Optics for Engineers
Chapter 12

Charles A. DiMarzio
Northeastern University

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Radiometry

- Power is Proportional to
  - Area of Aperture Stop
  - Area of Field Stop
  - “Brightness” of the Source (Radiance)
Radiometry and Photometry

**Note:** “Spectral X:” $X_\nu$, units $/\text{Hz}$ or $X_\lambda$, $/\mu\text{m}$.

**Rad. to Phot:**

$683 \ \text{lm/W} \times y(\lambda)$.

$y(555\text{nm}) \approx 1.$

| $\phi$ Radiant Flux, Watt $= W$ |
| Luminous Flux, lumen $= \text{Im}$ |
| $\partial/\partial A \rightarrow$ |

| $M$ Radiant Exitance, $W/\text{m}^2$ |
| Luminous Exitance, $\text{Im}/\text{m}^2 = \text{lux} = \text{lx}$ |

| $\downarrow \partial/\partial \Omega \downarrow$ |
| $\downarrow \partial/\partial \Omega \downarrow$ |

| $E$ Irradiance, $W/\text{m}^2$ |
| Illuminance, $\text{Im}/\text{m}^2 = \text{lux}$ |

$1\text{Ft.Candle} = 1\text{Im}/\text{ft}^2$

| $\leftarrow \cdot/R^2$ |
| $\partial/\partial A \rightarrow$ |

| $I$ Radiant Intensity, $W/\text{sr}$ |
| Luminous Intensity, $\text{Im}/\text{sr} = \text{cd}$ |

| $L$ Radiance, $W/\text{m}^2/\text{sr}$ |
| Luminance, $\text{nit} = \text{Im}/\text{m}^2/\text{sr}$ |

1Lambert $= 1\text{Im}/\text{cm}^2/\text{sr}/\pi$

1mLambert $= 1\text{Im}/\text{m}^2/\text{sr}/\pi$

1FtLambert $= 1\text{Im}/\text{ft}^2/\text{sr}/\pi$

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# Radiometric Quantities

<table>
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<th>Symbol</th>
<th>Equation</th>
<th>SI Units</th>
</tr>
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<tr>
<td>Radiant Energy</td>
<td>$Q$</td>
<td></td>
<td>Joules</td>
</tr>
<tr>
<td>Radiant Energy Density</td>
<td>$w$</td>
<td>$w = \frac{dQ}{dV^3}$</td>
<td>Joules/m$^3$</td>
</tr>
<tr>
<td>Radiant Flux or Power</td>
<td>$P$ or $\Phi$</td>
<td>$\Phi = \frac{dQ}{dt}$</td>
<td>W</td>
</tr>
<tr>
<td>Radiant Exitance</td>
<td>$M$</td>
<td>$M = \frac{d\Phi}{dA}$</td>
<td>W/m$^2$</td>
</tr>
<tr>
<td>Irradiance</td>
<td>$E$</td>
<td>$E = \frac{d\Phi}{dA}$</td>
<td>W/m$^2$</td>
</tr>
<tr>
<td>Radiant Intensity</td>
<td>$I$</td>
<td>$I = \frac{d\Phi}{d\Omega}$</td>
<td>W/sr</td>
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<tr>
<td>Radiance</td>
<td>$L$</td>
<td>$lm = \frac{d\Phi}{dA \cos \theta d\Omega}$</td>
<td>W/m$^2$</td>
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<tr>
<td>Fluence</td>
<td>$\Psi$</td>
<td>$\frac{dQ}{dA}$</td>
<td>J/m$^2$</td>
</tr>
<tr>
<td>Fluence Rate</td>
<td>$F$</td>
<td>$\frac{d\Psi}{dt}$</td>
<td>J/m$^2$</td>
</tr>
<tr>
<td>Emissivity</td>
<td>$\epsilon$</td>
<td>$\epsilon = \frac{M}{M_{1B}}$</td>
<td>Dimensionless</td>
</tr>
<tr>
<td>Spectral ()</td>
<td>$\nu$</td>
<td>$\frac{d\epsilon}{d\nu}$</td>
<td>($)/\text{Hz}$</td>
</tr>
<tr>
<td></td>
<td>or $\lambda$</td>
<td>$\frac{d\epsilon}{d\lambda}$</td>
<td>($)/\mu\text{m}$</td>
</tr>
<tr>
<td>Luminous Flux or Power</td>
<td>$P$ or $\Phi$</td>
<td>$\Phi$</td>
<td>lm</td>
</tr>
<tr>
<td>Luminous Exitance</td>
<td>$M$</td>
<td>$M = \frac{d\Phi}{dA}$</td>
<td>lm/m$^2$</td>
</tr>
<tr>
<td>Illuminance</td>
<td>$E$</td>
<td>$E = \frac{d\Phi}{dA}$</td>
<td>lm/m$^2$</td>
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<td>$I$</td>
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<td>Luminance</td>
<td>$L$</td>
<td>$L = \frac{d\Phi}{dA \cos \theta d\Omega}$</td>
<td>lm/m$^2$</td>
</tr>
<tr>
<td>Spectral Luminous Efficiency</td>
<td>$V(\lambda)$</td>
<td>$V(\lambda)$</td>
<td>Dimensionless</td>
</tr>
<tr>
<td>Color Matching Functions</td>
<td>$x(\lambda)$, $y(\lambda)$, $z(\lambda)$</td>
<td>Dimensionless</td>
<td></td>
</tr>
</tbody>
</table>
Irradiance

