Electrical Engineering
Week 9

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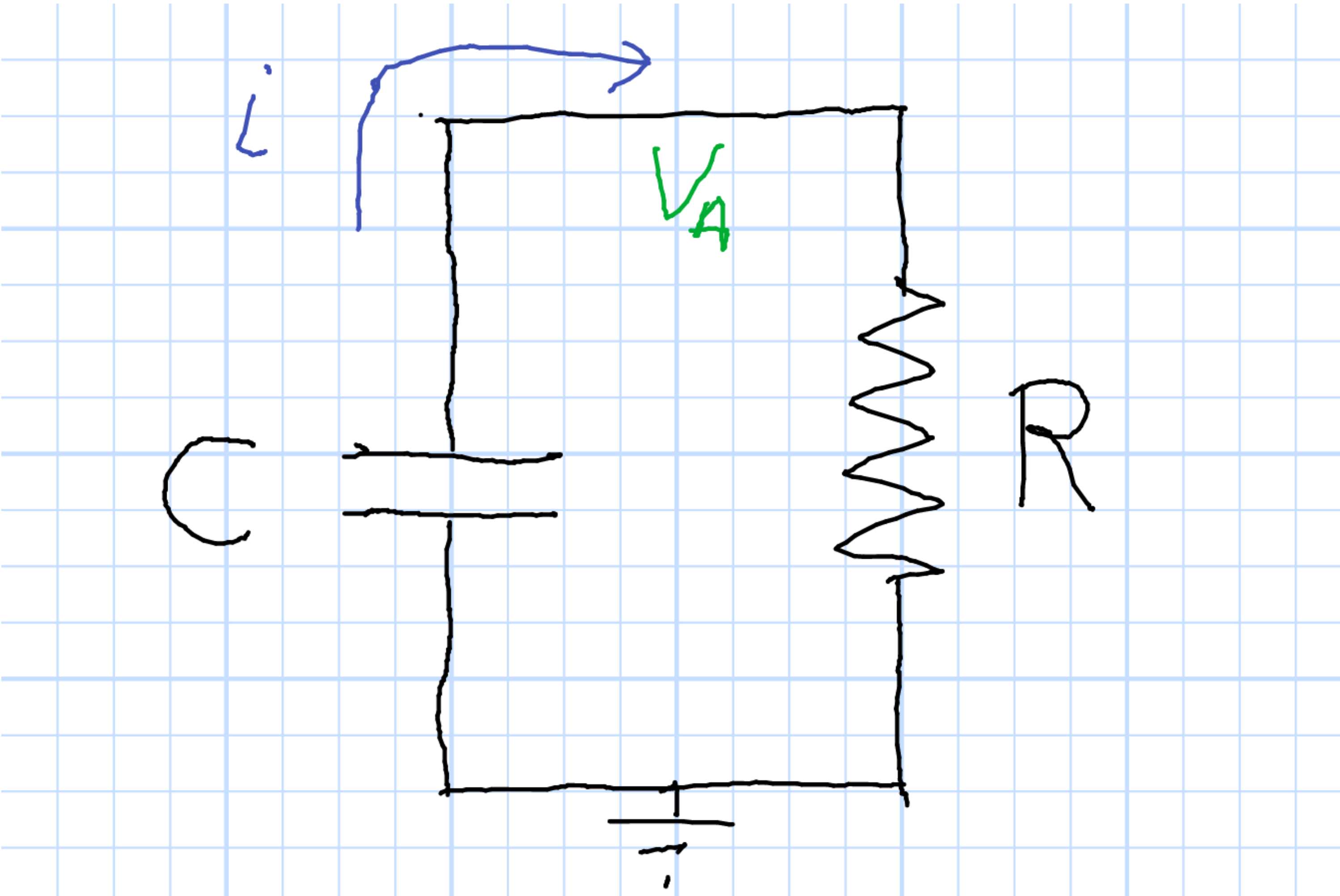
Week 9 Agenda: First–Order Circuits

- RC Circuits
- Boundary Conditions
- Steady State Solutions
- Charge and Discharge a Capacitor
- RL Circuits
- Some Examples

Time–Varying Sources

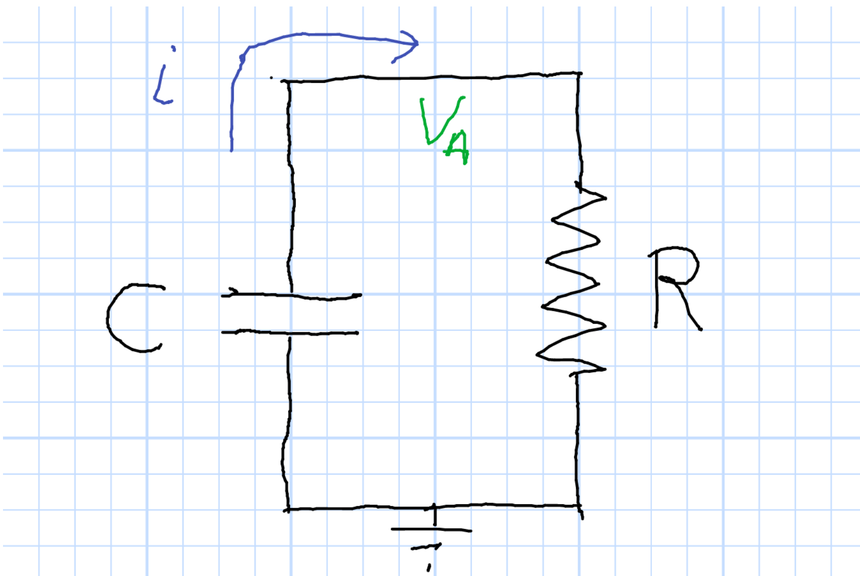
- Transient Analysis (Now)
  - Differential Equations
  - First Order for RL, RC
  - Second Order for RLC
  - Circuits Usually Involve Switches
  - Transient and Steady–State Solutions
- Sinusoidal Solution (Later)
  - Phasor Analysis ($\frac{d}{dt} = j2\pi f$)
  - Complex Impedance
  - “Easy” Solutions
  - Fourier Series and Transforms

