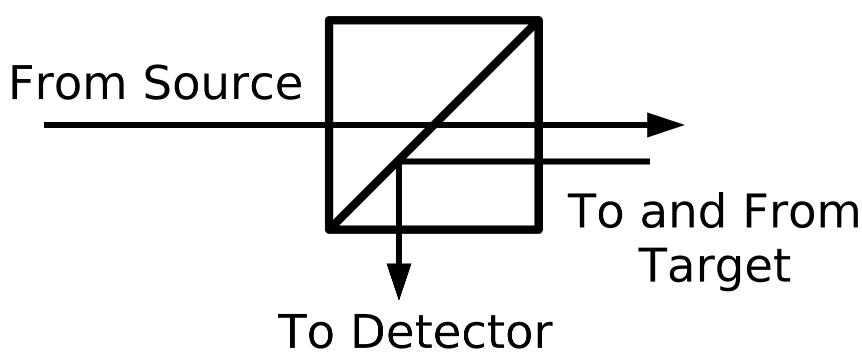
- Polarization Analysis

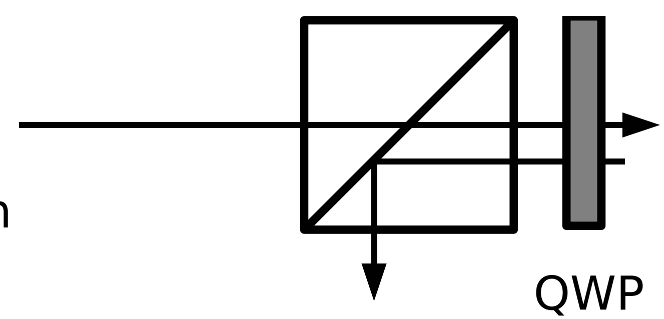
$$\mathcal{J}_{tr} = \mathcal{R}_{pbs}\mathcal{Q}_{45}\mathcal{F}_{target}\mathcal{Q}_{45}\mathcal{T}_{pbs}$$

- $\widehat{p}$–Polarized Source
- $\mathcal{F}_{target} = f$ (scalar) (Target Keeps Polarization)

$$\mathcal{J}_{tr}\hat{\mathbf{x}} = f\mathcal{R}_{pbs}\mathcal{Q}_{45}\mathcal{Q}_{45}\mathcal{T}_{pbs}\widehat{p}$$

$$= f\mathcal{R}_{pbs}\mathcal{H}_{45}\mathcal{T}_{pbs}\widehat{p}$$





T/R Switch Efficiency

- Round Trip

$$\mathcal{J}_{tr} = \mathcal{R}_{pbs}\mathcal{Q}_{45}\mathcal{F}_{target}\mathcal{Q}_{45}\mathcal{T}_{pbs}$$

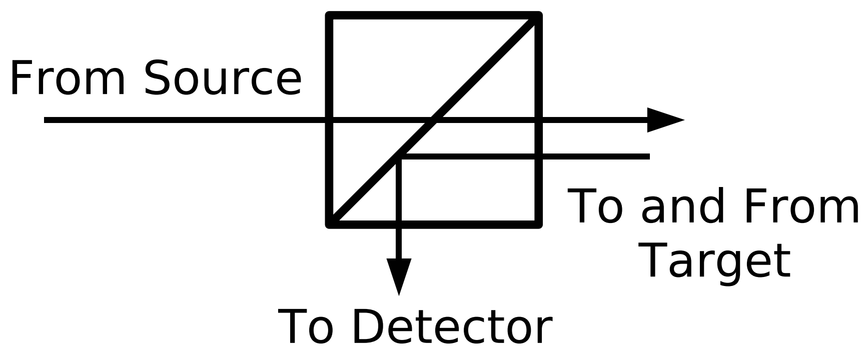
$$f\mathcal{R}_{pbs}\mathcal{H}_{45}\mathcal{T}_{pbs}$$

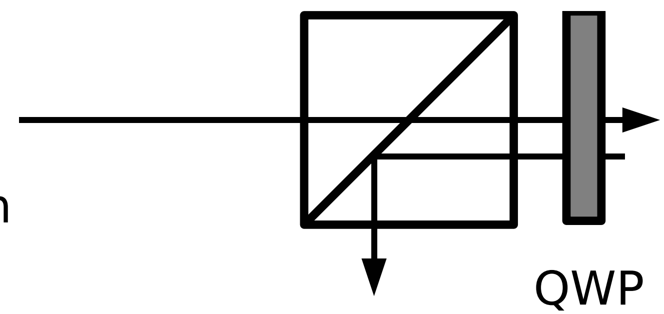
- Assumptions
  - 2% Insertion Loss (AR–Coated)
  - 5% Leakage of Wrong Polarization

$$\mathcal{J}_{tr}\hat{p} = f\begin{pmatrix}\sqrt{0.05} & 0\\ 0 & \sqrt{0.98}\end{pmatrix}\begin{pmatrix}0 & -1\\ -1 & 0\end{pmatrix}\begin{pmatrix}\sqrt{0.98} & 0\\ 0 & \sqrt{0.05}\end{pmatrix}\begin{pmatrix}1\\ 0\end{pmatrix}$$

$$= f\begin{pmatrix}0\\ 0.98\end{pmatrix}$$

(Leakage Only Matters if $\mathcal{F}_{target} \neq f$)





