Electrical Engineering
Week 13

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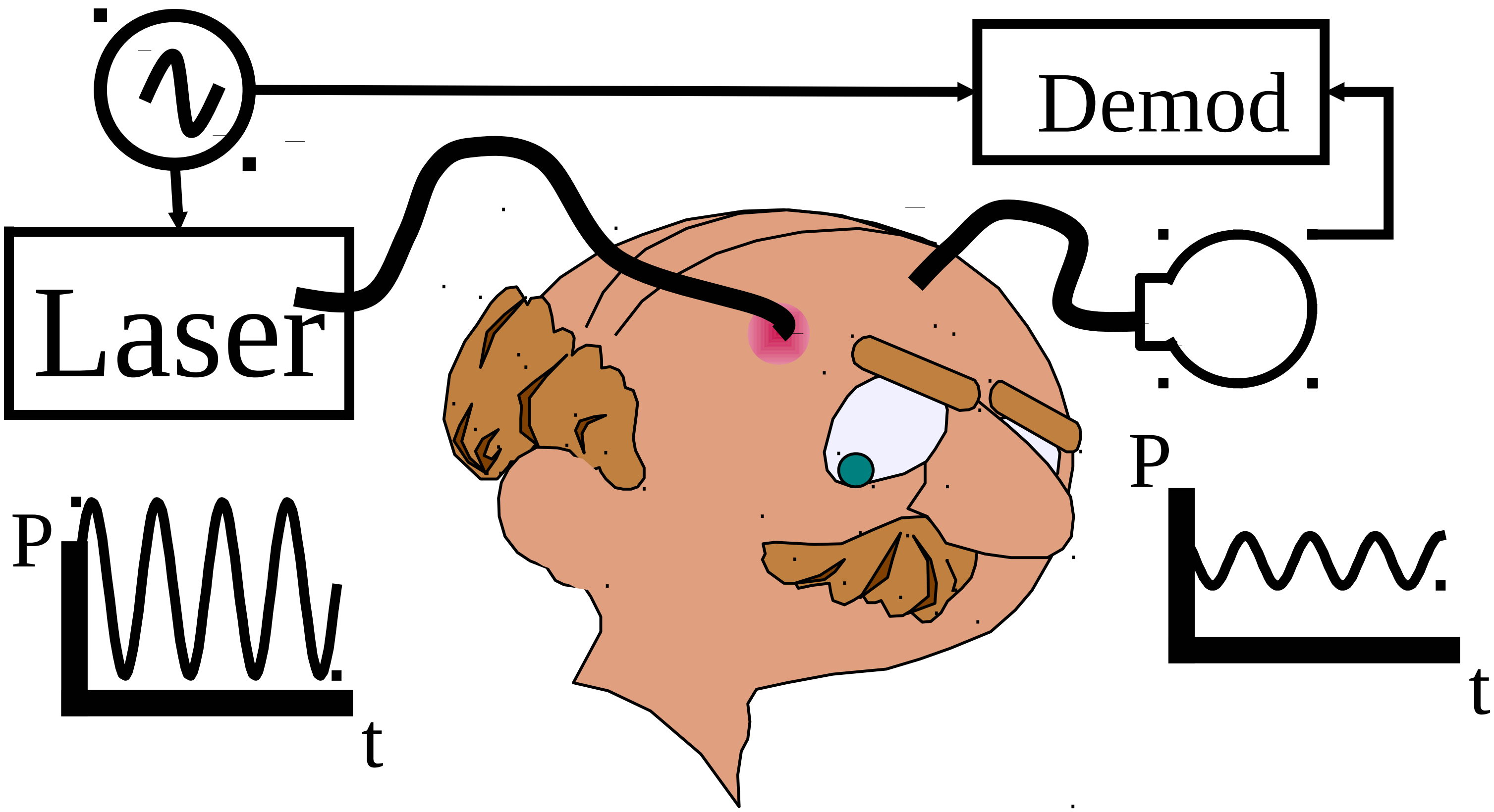
Week 13 Agenda: Fourier Analysis

- Motivation: Diffusive Optical Tomography
- Fourier Series
- Transfer Function
- Bels and deciBels
- High– and Low–Pass Filters
- Low–Pass Example
- Bode Plots
- Notch Filter