RC Circuit



$$i = C\frac{dv}{dt} \qquad i = -C\frac{dv_A}{dt} \qquad v_A = iR$$

RC Equations



$$i = -C\frac{dv_A}{dt}$$

$$i = \frac{v}{R}$$

- Differential Equation

$$v_A = -RC\frac{dv_A}{dt}$$

- Test Solution

$$v_A = k_1 e^{st} + k_2$$

- Substitute

$$k_1 e^{st} + k_2 = -RC\frac{d}{dt}\left(k_1 e^{st} + k_2\right)$$

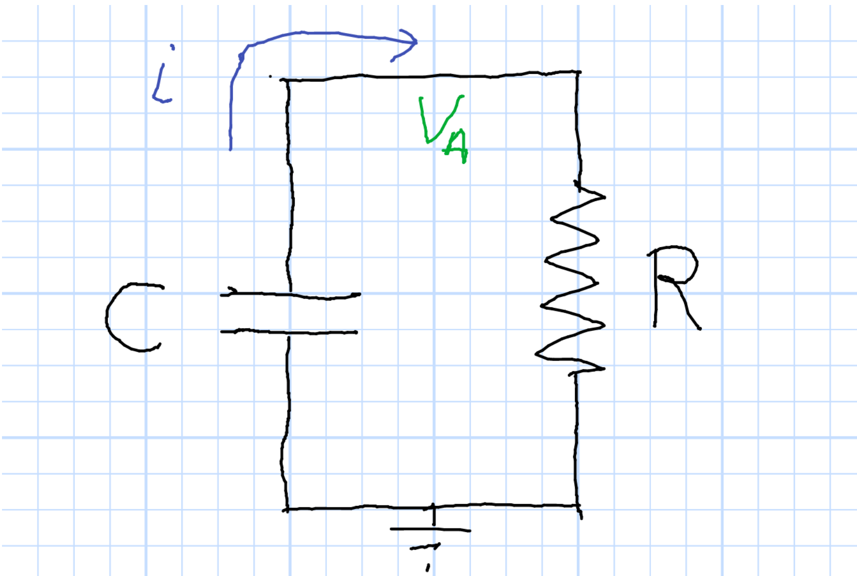
- Take the Derivative

$$k_1 e^{st} + k_2 = -RCsk_1 e^{st}$$

- Group

$$k_1\left(1 + RCs\right)e^{st} - k_2 = 0$$

RC Solution



$$i = -C\frac{dv_A}{dt}$$

$$i = \frac{v}{R}$$

$$v_A = k_1 e^{st} + k_2$$

- From Previous Page

$$k_1 \left(1 + RCs\right) e^{st} - k_2 = 0$$

- True for All Time

$$k_2 = 0 \qquad s = -\frac{1}{RC}$$

- Solution

$$v_A = k_1 e^{-t/(RC)} + 0$$

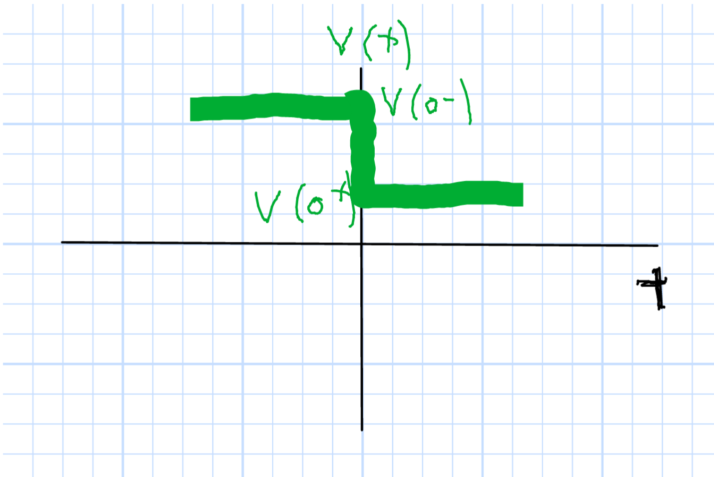
- Time Constant

$$v_A = k_1 e^{-t/\tau}$$

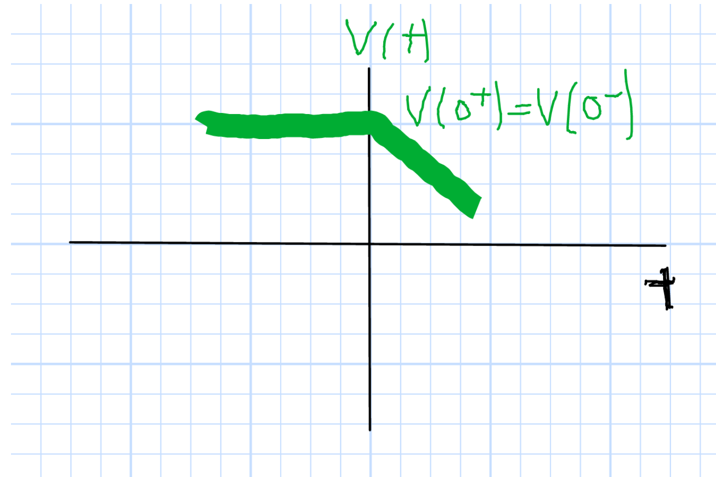
$$\tau = RC$$

- Still One Unknown (k_1)