- Poynting Vector for Coherent Wave

\[ dS \approx \frac{dP}{4\pi r^2} (\sin \theta \cos \xi \hat{x} + \sin \theta \sin \xi \hat{y} + \cos \theta \hat{z}) \]

- Irradiance (Projected Area, \( A_{proj} = A \cos \theta \))

\[ dE = \frac{d^2P}{dA'} = \frac{dP}{4\pi r^2} \]
Irradiance and Radiant Intensity

- Solid Angle
  \[ \Omega = \frac{A'}{r^2} \]

- Intensity from Irradiance (Unresolved Source, \( A \))
  \[ I = Er^2 \quad E = \frac{I}{r^2} \]

\[
dI = \frac{d^2P}{d\Omega} = \frac{d^2P}{d\frac{A'}{r^2}} = \frac{dP}{(4\pi r^2)}
\]
Intensity and Radiance

- Resolved Source: Many Unresolved Sources Combined

\[ I = \int dI = \int \frac{\partial I}{\partial A} dA \]

- Radiance

\[ L(x, y, \theta, \zeta) = \frac{\partial I(\theta, \zeta)}{\partial A} \]

- On Axis \((x' = y' = 0)\)

\[ L(x, y, \theta, \zeta) = \frac{\partial I(\theta, \zeta)}{\partial A} = \frac{\partial^2 P}{\partial A \partial \Omega} \]

- Off Axis (Projected Area)

\[ L = \frac{\partial I}{\partial A \cos \theta} = \frac{\partial^2 P}{\partial A \partial \Omega \cos \theta} \]
Radiant Intensity and Radiant Flux

- Flux or Power, $P$ or $\Phi$
- Integrate Intensity

\[ P = \Phi = \int \int I(\theta, \phi) \sin \theta d\theta d\zeta \]

- Constant Intensity

\[ P = \Phi = I\Omega \]
\[ \Omega = \int \int \sin \theta' d\theta' d\zeta \]
\[ \Omega = \int_0^{2\pi} \int_0^\Theta \sin \theta' d\theta' d\zeta = 2\pi (1 - \cos \Theta) = 2\pi \left( 1 - \sqrt{1 - \sin^2 \Theta} \right) \]

\[ \Omega = 2\pi \left( 1 - \sqrt{1 - \left( \frac{NA}{n} \right)^2} \right) \]
Radiance and Radiant Exitance

- Radiant Exitance from a Source (Same Units as Irradiance)

\[
M(x, y) = \int \int \left[ \frac{\partial^2 P}{\partial A \partial d\Omega} \right] \sin \theta d\theta d\zeta
\]

- Radiant Exitance from Radiance

\[
M(x, y) = \int \int L(x, y, \theta, \zeta) \cos \theta \sin \theta d\theta d\zeta
\]

- Radiance from Radiant Exitance

\[
L(x, y, \theta, \zeta) = \frac{\partial M(x, y)}{\partial \Omega} \frac{1}{\cos \theta}
\]
Radiance and Radiant Exitance: Special Cases

- Constant \( L \) and Small Solid Angle

\[
M(x, y) = L\Omega \cos \theta
\]

\[
\Omega = 2\pi \left(1 - \sqrt{1 - \left(\frac{NA}{n}\right)^2}\right)
\]

or

\[
\Omega \approx \pi \left(\frac{NA}{n}\right)^2
\]

- Constant \( L \), over Hemisphere

\[
M(x, y) = \int_0^{2\pi} \int_0^{\pi/2} L \cos \theta \sin \theta d\theta d\zeta = 2\pi L \frac{\sin^2 \frac{\pi}{2}}{2}.
\]

\[
M(x, y) = \pi L \quad \text{(Lambertian Source)}
\]
Radiant Exitance and Flux

- Power or Flux from Radiant Exitance

\[ P = \Phi = \int \int M(x, y) \, dx \, dy, \]

- Radiant Exitance from Power

\[ M(x, y) = \frac{\partial P}{\partial A} \]
The Radiance Theorem in Air

- Solid Angle two ways
  \[ \Omega = \frac{A'}{r^2} \quad \Omega' = \frac{A}{r^2} \]

- Power Increment
  \[ dP = L dA d\Omega = L dA \frac{dA'}{r^2} \cos \theta \quad dP = L dA' d\Omega' = L dA' \frac{dA}{r^2} \cos \theta \]
Using the Radiance Theorem: Examples Later

- Radiance is Conserved in a Lossless System (in Air)
- Losses Are Multiplicative
  - Fresnel Reflections and Absorption
- Radiance Theorem Simplifies Calculation of Detected Power
  - Determine Object Radiance
  - Multiply by Scalar, \( T_{\text{total}} \), for Loss
  - Find Exit Window (Of a Scene or a Pixel)
  - Find Exit Pupil
  - Compute Power

\[
P = L_{\text{object}} T_{\text{total}} A_{\text{exit window}} \Omega_{\text{exit pupil}}
\]
Etendue and the Radiance Theorem

- Abbe Invariant:
  \[ n' x' d\alpha' = nx d\alpha \]

- Etendue
  \[ n^2 A\Omega = (n')^2 A'\Omega' \]

- Power Conservation
  \[
  \int \int \int \int L d^2 A d^2 \Omega = \int \int \int \int L' d^2 A' d^2 \Omega' \\
  \int \int \int \int \frac{L}{n^2} n^2 d^2 A d^2 \Omega = \int \int \int \int \frac{L'}{(n')^2} d^2 A' d^2 \Omega'
  \]

Radiance Theorem: \[ \frac{L}{n^2} = \frac{L'}{(n')^2} \]