Diffusive Optical Imaging



Light Source can be DC, Sinusoidal, or Pulsed

Fourier Series

- Sinusoids From Last Week

$$v(t) = \text{Re}\left(\mathbf{V}e^{j\omega t}\right) = \frac{\mathbf{V}}{2}e^{j\omega t} + \frac{\mathbf{V}^*}{2}e^{-j\omega t}$$

- General Periodic Function

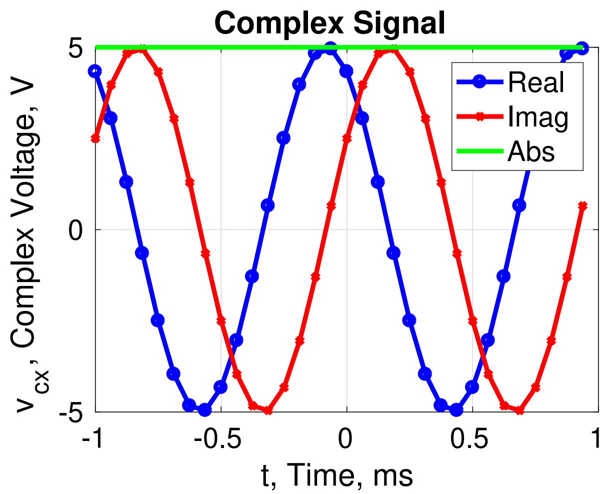
$$v(t) = \sum_{n=-\infty}^{\infty}\left(\frac{\mathbf{V}_n}{2}e^{jn\omega t}\right)$$

- Constraint for Real Functions of Time

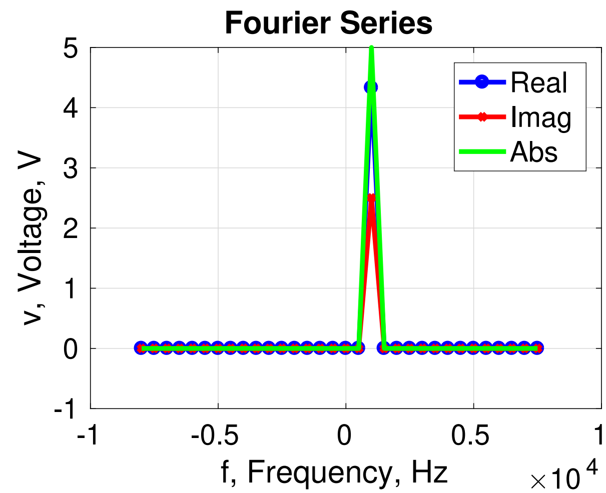
$$\mathbf{V}_{-n} = \mathbf{V}^*_{+n}$$

- Given a Periodic $v(t)$, We Can Find $\mathbf{V}_n$ for All n.

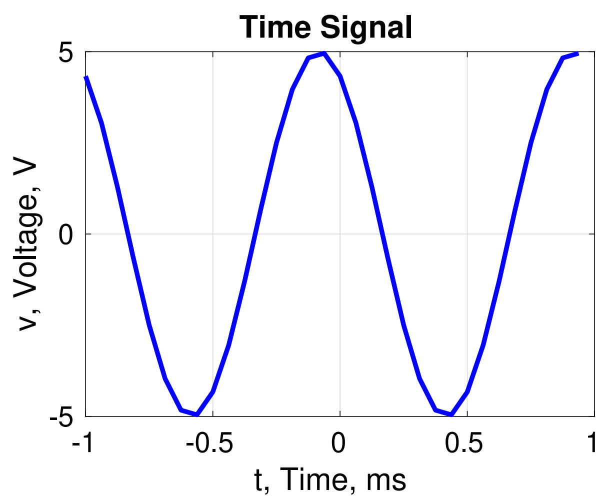
Fourier Series of Complex and Real Signals