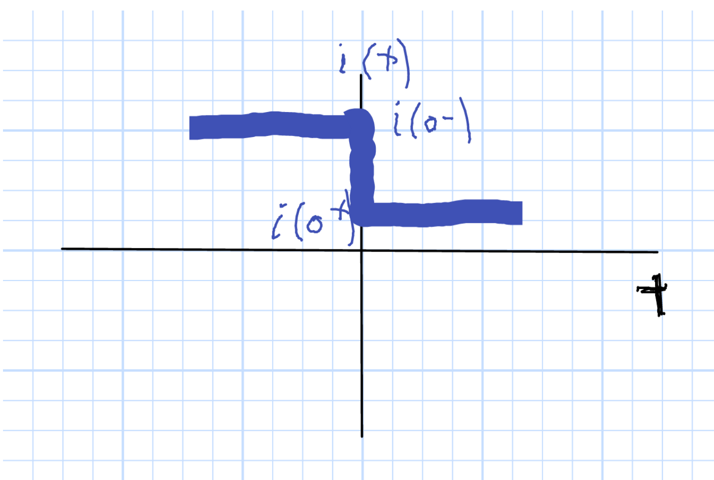
General Boundary Conditions



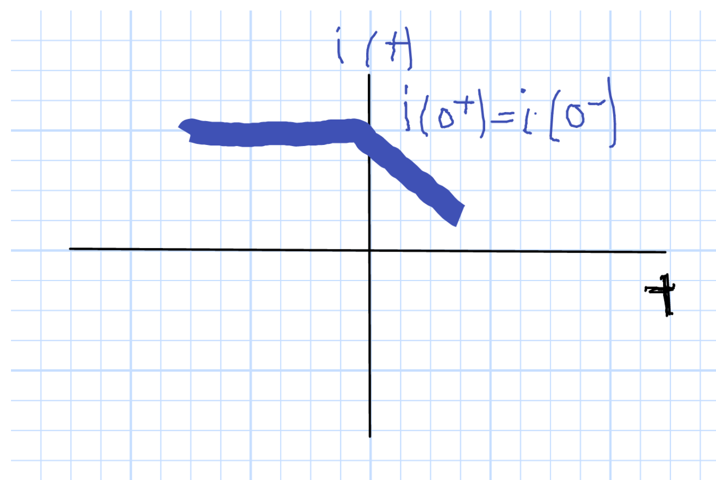
Not Valid for Capacitors



OK

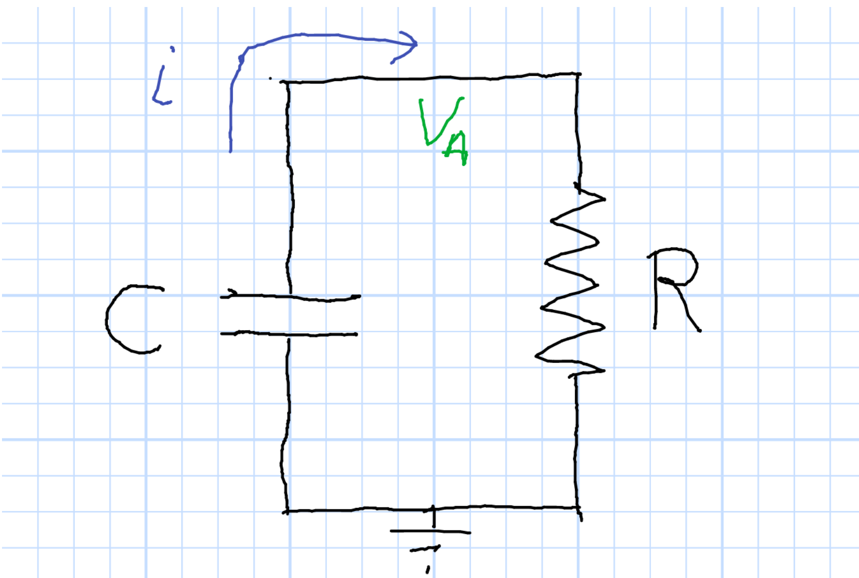


Not Valid for Inductors



OK

Initial Conditions



$$i = -C\frac{dv_A}{dt}$$

$$i = \frac{v}{R}$$

$$v_A = k_1 e^{st} + k_2$$

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$$v_A = k_1 e^{-t/\tau}$$

$$\tau = RC$$

- Original Voltage $V\left(0^-\right)$
- Boundary Condition

$$V\left(0^+\right) = V\left(0^-\right)$$

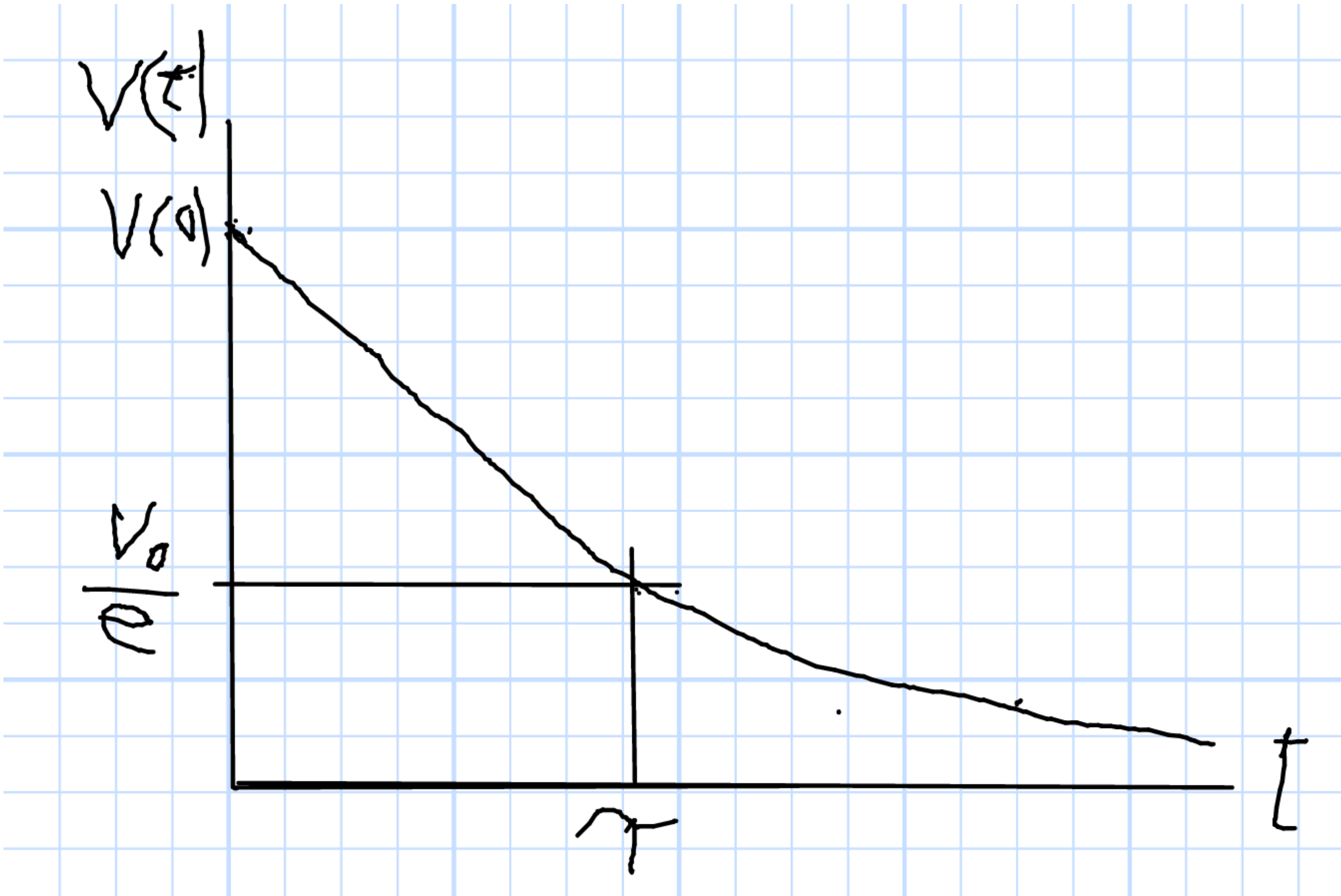
- At $t = 0$

$$k_1 e^{-0/\tau} = k_1 = V\left(0^+\right)$$

- Solution

$$v_A = V\left(0^-\right) e^{-t/\tau}$$

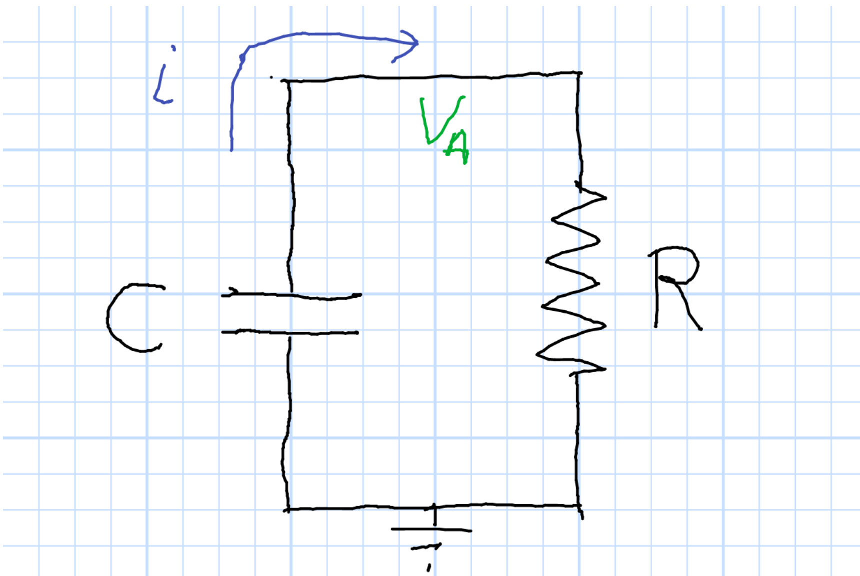
Exponential Solutions



$$v_a = v_a(0)\,e^{-t/\tau} \qquad v_a(\tau) = v_a(0) \times \frac{1}{e} \approx v_a(0) \times 0.3679$$

$$v_a(2\tau) \approx v_a(0) \times 0.1353 \qquad v_a(10\tau) \approx v_a(0) \times 4.540 \times 10^{-5}$$

Steady–State Solution



$$i = -C\frac{dv_A}{dt}$$

$$i = \frac{v}{R}$$

$$v_A = k_1 e^{st} + k_2$$

- Steady State

$$t \rightarrow \infty$$

- Anything that is going to happen has happened

$$\frac{d\text{Anything}}{dt} = 0$$

- Transient Solution is Zero

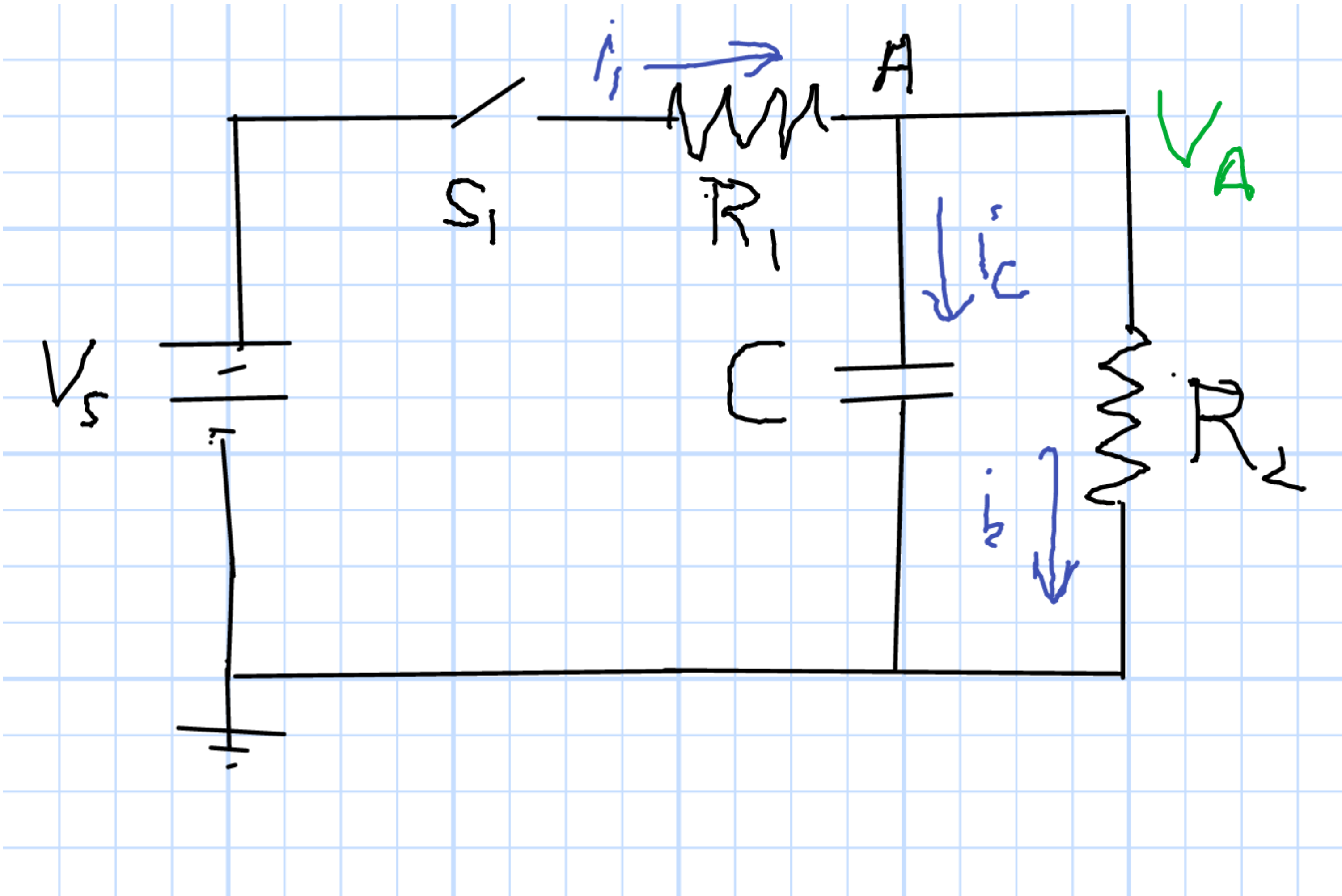
$$\frac{dv_A}{dt} = 0 \qquad i = 0$$

- Solution

$$k_2 = 0 = v_{A\infty}$$

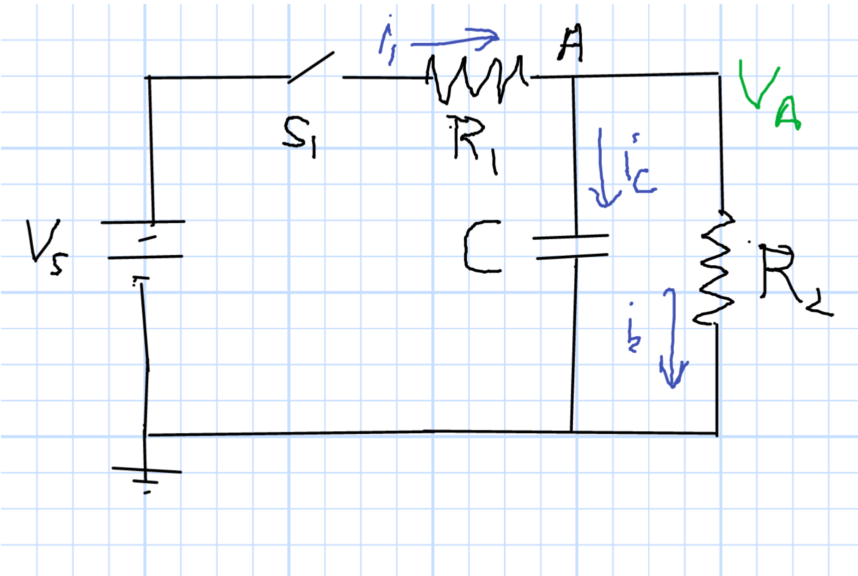
Q: Why would the steady–state solution not be zero?