- Basic Radiance, \( L/n^2 \), Conserved (Thermodynamics Later)
Radiance Theorem Example: Translation

- Translation Matrix Equation

\[
\begin{pmatrix}
  dx_2 \\
  d\alpha_2
\end{pmatrix}
= \begin{pmatrix}
  1 & z_{12} \\
  0 & 1
\end{pmatrix}
\left[
\begin{pmatrix}
  x_1 + dx_1 \\
  \alpha_1 + d\alpha_1
\end{pmatrix}
- \begin{pmatrix}
  x_1 \\
  \alpha_1
\end{pmatrix}
\right]
= \begin{pmatrix}
  dx_1 \\
  d\alpha_1
\end{pmatrix}
\]

\[dx_2 d\alpha_2 = dx_1 d\alpha_1\]
Radiance Theorem Example: Imaging

- Imaging Matrix Equation

\[
\mathcal{M}_{SS'} = \begin{pmatrix} m & 0 \\ ? & \frac{n}{n'm} \end{pmatrix}
\]
Radiance Theorem Example: Dielectric Interface

- Dielectric Interface Matrix Equation
  \[
  \begin{pmatrix}
  1 & 0 \\
  0 & \frac{n}{n'}
  \end{pmatrix}
  \]

- Power Increment
  \[
  d^2 \Phi = L_1 dA \cos \theta_1 d\Omega_1 = L_2 dA \cos \theta_2 d\Omega_2
  \]
  \[
  L_1 dA \cos \theta_1 \sin \theta_1 d\theta_1 d\zeta_1 = L_2 dA \cos \theta_2 \sin \theta_2 d\theta_2 d\zeta_2
  \]

- Snell’s Law and Derivative
  \[
  n_1 \sin \theta_1 = n_2 \sin \theta_2
  \]
  \[
  n_1 \cos \theta_1 d\theta_1 = n_2 \cos \theta_2 d\theta_2
  \]
  \[
  \frac{\sin \theta_1}{\sin \theta_2} \frac{d\theta_1}{d\theta_2} = \frac{n_2}{n_1} \frac{n_2 \cos \theta_2}{n_1 \cos \theta_1}
  \]

- Radiance Theorem
  \[
  \frac{L}{n^2} = \frac{L'}{(n')^2}
  \]
Radiance Theorem Example: 1/2 Telecentric Relay

- Fourier–Optics Matrix Equation
  \[ \mathcal{M}_{FF'} = \begin{pmatrix} 0 & f \\ \frac{1}{f} & 0 \end{pmatrix} \]
  \[ \det \mathcal{M}_{FF'} = 1 = n'/n \quad n^2 A\Omega = (n')^2 A'\Omega' \]

- Radiance Theorem
  \[ \frac{L}{n^2} = \frac{L'}{(n')^2} \]
Idealized Example: 100W Lamp

- $P = 100W$, Uniform in Angle
- Tungsten Filament: Area $A = (5\text{mm})^2$, Distance, $R$
- Radiant Exitance

\[ M = \frac{100W}{(5\text{mm})^2} = 4W/\text{mm}^2 \]

- Radiance (Uniform in Angle)

\[ L = \frac{M}{(4\pi)} \approx 0.32W/\text{mm}^2/\text{sr} \]

- Intensity

\[ I = \frac{100W}{(4\pi)} \approx 7.9W/\text{sr} \]

- Receiver: Area $A'$: Received Power and Irradiance

\[ P A'/\left(4\pi r^2\right) = I A'/r^2 \quad E = \frac{I}{r^2} \approx 8W/\text{sr}/(10m)^2 = 0.08W/\text{m}^2 \]
100W Lamp at the Receiver

- Using Source Radiance at Receiver

\[ E = L \frac{A}{r^2} = L\Omega' \]

- Irradiance Same as Previous Page

\[ E \approx 0.32\text{W/mm}^2/\text{sr} \times \frac{(5\text{mm})^2}{(10\text{m})^2} \approx 0.08\text{W/m}^2 \]

- Useful if \( r \) and \( A \) are Not Known
  - Only Depends on \( \Omega' \) (Measured at Receiver)
Practical Example: Imaging the Moon

- Known Moon Radiance $L_{moon} = 13 \text{W/m}^2/\text{sr}$
- Calculation
  - Jones matrices for Transmission ($0.25$), other multiplicative losses ($12$ Lenses: $(0.96^2)^{12}$).
    \[
    L = 13 \text{W/m}^2/\text{sr} \times 0.25 \times 0.96^{24} = 13 \text{W/m}^2/\text{sr} \times 0.25 \times 0.367 = 1.2 \text{W/m}^2/\text{sr}
    \]
  - Exit Pupil ($NA = 0.1$) and exit window (Pixel: $d = 10 \mu\text{m}$).
  - Irradiance, $E = L\Omega$ and Power on a Pixel, $P = EA = LA\Omega$
    \[
    P = 1.2 \text{W/m}^2/\text{sr} \times 2\pi \left( 1 - \sqrt{1 - NA^2} \right) \times (10 \times 10^{-6} \text{m})^2
    \]
    \[
    P = 1.2 \text{W/m}^2/\text{sr} \times 0.315\text{sr} \times 10^{-10}\text{m}^2 = 3.8 \times 10^{-12}\text{W}
    \]
  - If desired, multiply by time ($1/30\text{sec}$) to obtain energy.
    * About One Million Photons (and Electrons)
  - Alternative to Solve in Object Space
    (Need Pixel Size on Moon)
Radiometry Summary

- Five Radiometric Quantities: Radiant Flux $\Phi$ or Power $P$, Radiant Exitance, $M$, Radiant Intensity, $I$, Radiance, $L$, and Irradiance, $E$, Related by Derivatives with Respect to Projected Area, $A \cos \theta$ and Solid Angle, $\Omega$.