T/R Switch Narcissus Rejection

- Round Trip (Source to Source)

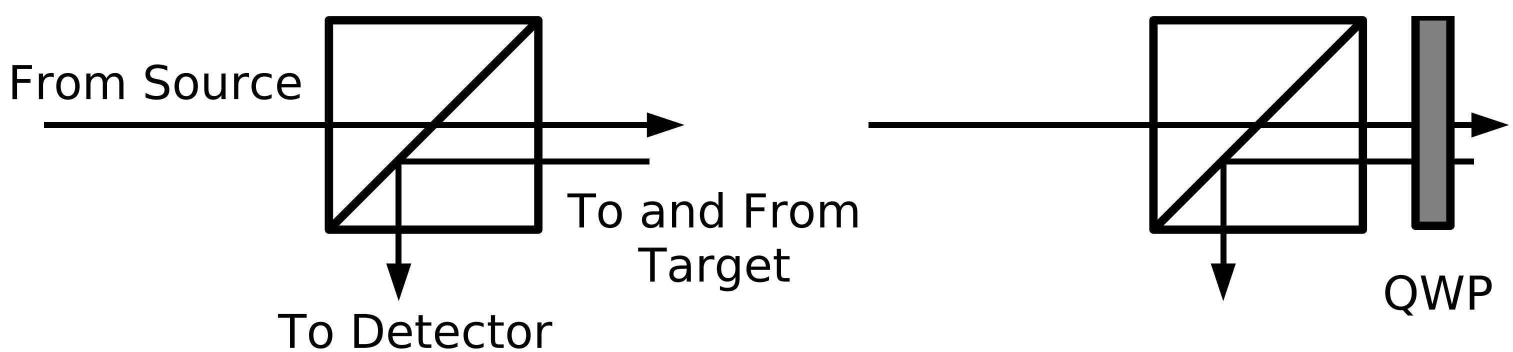
$$\mathcal{J}_{tt} = \mathcal{T}_{pbs}\mathcal{Q}_{45}\mathcal{F}_{target}\mathcal{Q}_{45}\mathcal{T}_{pbs} = f\mathcal{T}_{pbs}\mathcal{H}_{45}\mathcal{T}_{pbs}$$

$$\mathcal{J}_{tt}\hat{p} = f\begin{pmatrix}\sqrt{0.98} & 0\\ 0 & \sqrt{0.05}\end{pmatrix}\begin{pmatrix}0 & -1\\ -1 & 0\end{pmatrix}\begin{pmatrix}\sqrt{0.98} & 0\\ 0 & \sqrt{0.05}\end{pmatrix}\begin{pmatrix}1\\ 0\end{pmatrix}$$

$$= f\begin{pmatrix}0.05\\ 0\end{pmatrix}$$

(Minimizes Laser Instability)

(Assumes $\mathcal{F}_{target} = f$)



Coherency Matrices

- Remember the Inner Product

$$\mathbf{E}^\dagger\mathbf{E} = \begin{pmatrix} E_x^* & E_y^* \end{pmatrix} \begin{pmatrix} E_x \\ E_y \end{pmatrix} = |E_x|^2 + |E_y|^2$$

- Consider the Outer Product

$$\mathbf{E}\mathbf{E}^\dagger = \begin{pmatrix} E_x \\ E_y \end{pmatrix} \begin{pmatrix} E_x^* & E_y^* \end{pmatrix} = \begin{pmatrix} E_xE_x^* & E_xE_y^* \\ E_yE_x^* & E_yE_y^* \end{pmatrix}$$

- Expectation Value (Matrix Describes Field Statistics)

$$\boxed{\mathcal{C} = \left\langle \mathbf{E}\mathbf{E}^\dagger \right\rangle = \begin{pmatrix} < E_xE_x^* > & < E_xE_y^* > \\ < E_yE_x^* > & < E_yE_y^* > \end{pmatrix} = \begin{pmatrix} a & b \\ b^* & c \end{pmatrix}}$$