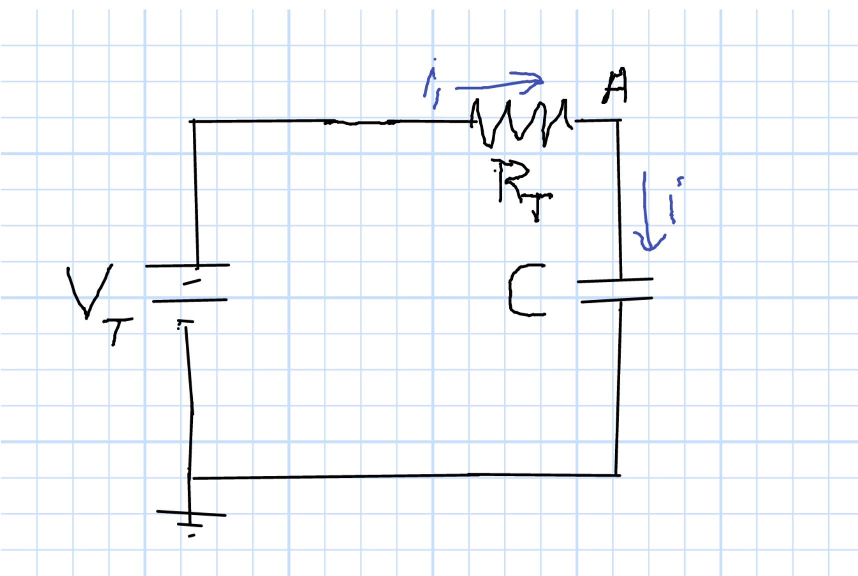
Charge and Discharge



Close S_1 at $t = 0$. Open S_1 at $t = t_1$. What will happen?

Charge!





Thévenin Equivalent Charging

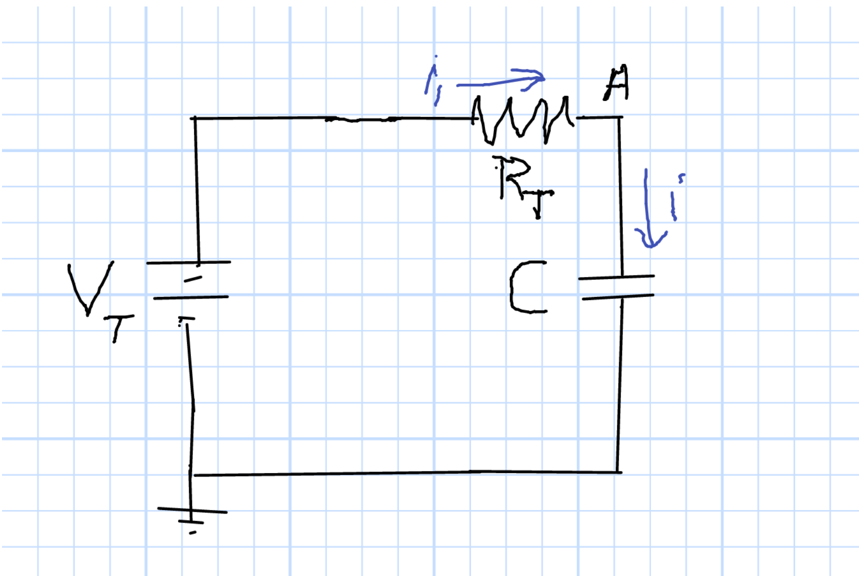
$$v_T = v_S \frac{R_2}{R_1 + R_2}$$

$$R_T = R_1 \parallel R_2$$

Assume

$$v_A(0) = 0$$

Charging Equations



$$i = \frac{v_T - v_A}{R_T} = C\frac{dv_A}{dt}$$

$$v_T - v_A = R_T C\frac{dv_A}{dt}$$

$$v_A + R_T C\frac{dv_A}{dt} = v_T$$

$$v_T = v_S\frac{R_2}{R_1 + R_2}$$

$$R_T = R_1 \parallel R_2$$

$$v_A(0) = 0$$

Proposed Solution

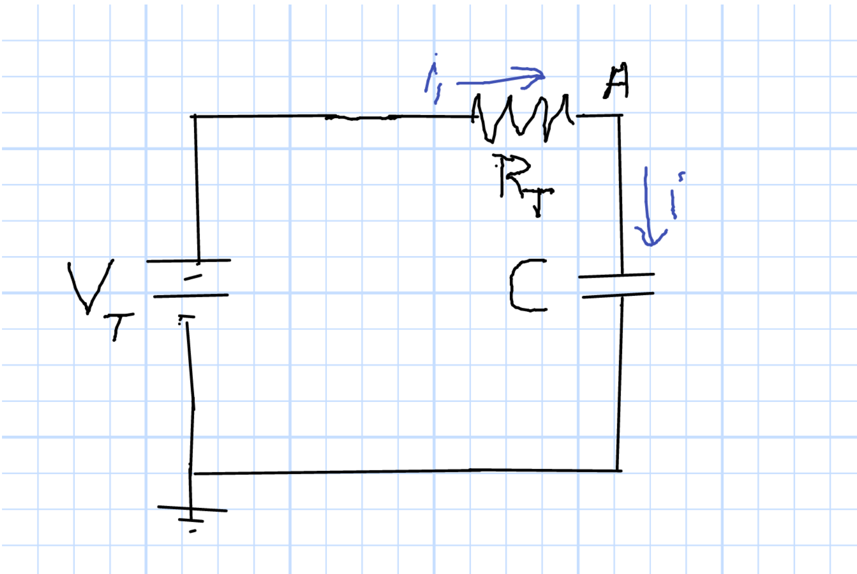
$$v_A = k_1 e^{st} + k_2$$

$$k_1 e^{st} + k_2 + R_T C\frac{d}{dt}\left(k_1 e^{st} + k_2\right) = v_T$$

$$k_1 e^{st}\left(1 + R_T C s\right) + k_2 = v_T$$

$$s = \frac{-1}{R_T C} \qquad k_2 = v_T$$

Charging Solution



$$v_T = v_S \frac{R_2}{R_1 + R_2}$$

$$R_T = R_1 \parallel R_2$$

$$v_A(0) = 0$$

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$$v_A = k_1 e^{st} + k_2$$

$$s = \frac{1}{R_T C} \qquad k_2 = v_T$$

$$v_A = k_1 e^{-t/(R_T C)} + v_T$$

- Initial Condition

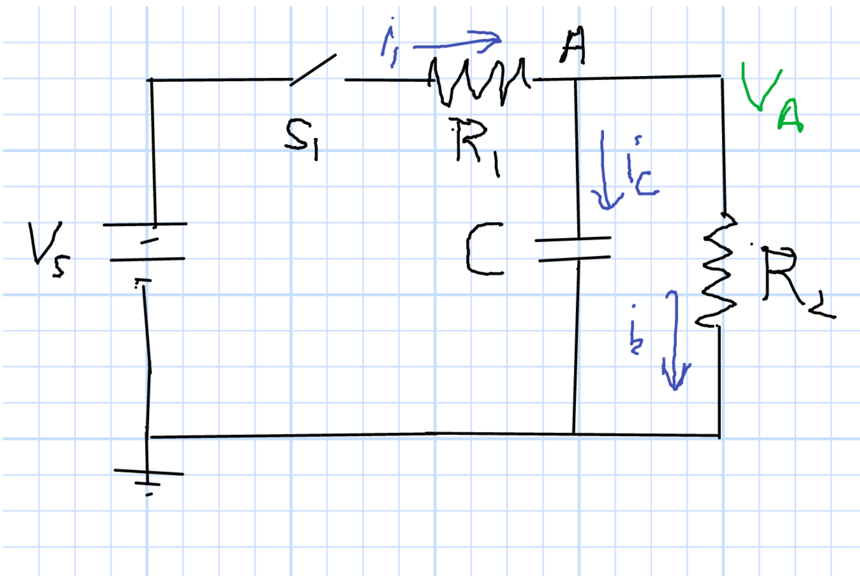
$$v_A(0) = k_1 + v_T$$

$$v_A(0) = 0 \qquad k_1 = -v_T$$

- Solution

$$v_A = v_T \left(1 - e^{-t/(R_T C)}\right)$$

Charging Result



$$v_T = v_S \frac{R_2}{R_1 + R_2}$$

$$R_T = R_1 \parallel R_2$$

$$v_A(0) = 0$$

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$$v_A = v_T \left(1 - e^{-t/(R_T C)}\right)$$