- Basic Radiance, $L/n^2$, Conserved, with the Exception of Multiplicative Factors.

- Power Calculated from Numerical Aperture and Field Of View in Image (or Object) Space, and the Radiance.

- Losses Are Multiplicative.

- Finally “Intensity” is Not “Irradiance.”
Spectral Radiometry Definitions

- Any Radiometric Quantity Resolved Spectrally
  - Put the Word Spectral in Front
  - Use a Subscript for Wavelength or Frequency
  - Modify Units

- Example: Radiance, \( L \), Spectral Radiance (Watch Units)
  \[
  L_\nu = \frac{dL}{d\nu} \text{ W/m}^2/\text{sr}/\text{THz} \quad \text{or} \quad L_\lambda = \frac{dL}{d\lambda} \text{ W/m}^2/\text{sr}/\mu\text{m}
  \]

- Spectral Fraction
  \[
  f_\lambda (\lambda) = \frac{X_\lambda (\lambda)}{X} \quad \text{for} \quad X = \Phi, M, I, E, \text{ or } L
  \]
# Spectral Radiometric Quantities

<table>
<thead>
<tr>
<th>Quantity</th>
<th>Units</th>
<th>( d/d\nu )</th>
<th>Units</th>
<th>( d/d\lambda )</th>
<th>Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>Radiant Flux, ( \Phi )</td>
<td>W</td>
<td>Spectral Radiant Flux, ( \Phi_\nu )</td>
<td>W/Hz</td>
<td>Spectral Radiant Flux, ( \Phi_\lambda )</td>
<td>W/(\mu m)</td>
</tr>
<tr>
<td>Power, ( P )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Radiant Exittance, ( M )</td>
<td>W/(m^2)</td>
<td>Spectral Radiant Exittance, ( M_\nu )</td>
<td>W/(m^2/Hz)</td>
<td>Spectral Radiant Exittance, ( M_\lambda )</td>
<td>W/(m^2/\mu m)</td>
</tr>
<tr>
<td>Radiant Intensity, ( I )</td>
<td>W/sr</td>
<td>Spectral Radiant Intensity, ( I_\nu )</td>
<td>W/sr/Hz</td>
<td>Spectral Radiant Intensity, ( I_\lambda )</td>
<td>W/sr/(\mu m)</td>
</tr>
<tr>
<td>Radiance, ( L )</td>
<td>W/(m^2/sr)</td>
<td>Spectral Radiance, ( L_\nu )</td>
<td>W/(m^2/sr/Hz)</td>
<td>Spectral Radiance, ( L_\lambda )</td>
<td>W/(m^2/sr/\mu m)</td>
</tr>
</tbody>
</table>
Spectral Fraction: Sunlight on Earth

- 5000K Black Body (Discussed Later) Defines $M_\lambda$

  \[ f_\lambda (\lambda) = \frac{M_\lambda (\lambda)}{M} \quad M = \int_0^\infty M_\lambda d\lambda \]

- Compute Spectral Irradiance with Known $E = 1000\text{W/m}^2$

  \[ E_\lambda (\lambda) = E f_\lambda (\lambda), \quad \text{with} \quad f_\lambda (\lambda) = \frac{M_\lambda (\lambda)}{M} = \frac{M_\lambda (\lambda)}{\int_0^\infty M_\lambda (\lambda) d\lambda} \]

In Practice:
Use the spectral fraction with any radiometric quantity

* Solar Spectral Irradiance is

  \[ E_\lambda = \frac{L_\lambda \Omega'}{r^2} e^{-\alpha} \, dr \]

  but we need to know a lot to compute it.
Spectral Fraction: Molecular Tag

- Excitation: Argon 488nm
- Green Emission Power, $\Phi$
  
  $$\Phi_\lambda(\lambda) = \Phi f_\lambda(\lambda),$$
  
  $$f_\lambda(\lambda) = \frac{S_\lambda(\lambda)}{\int_0^\infty S_\lambda(\lambda) \, d\lambda}$$
  
  - Dash–Dot on Plot
  - Obtained from Experiment
    (Radiometric Calibration not Needed)
  - Available from Vendor

- Fluorescein Spectra
  - Dash: Absorption Spectrum
  - Solid: Emission Spectrum
  - Dash–Dot: Spectral Fraction for Emission (nm$^{-1}$)
Molecular Tag in Epi–Fluorescence

Narrow Signal
Bad Rejection

Narrow Filter
Lost Light

Filter
Source

λ

CCD Camera
Em
Dichroic
Ex
Objective Lens
Sample

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Spectral Matching Factor (1)

- Reflective Filter (Ch. 8)
- Spectral Radiance Output of Filter

\[ L'_\lambda (\lambda) = L_\lambda (\lambda) R (\lambda) \]

- Total Radiance Output

\[ L' = \int_{0}^{\infty} L'_\lambda (\lambda) d\lambda = \int_{0}^{\infty} L_\lambda (\lambda) R (\lambda) d\lambda \]
Spectral Matching Factor (2)

- Total Radiance (Previous Page)

\[ L' = \int_{0}^{\infty} L'_\lambda (\lambda) \, d\lambda = \int_{0}^{\infty} L_\lambda (\lambda) \, R(\lambda) \, d\lambda \]

- Define Reflectance for This Filter; \( R = L'/L \)
  - One Number as Opposed to \( R(\lambda) \)
  - Relate to Transmission at Spectral Maximum
  - Characterize SMF for Given Input Spectrum

\[
R = R_{max} \int_{0}^{\infty} \frac{R(\lambda)}{R_{max}} f_\lambda (\lambda) \, d\lambda = R_{max}SMF
\]

\[ SMF = \frac{R}{R_{max}} \]
Summary of Spectral Radiometry

- For every radiometric quantity, there exists a spectral radiometric quantity. In the name, no distinction is made between frequency and wavelength derivatives.