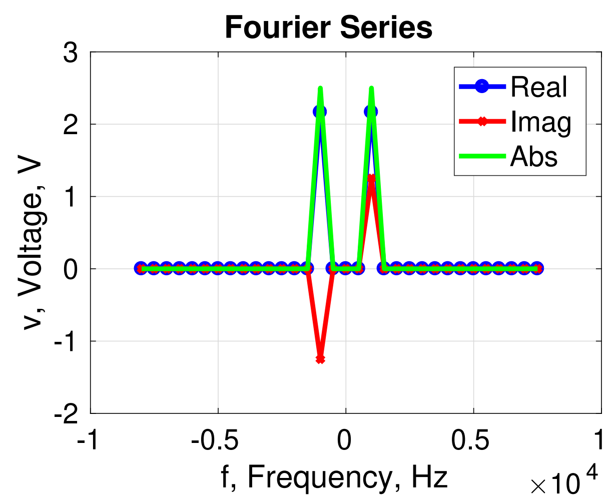
$Ve^{j30\pi/180}e^{j\omega t}$



$V(-f) = 0$

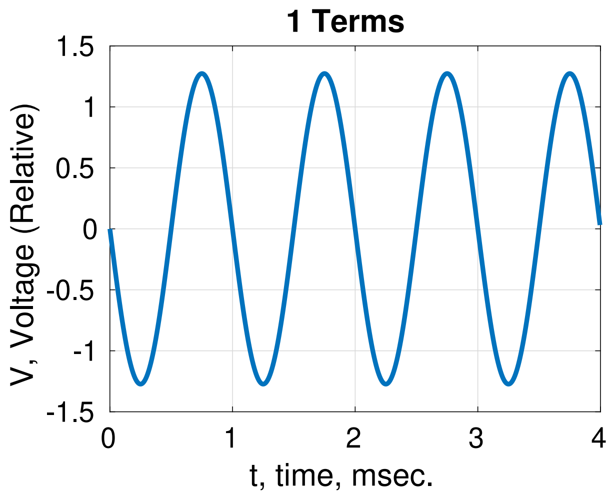


$\cos(\omega t + 30°)$

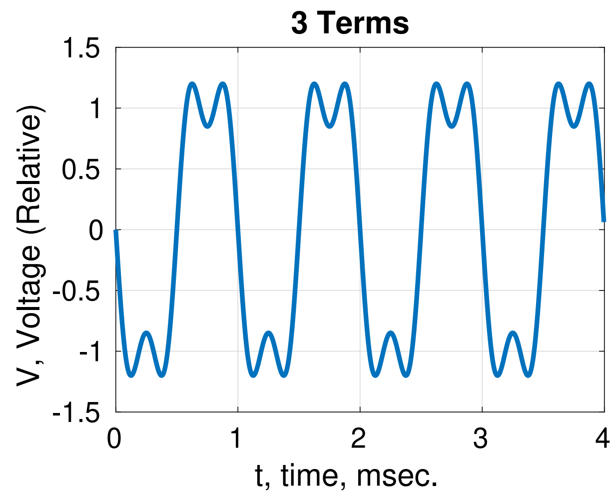