- Use v_T and R_T

$$v_A = v_s \frac{R_2}{R_1 + R_2} \times$$

$$\left\{1 - e^{-t/((R_1 \| R_2)C]}\right\}$$

- Assume $R_1 \ll R_2$

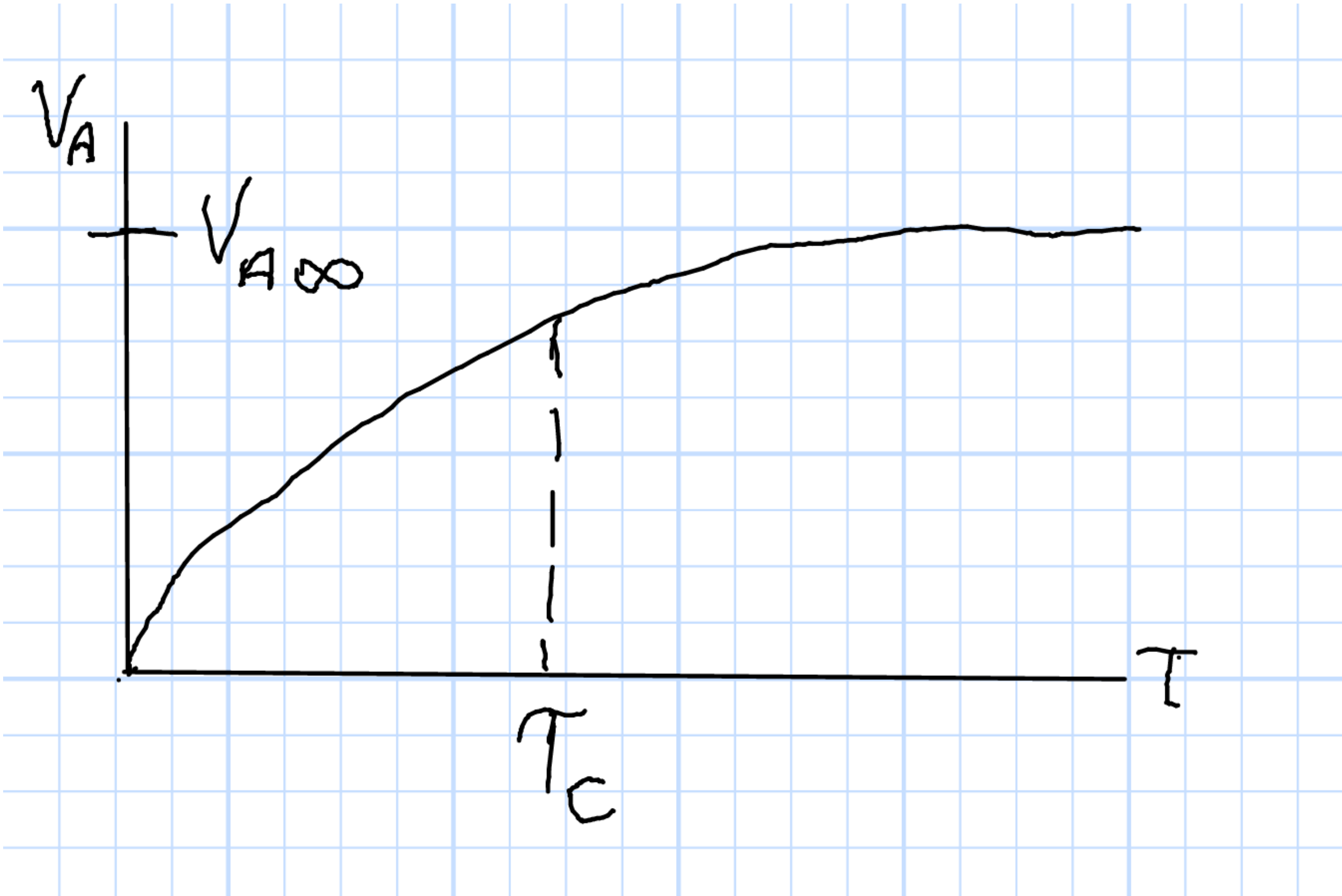
$$v_A \approx v_s \times \left\{1 - e^{-t/(R_1 C]}\right\}$$

- Example

$$R_1 = 100\Omega \qquad C = 100\mu\text{F}$$

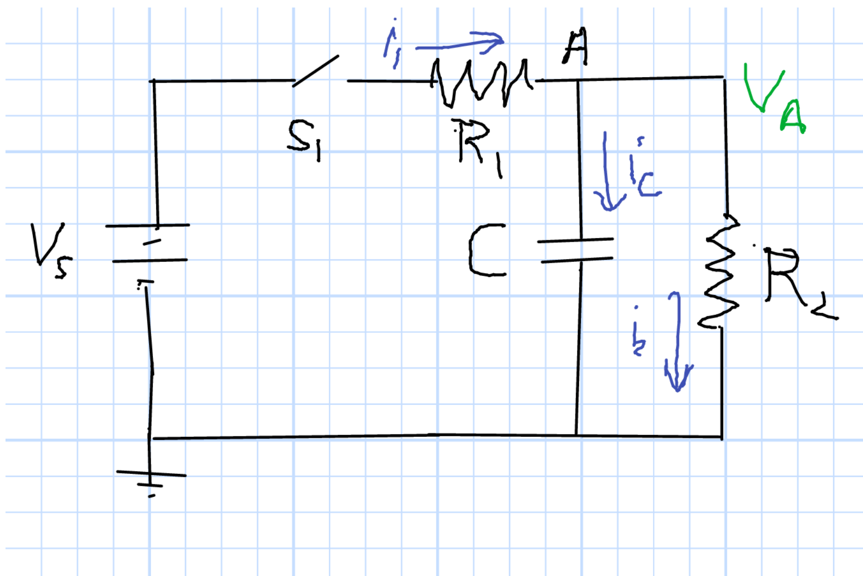
$$\tau \approx R_1 C = 10\text{ms}$$

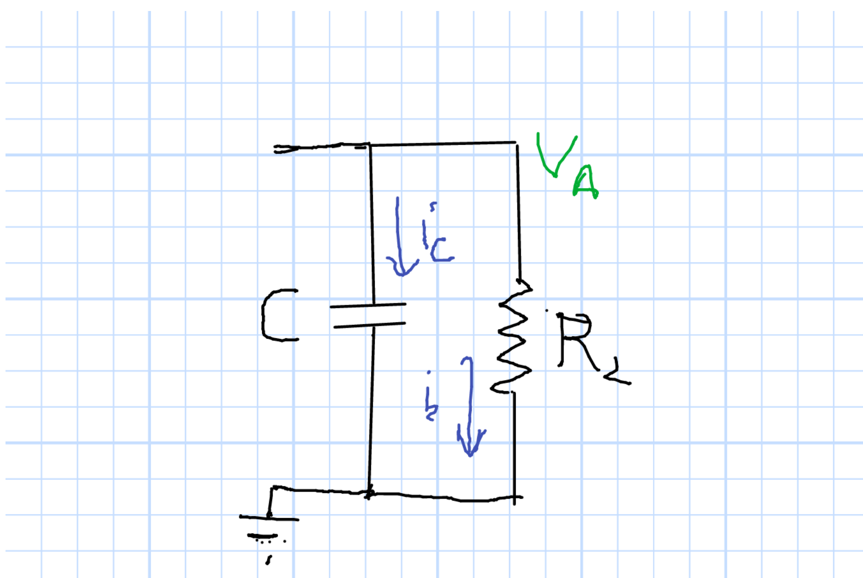
Charging Voltage



$v_{A\infty} = v_T \approx v_s \qquad \tau_C \approx R_1 C \qquad v_A(\tau_C) \approx (1 - 0.3679)\, v_s = 0.6321 v_s$

Discharge!