- The notation: subscript $\nu$ for frequency or $\lambda$ for wavelength.

- The units are the original units divided by frequency or wavelength units.

- Spectral Fraction, $f_\lambda$ can be applied to any of the radiometric quantities.

- The spatial derivatives (area and angle) are valid for the spectral quantities, wavelength–by–wavelength.

- The behavior of filters is more complicated, and is usually treated with the spectral matching factor.
Photometry and Colorimetry

- Spectral Luminous Efficiency, $\bar{y}(\lambda)$
- Source **Spectral** Radiance, $L_\lambda(\lambda, x, y)$
- Eye Response

$$Y(x, y) = \int_0^\infty \bar{y}(\lambda) L_\lambda(\lambda, x, y) \, d\lambda$$

- Four LEDs: Equal Radiance
  - Blue, 400 Appears Weak
  - Green, 550 Appears Strong
  - Red, 630 Moderately Weak
  - IR, 980 Invisible
Lumens

- Power or Radiant Flux (Watts)
  \[ P = \int_{0}^{\infty} P_{\lambda} (\lambda) \]

- Eye Response
  \[ Y = \int_{0}^{\infty} \bar{y} (\lambda) P_{\lambda} (\lambda) d\lambda \]

- Luminous Flux (Lumens, Subscript \( V \) for Clarity)
  \[ P_{(V)} = \frac{683 \text{ lumens/Watt}}{\text{max} (\bar{y})} \int_{0}^{\infty} \bar{y} (\lambda) P_{\lambda} (\lambda) d\lambda \]

- Luminous Efficiency
  \[ \frac{P_{(V)}}{P} = 683 \text{ lumens/Watt} \int_{0}^{\infty} \frac{\bar{y} (\lambda)}{\text{max} (\bar{y})} \frac{P_{\lambda} (\lambda)}{P} d\lambda \]
Some Typical Radiance and Luminance Values

<table>
<thead>
<tr>
<th>Object</th>
<th>W/m²/sr</th>
<th>nits = lm/m²/sr</th>
<th>Footlamberts</th>
<th>lm/W</th>
</tr>
</thead>
<tbody>
<tr>
<td>Minimum Visible</td>
<td>7 ×10⁻¹⁰</td>
<td>5 ×10⁻⁷</td>
<td>1.5×10⁻⁷</td>
<td>683</td>
</tr>
<tr>
<td>Dark Clouds</td>
<td>0.2</td>
<td>40</td>
<td>12</td>
<td>190</td>
</tr>
<tr>
<td>Lunar disk</td>
<td>13</td>
<td>2500</td>
<td>730</td>
<td>190</td>
</tr>
<tr>
<td>Clear Sky</td>
<td>27</td>
<td>8000</td>
<td>2300</td>
<td>300</td>
</tr>
<tr>
<td>Bright Clouds</td>
<td>130</td>
<td>2.4×10⁴</td>
<td>7 ×10³</td>
<td>190</td>
</tr>
<tr>
<td></td>
<td>300</td>
<td></td>
<td></td>
<td>82</td>
</tr>
<tr>
<td>Solar disk</td>
<td>4.8×10⁶</td>
<td>7 ×10⁸</td>
<td>2.6×10⁷</td>
<td>190</td>
</tr>
<tr>
<td></td>
<td>1.1×10⁷</td>
<td></td>
<td>×10⁷</td>
<td>82</td>
</tr>
</tbody>
</table>
Tristimulus Values: Three is Enough

- $X$, $Y$, $Z$

\[
X = \int_0^\infty \bar{x}(\lambda) L_\lambda(\lambda) \, d\lambda
\]

\[
Y = \int_0^\infty \bar{y}(\lambda) L_\lambda(\lambda) \, d\lambda
\]

\[
Z = \int_0^\infty \bar{z}(\lambda) L_\lambda(\lambda) \, d\lambda
\]

- **Example: 3 Lasers**

  Krypton \quad \lambda_{\text{red}} = 647.1 \text{nm}

\[
X_R = \int_0^\infty \bar{x}(\lambda) \delta(\lambda - 647.1\text{nm}) \, d\lambda = \bar{x}(647.1\text{nm})
\]

\[
X_R = 0.337 \quad Y_R = 0.134 \quad Z_R = 0.000
\]

Krypton \quad \lambda_{\text{green}} = 530.9 \text{nm}

\[
X_G = 0.171 \quad Y_G = 0.831 \quad Z_G = 0.035
\]

Argon \quad \lambda_{\text{blue}} = 457.9 \text{nm}

\[
X_B = 0.289 \quad Y_B = 0.031 \quad Z_B = 1.696
\]
Chromaticity Coordinates (1)

- Three Laser Powers, $R$, $G$, and $B$, Watts
- Tristimulus Values

$$
\begin{pmatrix}
X \\
Y \\
Z 
\end{pmatrix} = \begin{pmatrix}
X_R & X_G & X_B \\
Y_R & Y_G & Y_B \\
Z_R & Z_G & Z_B 
\end{pmatrix} \begin{pmatrix}
R \\
G \\
B 
\end{pmatrix}
$$