$V(-f) = V^*(f)$

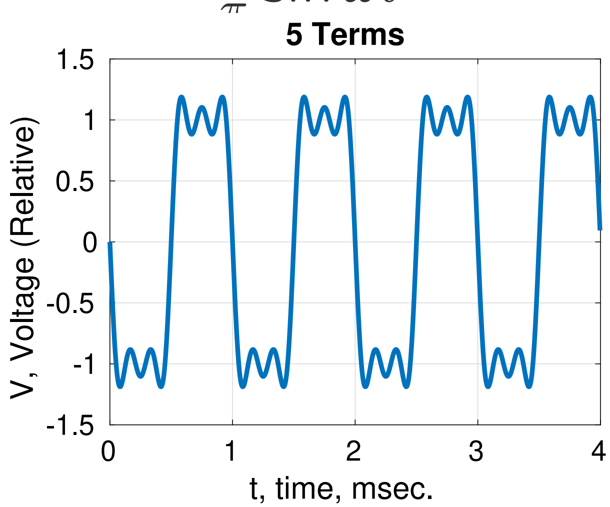
Square Wave



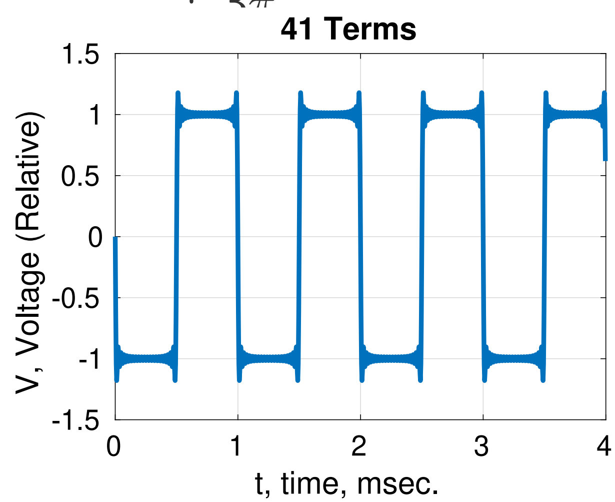
$$\frac{4}{\pi}\sin\omega t$$



$$+\frac{4}{3\pi}\sin 3\omega t$$