Real Powers: $\begin{matrix} a =< E_xE_x^* > \\ c =< E_yE_y^* > \end{matrix}$ Correlation: $b =< E_xE_y^* >$

Coherency Matrix Examples

- $\hat{x}$ Polarization

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad a = 1,\ b = c = 0$$

- 45–Degree Polarization

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad a = b = c = 1/2$$

- Right–Circular Polarization

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ j \end{pmatrix} \qquad a = c = 1/2 \text{ and } b = j/2$$

- Unpolarized (Randomly Polarized) Light

$$a = c = 1/2 \qquad b = 0$$

Devices and Coherency Matrices

- Jones Matrix Acting on Field

$$\mathbf{E}_{out} = \mathcal{J}\mathbf{E}_{in}$$

- Adjoint Equation (Same Information)

$$\mathbf{E}_{out}^{\dagger} = \mathbf{E}_{in}^{\dagger}\mathcal{J}^{\dagger}$$

- Combination

$$\boxed{\left\langle \mathbf{E}_{out}\mathbf{E}_{out}^{\dagger} \right\rangle = \left\langle \mathcal{J}\mathbf{E}_{in}\mathbf{E}_{in}^{\dagger}\mathcal{J}^{\dagger} \right\rangle}$$

- If $\mathcal{J}$ is Constant (Big If)

$$\boxed{\mathcal{C}_{out} = \mathcal{J}\mathcal{C}\mathcal{J}^{\dagger}}$$

Coherency Matrix Application

- Sunlight (Nearly Unpolarized) on Water

$$\mathcal{C}_{out} = \begin{pmatrix} \rho_p & 0 \\ 0 & \rho_s \end{pmatrix} \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix} \begin{pmatrix} \rho_p^* & 0 \\ 0 & \rho_s^* \end{pmatrix} = \frac{1}{2} \begin{pmatrix} R_p & 0 \\ 0 & R_s \end{pmatrix}$$

$R_s > R_p$

Stokes Vectors

- Equation (Different Notations in Different Texts)

$$\begin{pmatrix} I \\ M \\ C \\ S \end{pmatrix} = \begin{pmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{pmatrix} = \begin{pmatrix} a+c \\ a-c \\ b+b^* \\ \left(b-b^*\right)/j \end{pmatrix}$$

- Meanings (Four Real Numbers)
  - I is Total Power (Always Positive)
  - M is Preference for $\hat{x}$ over $\hat{y}$ ($-I$ to $+I$)
  - C is Preference for 45–Degree over -45–Degree
  - S is Preference for RHC over LHC

Example Stokes Vectors

$$\mathbf{E}_x = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \qquad \mathbf{E}_y = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} \qquad \mathbf{E}_{45} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\mathbf{E}_{-45} = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix} \qquad \mathbf{E}_{RHC} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \qquad \mathbf{E}_{LHC} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}$$

$$\mathbf{E}_{unpolarized} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad \mathbf{E}_{sun-reflected-from-water} = \begin{pmatrix} 1 \\ R_s - R_p \\ 0 \\ 0 \end{pmatrix}$$

Degree of Polarization

- Definition

$$\boxed{V = \frac{\sqrt{M^2 + C^2 + S^2}}{I} \leq 1}$$

- Meaning
  - $V = 1$ Means Complete Polarization
  - $V = 0$ Means Random Polarization

Q: What is V in terms of a, b, c?

Mueller (or Müller) Matrices

- 16 Real Numbers
  - 7 Independent
    (Phase lost)
- Compare Jones Matrices
  - 4 Complex Numbers
- $\hat{x}$ Polarizer

$$\mathcal{P}_x = \frac{1}{2}\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

- $\hat{y}$ Polarizer

$$\mathcal{P}_y = \frac{1}{2}\begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

- Polarization Randomizer

$$\mathcal{P}_x = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

- Recall

$$\left\langle \mathbf{E}_{out}\mathbf{E}_{out}^{\dagger}\right\rangle = \left\langle \mathcal{J}\mathbf{E}_{in}\mathbf{E}_{in}^{\dagger}\mathcal{J}^{\dagger}\right\rangle$$

$$\mathcal{C}_{out} = \left\langle \mathcal{J}\mathbf{E}_{in}\mathbf{E}_{in}^{\dagger}\mathcal{J}^{\dagger}\right\rangle$$

- Vary $\mathcal{J}$ to Make

$$\mathcal{C}_{out} = \mathcal{I}$$

- How?

Poncaré Sphere

- Normalized Stokes Parameters

$$\frac{1}{I}\begin{pmatrix} M \\ S \\ C \end{pmatrix}$$

- Radius V
  - Complete Polarization on Surface
  - Random at Center