- Chromaticity Coordinates (Normalized $X$, $Y$

$$
x = \frac{X}{X + Y + Z} \quad y = \frac{Y}{X + Y + Z}
$$

- Monochromatic Light

$$
x = \frac{\bar{x}(\lambda)}{\bar{x}(\lambda) + \bar{y}(\lambda) + \bar{z}(\lambda)} \quad y = \frac{\bar{y}(\lambda)}{\bar{x}(\lambda) + \bar{y}(\lambda) + \bar{z}(\lambda)}
$$
Chromaticity Coordinates (2)

Monochromatic Light (Boundary)

\[ x = \frac{\bar{x}(\lambda)}{\bar{x}(\lambda) + \bar{y}(\lambda) + \bar{z}(\lambda)} \]

\[ y = \frac{\bar{y}(\lambda)}{\bar{x}(\lambda) + \bar{y}(\lambda) + \bar{z}(\lambda)} \]

3 Lasers (Triangles) and Their Gamut (Black - -)

Phosphors (White)

Thermal Sources (Later)
Generating Colored Light

- Given $P(V)$, $x$, and $y$
- Required Tristimulus Values

\[
Y = \frac{P(V)}{y \times 683\text{lm/W}}
\]

\[
X = \frac{xP(V)}{y \times 683\text{lm/W}}
\]

\[
Z = \frac{(1-x-y)P(V)}{y \times 683\text{lm/W}}
\]

- Powers of Three Lasers

\[
\begin{pmatrix}
R \\
G \\
B
\end{pmatrix} = 
\begin{pmatrix}
X_R & X_G & X_B \\
Y_R & Y_G & Y_B \\
Z_R & Z_G & Z_B
\end{pmatrix}^{-1}
\begin{pmatrix}
X \\
Y \\
Z
\end{pmatrix}
\]

- Projector: 100lux “White”
- $x = y = 1/3$
- $400\text{lm on (2m)}^2 = 4\text{m}^2$
- $X = Y = Z = 400\text{lm/(1/3)}/683\text{lm/W}$
- $R = 1.2W$, $G = 0.50W$, $B = 0.33W$

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Another Example: Simulating Blue ““Laser”” Light

- Monochromatic Light from Blue Argon Laser Line
  \( \lambda = 488\text{nm} \quad X = 0.0605 \quad Y = 0.2112 \quad Z = 0.5630 \)

- Solution with Inverse Matrix
  \[
  R = -0.2429 \quad G = 0.2811 \quad B = 0.3261
  \]

- Old Bear–Hunters’ Saying:
  - Sometimes You Eat the Bear, …
  - and Sometimes the Bear Eats You.

- Best Compromise (Right Hue, Insufficient Saturation)
  \[
  R = 0 \quad G = 0.2811 \quad B = 0.3261
  \]
Summary of Color

- Three color–matching functions $\bar{x}$, $\bar{y}$, and $\bar{z}$
- Tristimulus Values: 3 Integrals $X$, $Y$, and $Z$ Describe Visual Response
- $Y$ Related to Brightness. 683 lm/W, Links Radiometric to Photometric Units
- Chromaticity Coordinates, $x$ and $y$, Describe Color. Gamut of Human Vision inside the Horseshoe Curve.
- Sources with Different Spectra May Appear Identical.
- Light Reflected or Scattered May Appear Different Under Different Sources Of Illumination.
- $R$, $G$, and $B$, to $X$, $Y$, $Z$ Conversion Can Be Inverted over Limited Gamut.
The Radiometer or Photometer

- Aperture Stop
- Field Stop
- Measured Power

\[ P = L A \Omega = L A' \Omega' \]

- Adjustable Stops (Match FOV)
- Sighting Scope?

- Calibration Required
- Spectrometer on Output?
- Spectral Filter?
- Photometric Filter?
- Computer
  - Radiance, \( P/(\Omega A) \)
  - Luminance (All Units)
  - Spectral Quantities

---

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Integrating Sphere

- Power from Intensity
- Integrate over Solid Angle
  - Goniometry
    * Information–Rich
    * Time–Intensive
    * Integrating Sphere
      - Easy
      - Single Measurement
- Applications
  - Wide–Angle Sources
  - Diffuse Materials
- Variations
  - Two Spheres
  - Spectroscopic Detector
  - More
Blackbody Radiation Outline

- Background
- Equations, Approximations
- Examples
- Illumination
- Thermal Imaging
- Polar Bears, Greenhouses

\[ \lambda, \text{Wavelength, } \mu\text{m} \]
\[ m_\lambda, \text{Spect. Radiant Exitance, W/m}^2/\mu\text{m} \]

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Blackbody Background

- Waves in a Cavity
  \[ \frac{\partial^2 E}{\partial x^2} + \frac{\partial^2 E}{\partial y^2} + \frac{\partial^2 E}{\partial z^2} = \frac{1}{c^2} \frac{\partial^2 E}{\partial t^2} \]

- Cavity Modes
  \[ \frac{2\pi \nu_x \ell}{c} = N_x \pi \]
  \[ \frac{2\pi \nu_y \ell}{c} = N_y \pi \]
  \[ \frac{2\pi \nu_z \ell}{c} = N_z \pi \]

- Mode Frequencies
  \[ \nu = \sqrt{\nu_x^2 + \nu_y^2 + \nu_z^2} \quad \nu_0 = \frac{c}{2\ell} \]
  \[ \nu = \nu_0 \sqrt{N_x^2 + N_y^2 + N_z^2} \]