- Thévenin Equivalent

$$v_T = 0 \qquad R_T = R_2$$

- Initial Voltage
- Open S_1 at $t = t_1$

$$v_A(t_1) = v_s \frac{R_2}{R_1 + R_2} \times \left\{1 - e^{-t_1/((R_1 \| R_2)C]}\right\}$$

- We've seen this before

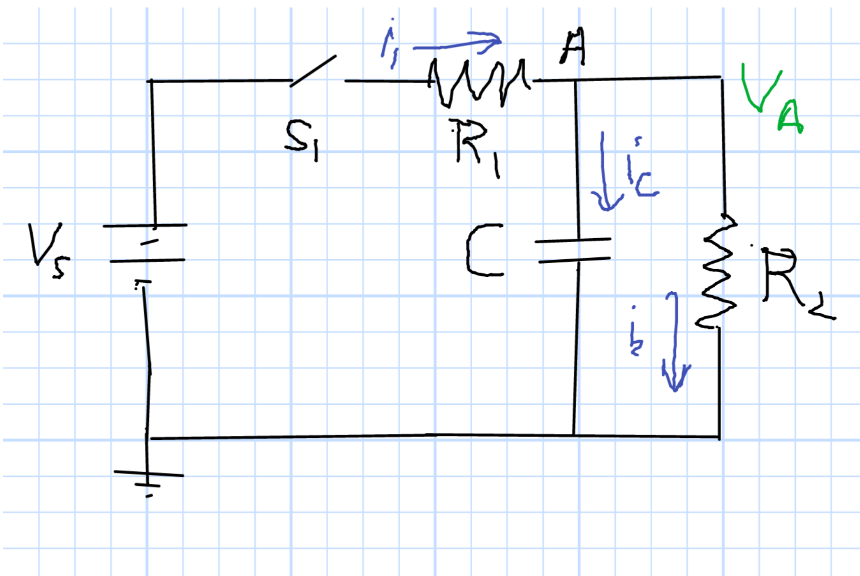
$$v_A = v_A(t_1)\, e^{-(t-t_1)/(R_2 C)}$$