- Boundary Conditions
  \[ E(x, y, z, t) = \sin \left( \frac{2\pi \nu_x x}{c} \right) \sin \left( \frac{2\pi \nu_y y}{c} \right) \sin \left( \frac{2\pi \nu_z z}{c} \right) \sin 2\pi \nu t \]
Counting Modes

- Mode Density (Sphere)
  \[ dN = \frac{4\pi\nu^2 d\nu}{c^3/(8\ell^3)} \]
  \[ N_\nu = \frac{dN}{d\nu} = \frac{4\pi\nu^2\ell^3}{c^3} \]

- Energy Distribution
  - Rayleigh–Jeans: Equal Energy, \( k_B T \) (Dash–Dot)
    \[ J_\nu = k_B T \frac{8\pi\nu^2\ell^3}{c^3} \]
  - Dashed
    \[ J_\nu = h \nu \exp\left(\frac{h\nu}{kT}\right) \frac{8\pi\nu^2\ell^3}{c^3} \]

- \( \times 2 \) for Polarization
  \[ N_\nu = \frac{8\pi\nu^2\ell^3}{c^3} \]

- Ultraviolet Catastrophe

Temperature = 5000K

\( \lambda \), Wavelength, \( \mu \text{ m} \)

\( Q \), Energy Density, J/Hz

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Planck Got it Right

Temperature = 5000K

- Planck
  - Mode Energy Quantized
    \[ J_N = N \hbar \nu \]
  - Mode Random
  - Average Energy Density
    \[ \bar{J} = \frac{\sum_{N=0}^{\infty} J_N P(J_N)}{\sum_{N=0}^{\infty} P(J_N)} \]

- Boltzmann Distribution
  \[ P(J_N) = e^{\frac{-J_N}{k_B T}} \]

- Average Energy
  \[ \bar{J} = \frac{\sum_{N=0}^{\infty} N \hbar \nu e^{\frac{-N \hbar \nu}{k_B T}}}{\sum_{N=0}^{\infty} e^{\frac{-N \hbar \nu}{k_B T}}} = \]
  \[ h \nu \frac{\sum_{N=0}^{\infty} Ne^{\frac{-N \hbar \nu}{k_B T}}}{\sum_{N=0}^{\infty} e^{\frac{-N \hbar \nu}{k_B T}}} \]
  \[ = \frac{e^{\frac{\hbar \nu}{k_B T}}}{(1-e^{\frac{\hbar \nu}{k_B T}})^2} = \]
  \[ h \nu \frac{1}{(1-e^{\frac{\hbar \nu}{k_B T}})} \]

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Planck’s Result (1)

- Previous Page
  \[ \frac{h\nu}{e^{h\nu/k_BT} - 1} \]
- \( \times \) Mode Density
  \[ N_\nu \bar{J} = \frac{8\pi \nu^2 \ell^3}{c^3} \frac{h\nu}{e^{h\nu/k_BT} - 1} \]
- Solid Curve (Previous Page)
- Energy Per Volume
  \[ \frac{d\bar{w}}{d\nu} = \frac{N_\nu \bar{J}}{\ell^3} = \frac{8\pi \nu^2}{c^3} \frac{h\nu}{e^{h\nu/k_BT} - 1} \]
- 1. Radiance Independent of Direction
- 2. Emission Balanced by Absorption
  \[ \bar{M}(T)_{emitted} = \epsilon_a \bar{E}(T)_{incident} \]
- 3. Balance at Every \( \lambda \)
- 4. Light Moves at \( c \)
  \[ w_\lambda = \frac{1}{c} \int_{sphere} Ld\Omega = \frac{4\pi l}{c} \]
- \( E \) from \( L \)
  \[ E_\lambda = \int \int L_\lambda \cos \theta \sin \theta d\theta d\zeta = \pi L_\lambda \]
Planck’s Result (2)

- $E$ from $w$: $E_\lambda = 4c w_\lambda$
- Wall Balance & $\epsilon_a = 1$: $\bar{M}(T)_{bb} = \bar{E}(T)_{\text{incident}}$
- Planck’s Law
  \[
  M_{\nu(bb)}(\lambda, T) = h\nu c \frac{2\pi \nu^2}{c^3} \frac{1}{e^{h\nu/k_B T} - 1} = \frac{2\pi h\nu^3}{c^2} \frac{1}{e^{h\nu/k_B T} - 1}
  \]
- Emissivity
  \[
  M_{\nu}(\lambda, T) = \epsilon M_{\nu(bb)}(\lambda, T)
  \]
- Detailed Balance

$$\epsilon(\lambda) = \epsilon_a(\lambda)$$
The Planck Equation

- Fractional Linewidth

\[ \left| \frac{dv}{v} \right| = \left| \frac{d\lambda}{\lambda} \right| \]

\[ M_\lambda = \frac{dM}{d\lambda} = \frac{dv \, dM}{d\lambda \, dv} = \frac{\nu \, dM}{\lambda \, d\nu} = \frac{c \, dM}{\lambda^2 \, d\nu} \]

- Planck Law vs. Wavelength

\[ M_\lambda (\lambda, T) = \frac{2\pi h c^2}{\lambda^5} \frac{1}{e^{h\nu/k_BT} - 1} \]
Useful Blackbody Equations

- **Wien Displacement Law**
  \[ \lambda_{\text{peak}}T = 2898\mu\text{m} \cdot \text{K} \]