- Example: $R_2 = 10\text{k}\Omega$

$$v_A = v_A(t_1)\, e^{-(t-t_1)/\tau_D}$$

$$\tau_D = 1\text{s}$$

Summary



- Charge

$$v_A = v_s \frac{R_2}{R_1 + R_2} \left\{ 1 - e^{-t/\tau_C} \right\}$$

- Charging Time Constant

$$\tau_C = (R_1 \parallel R_2)\, C$$

- End of Actual Charge

$$v_C = v_s \frac{R_2}{R_1 + R_2} \left\{ 1 - e^{-t_1/\tau_C} \right\}$$

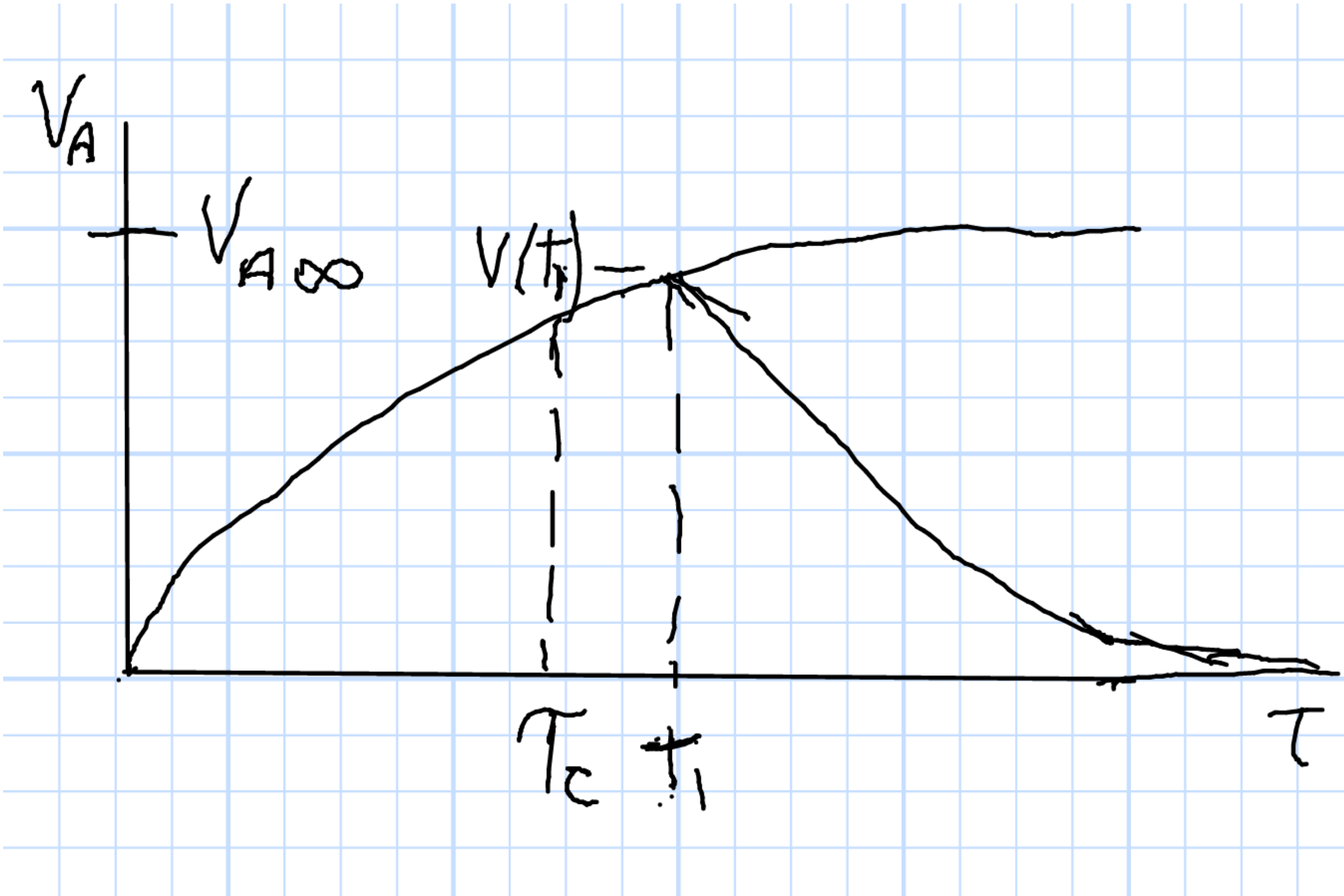
- Discharge

$$v_A = v_C\,(t_1)\, e^{-(t-t_1)/\tau_D}$$

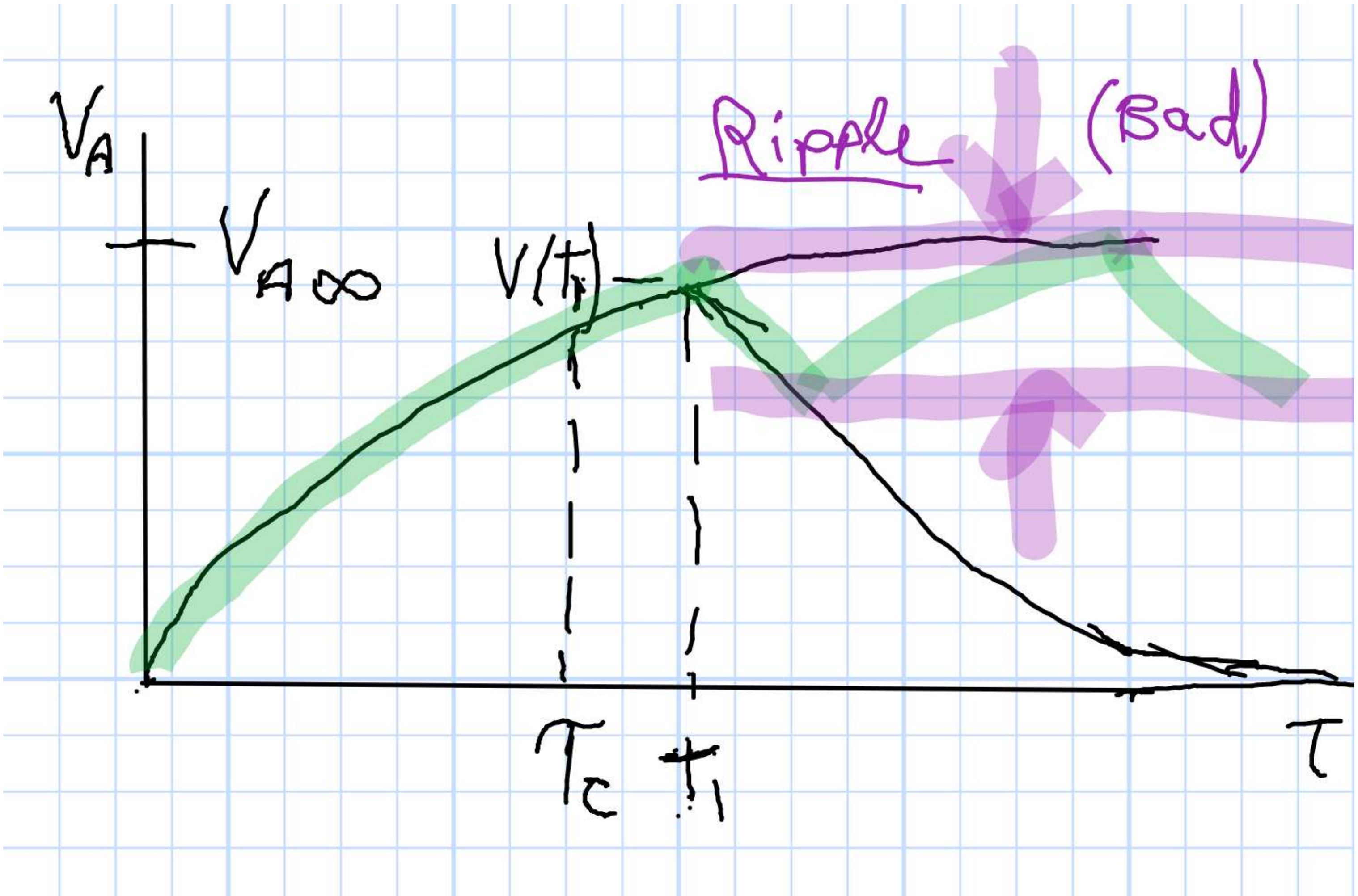
- Discharge Time Constant

$$\tau_D = R_2 C$$

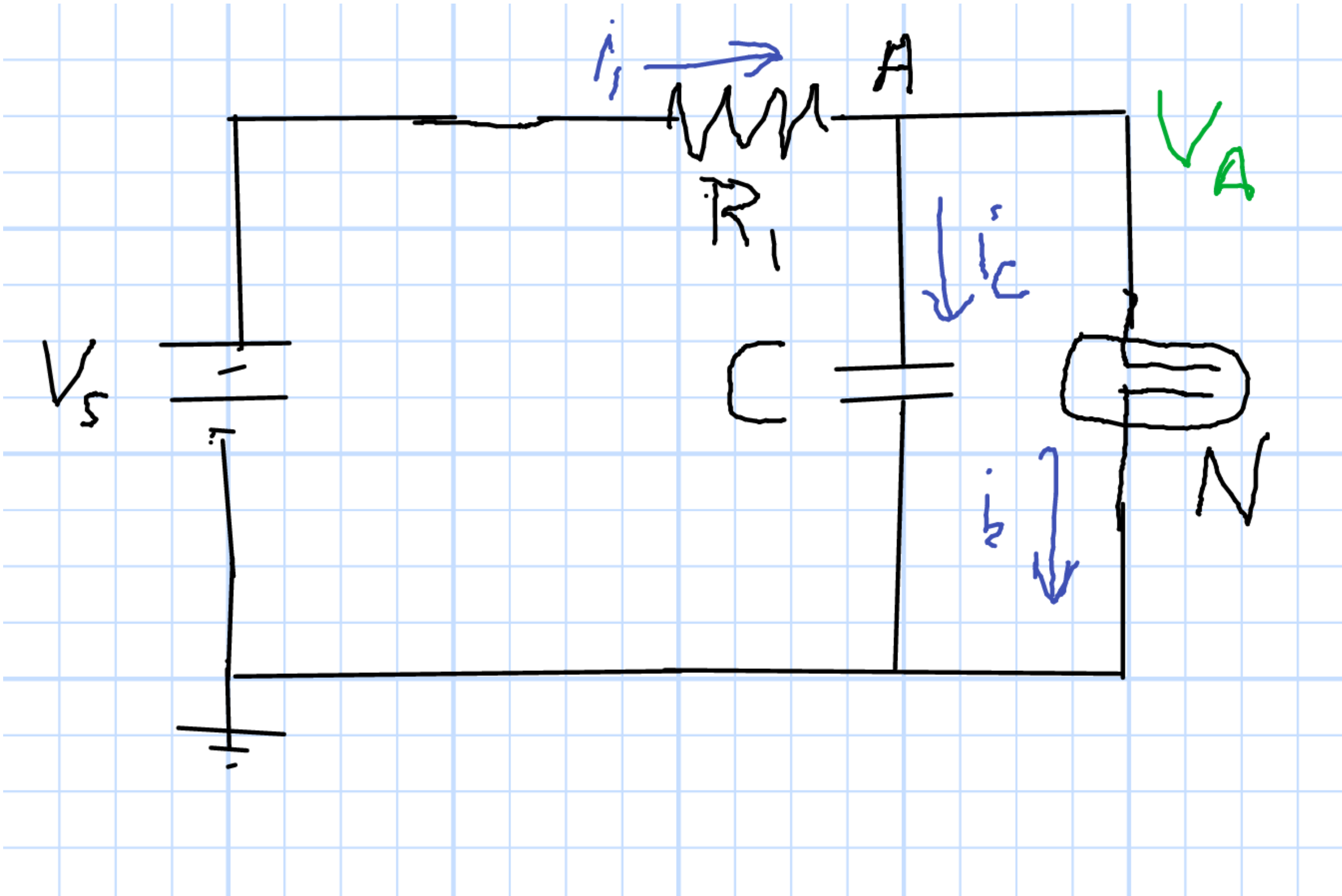
Charging and Discharging Voltage



Repeated Charging and Discharging Voltage

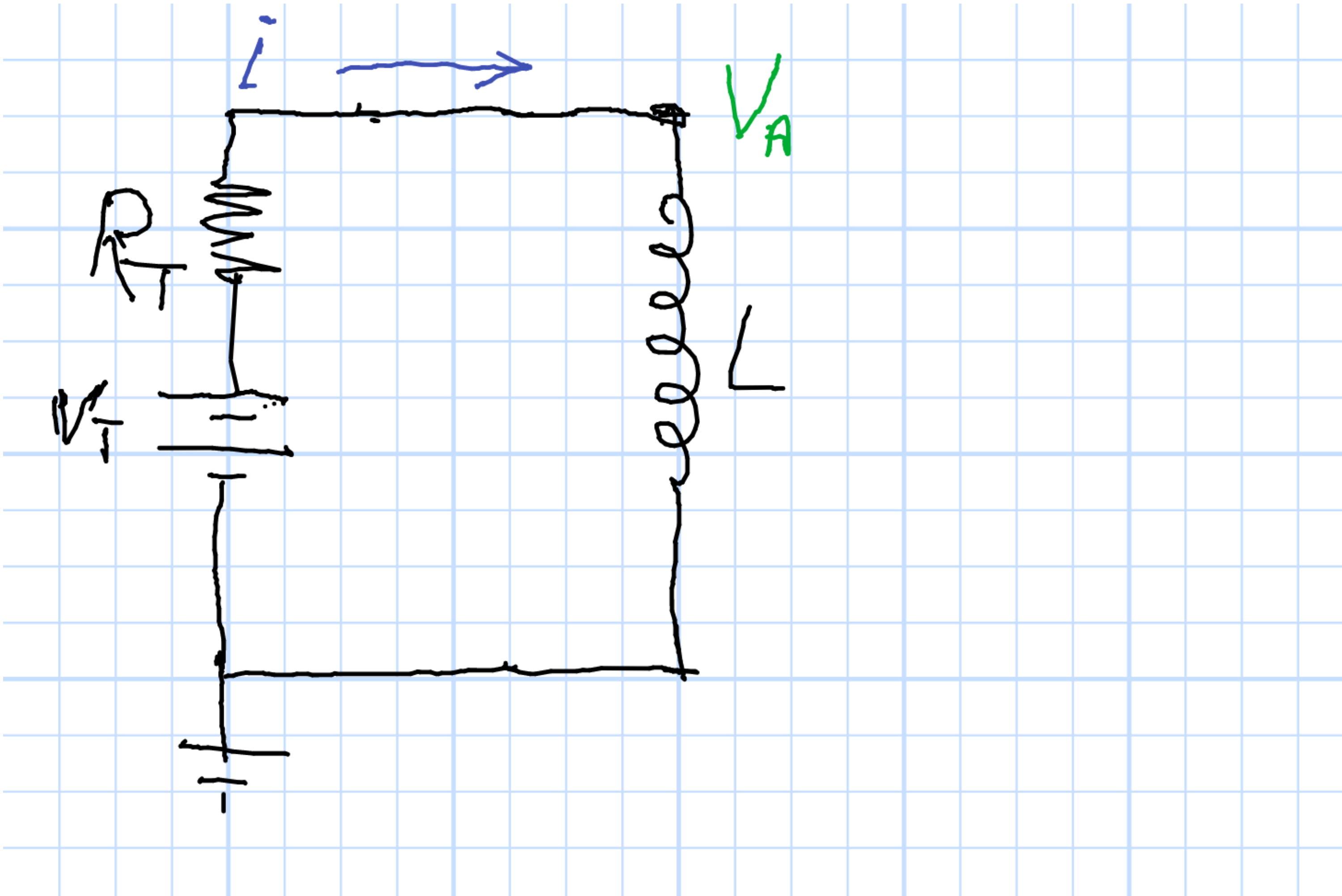


Neon Light Blinker

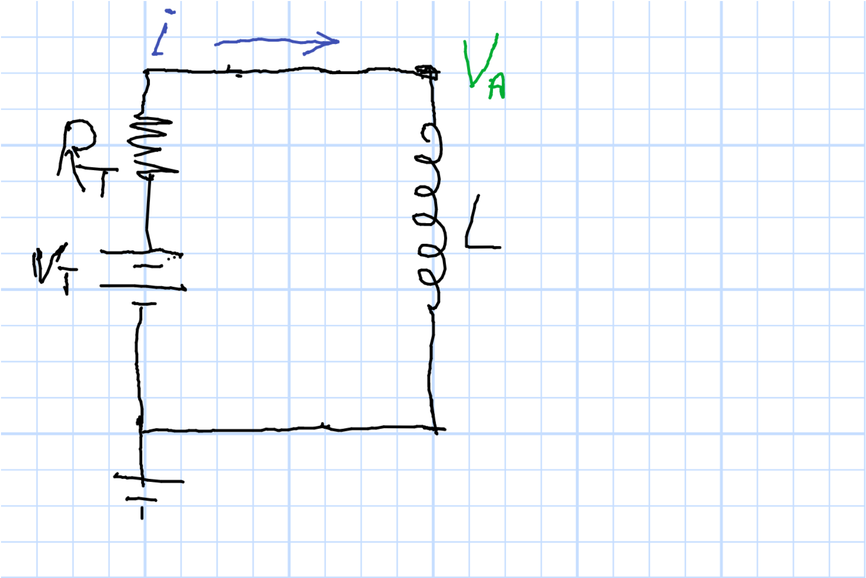


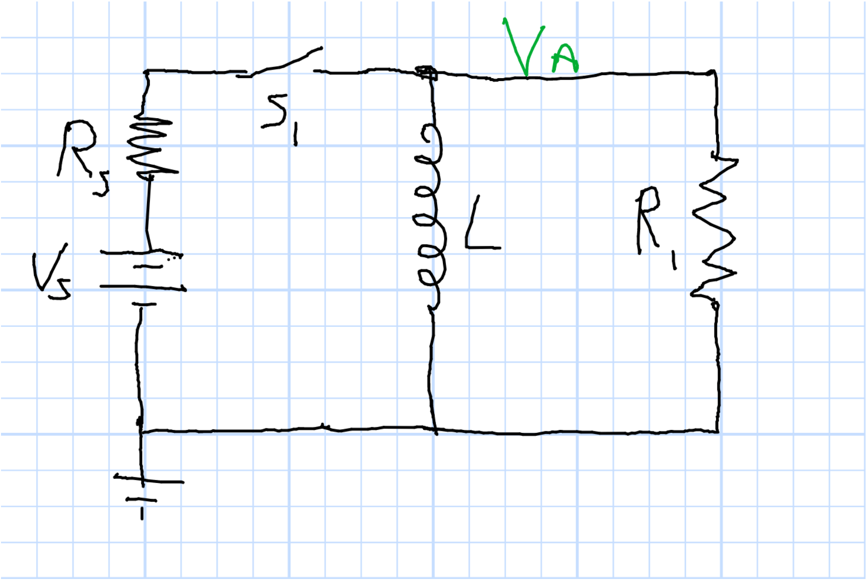
Neon Light is open for low voltage, low resistance for high voltage.
$R = 1\text{M}\Omega$, $C = 1\mu\text{F}$, $\tau = 1\text{s}$, $v_S \approx 100\text{V}$ or more.

RL Circuit



Simplified RL Circuit

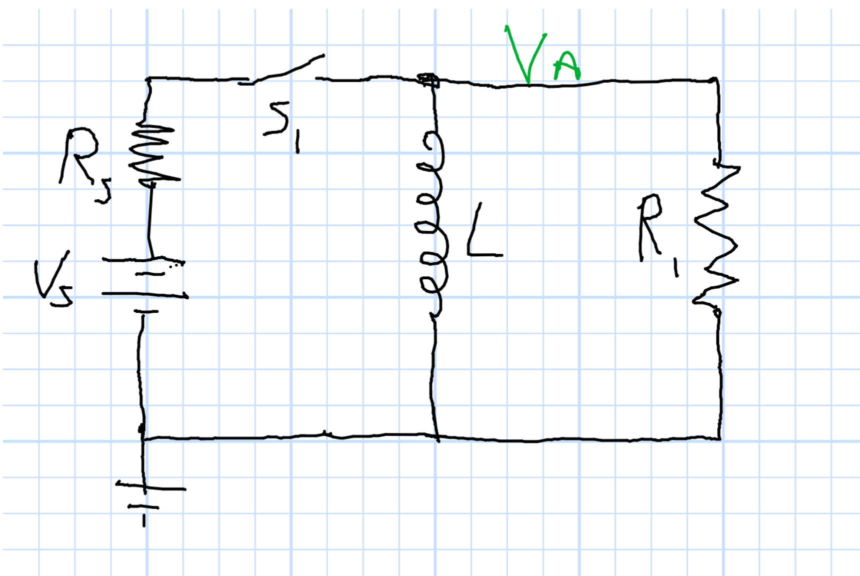




$$v_T = v_S \frac{R_1}{R_1 + R_s}$$

$$R_T = R_1 \parallel R_s$$

RL Equations



$$V_A = L\frac{di}{dt}$$

$$v_A = L\frac{d}{dt}\frac{v_T - v_A}{R_T}$$

Try

$$v_A = k_1 e^{st} + k_2$$

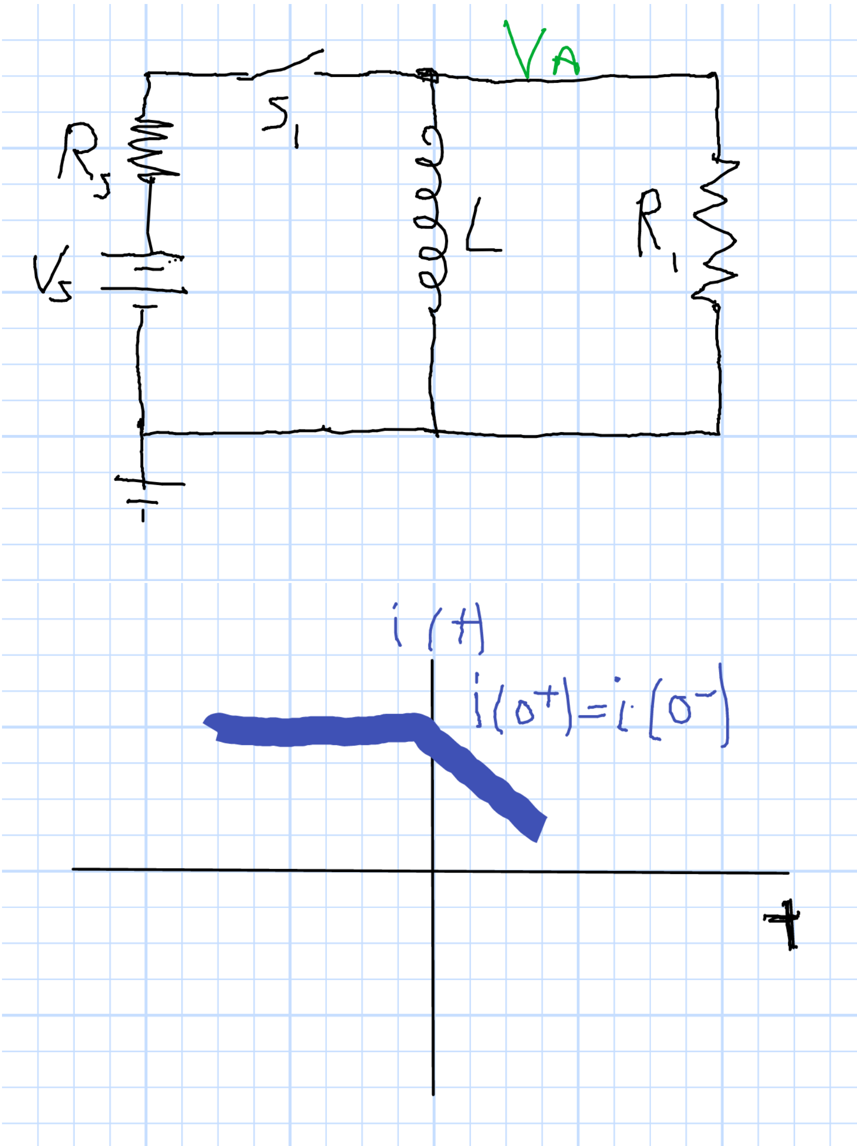
Solve

$$k_1 e^{st} + k_2 = -\frac{L}{R_T}\frac{dv_A}{dt} = -\frac{L}{R_T}k_1 s e^{st}$$

$$\left(1 + \frac{L}{R_T}s\right) k_1 e^{st} + k_2 = 0$$

$$s = -\frac{L}{R_T} \qquad k_2 = 0$$

RL Solution



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$$v_A = k_1 e^{st} + k_2$$

$$s = -\frac{L}{R_T} \qquad k_2 = 0$$

- Result

$$v_A = k_1 e^{-\frac{L}{R_T}t}$$

- Boundary Condition:
  $i(0) = 0,\ v_A(0) = v_T$

$$v_A = v_T e^{\frac{L}{R_T}t}$$

- Time Constant

$$v_A = v_T e^{-t/\tau} \qquad \tau = \frac{R_T}{L}$$

AC Coupled Amplifier