- **Stefan–Boltzmann Law**
  \[ M(T) = \frac{2\pi^5 k^4 T^4}{15h^3 c^2} = \sigma_e T^4 \]

- **Stefan–Boltzmann Constant**
  \[ \sigma_e = 5.67032 \times 10^{-12} \text{W/cm}^2/\text{K}^4 = 5.67032 \times 10^{-8} \text{W/m}^2/\text{K}^4 \]
Some Examples: Thermal Equilibrium

- Earth Temperature
  - Heating
    * 1kw/m²
    * Half of Surface
  - Cooling
    * Radiation
    \[ M(T) = \sigma_e T^4 \]
  - Result
    \[ T \approx 306K \]

- Body Temperature
  - Cooling (310K)
    \[ M(T) = \sigma_e T^4 = 524W/m^2 \]
  - Heating (Room=295K)
    \[ M(T) = \sigma_e T^4 = 429W/m^2 \]
  - Excess Cooling
    \[ \Delta M \times A \approx 100W \]
    \[ \approx 2000kCal/day \]
    (Many Approximations)

Because \( d\left(T^4\right) = 4T^3dT \), small changes in temperature have big effects on heating or cooling.
Temperature and Color

- Radiant Exitance

\[ M(R) = \int_0^\infty M(R)\lambda d\lambda \]

- Luminous Exitance

\[ M(V) = \int_0^\infty M(V)\lambda d\lambda \]

\[ M(V) = 683\text{lm}/\text{W} \times \int_0^\infty \bar{y}M(R)\lambda d\lambda \]

- Tristimulus Values

\[ \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \int_0^\infty \begin{pmatrix} \bar{x} \\ \bar{y} \\ \bar{z} \end{pmatrix} M(R)\lambda d\lambda \]

- Integrands for

2000 3000, 3500K

\[ \lambda, \text{Wavelength, nm} \]

\[ x_s, y_s, z_s \]
Chromaticity Coordinates of Blackbody

- Chromaticity

\[ x = \frac{X}{X + Y + Z} \]

\[ y = \frac{Y}{X + Y + Z} \]
Solar Spectrum

- Exo–Atmospheric
  - 6000K, 1480W/m²
  
  \[ E_\lambda (\lambda) = 1560\text{W/m}^2 \times \]

  \[ f_\lambda (\lambda, 6000\text{K}) \]

- Sea Level
  - 5000K, 1000W/m²
  
  \[ E_\lambda (\lambda) = 1250\text{W/m}^2 \times \]

  \[ f_\lambda (\lambda, 5000\text{K}) \]

Constants are higher than total irradiance to account for absorption in certain regions of the spectrum.
Outdoor Radiance

- **Sunlit Cloud**
  \[ E_{\text{incident}} = 1000\text{W/m}^2 \]
  \[ 5000\text{K} \]
  - White Cloud
    \[ M_{\text{cloud}} = E_{\text{incident}} \]
  - Lambertian
    \[ L_{\text{cloud}} = M_{\text{cloud}}/\pi \]

- **Blue Sky**: \[ M_{\lambda} = M_{\text{sky}} f_{\lambda} (5000\text{K}) \times \lambda^{-4} \]

- **Radiant Exitance**, \( M \) of Object Surface Illuminated with \( E \)
  \[ M_{\lambda} (\lambda) = R (\lambda) E_{\lambda} (\lambda) \]

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Thermal Illumination

- Thermal Sources (e.g. Tungsten)

\[ M(R) = \int_0^\infty M(R)\lambda d\lambda \]
\[ M(V) = 683\text{lm/W} \int_0^\infty \bar{y}M(R)\lambda d\lambda \]

- Luminous Efficiency

\[ M(V)/M(R) = 683\text{lm/W} \frac{\int_0^\infty \bar{y}M(R)\lambda d\lambda}{\int_0^\infty M(R)\lambda d\lambda} \]

- Highest Efficiency

\[ \lambda = 555\text{nm} \]

- Bad Color
Illumination

Thanks to Dr. Joseph F. Hetherington

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IR Thermal Imaging

- Infrared Imaging
  \[
  \int_{\lambda_1}^{\lambda_2} \rho(\lambda) \epsilon(\lambda) M_\lambda(\lambda, T) \, d\lambda
  \]
  - \( \rho \) is Responsivity
  - \( \epsilon \) is Emissivity
- Maximize Sensitivity
  \[
  \frac{\partial M_\lambda(\lambda, T)}{\partial T}
  \]
- Work in Atmospheric Pass Bands
- Shorter Wavelength for Higher Temperatures
- Hard to Calibrate (Emissivity, etc.)
- Reflectance and Emission

\[ \Delta M_\lambda / \Delta T \]
Polar Bears

- Thermal Equilibrium
  - Heating by Sun
    * High Temperature
    * High Radiance
    * Small Solid Angle
  - Cooling to Surroundings
    * Body Temperature
    * Low Radiance
    * $\Omega = 2\pi$
  - Extra Heat from Metabolism
- Short-Pass Filter
  - Pass Visible

$E_{\text{incident}} = 50\text{W/m}^2, 200\text{W/m}^2$
600W/m$^2$, top to bottom

Heating (—), Cooling (---), Net (···)
Wavelength Filtering

- Bare Bear: No Filter
- 800nm Short Pass
- 2.5nm Short Pass Like Glass
- Best Bear is a Dynamic Filter

- Net Cooling for Two Best
- Dynamic Filter Follows Zero Crossing of Net $M_\lambda$
- Perfect: No Cooling (Impossible: Nothing Out Means Nothing In)

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