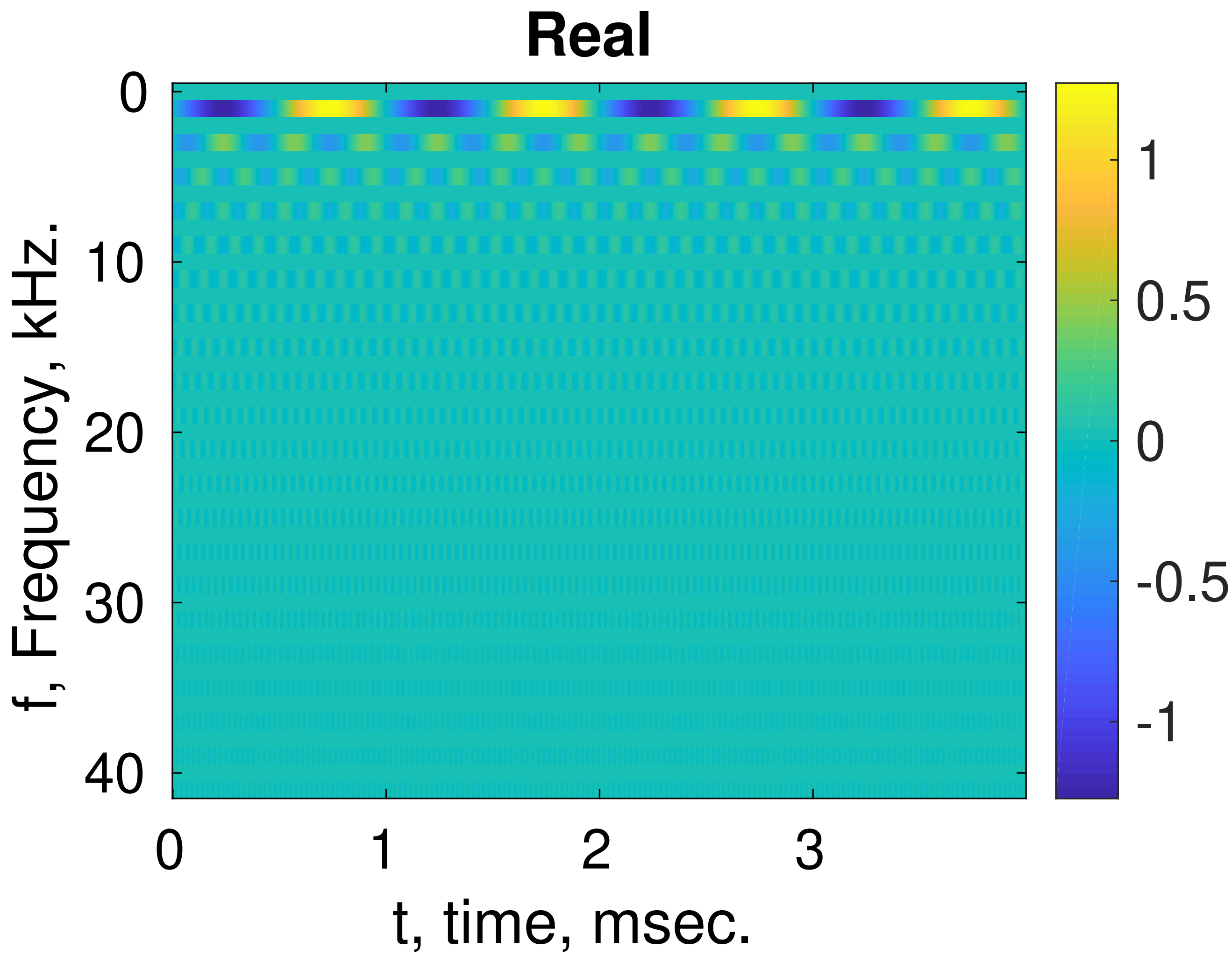


$$+\frac{4}{5\pi}\sin 5\omega t$$



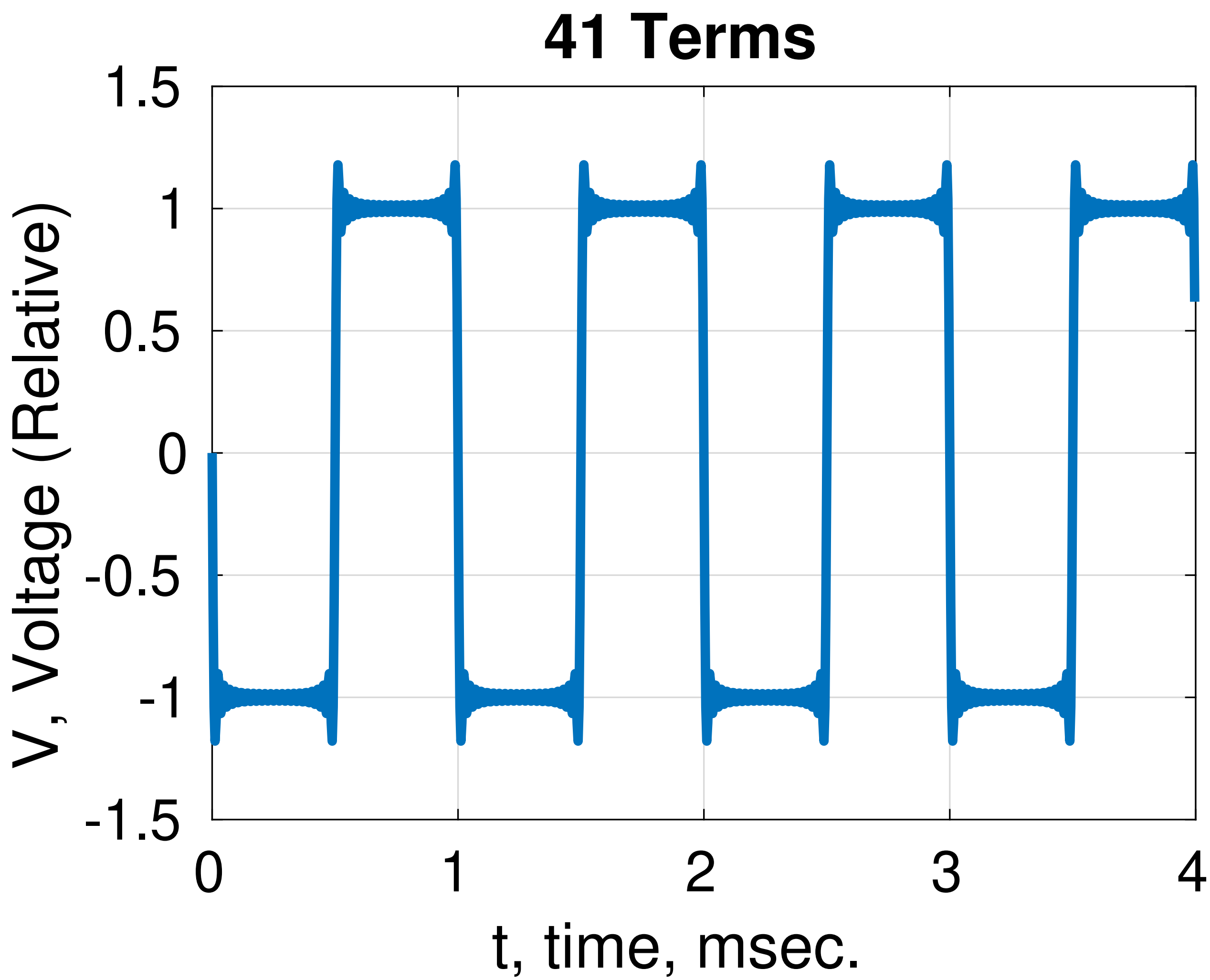
$$+\cdots+\frac{4}{41\pi}\sin 41\omega t$$

Harmonics Added



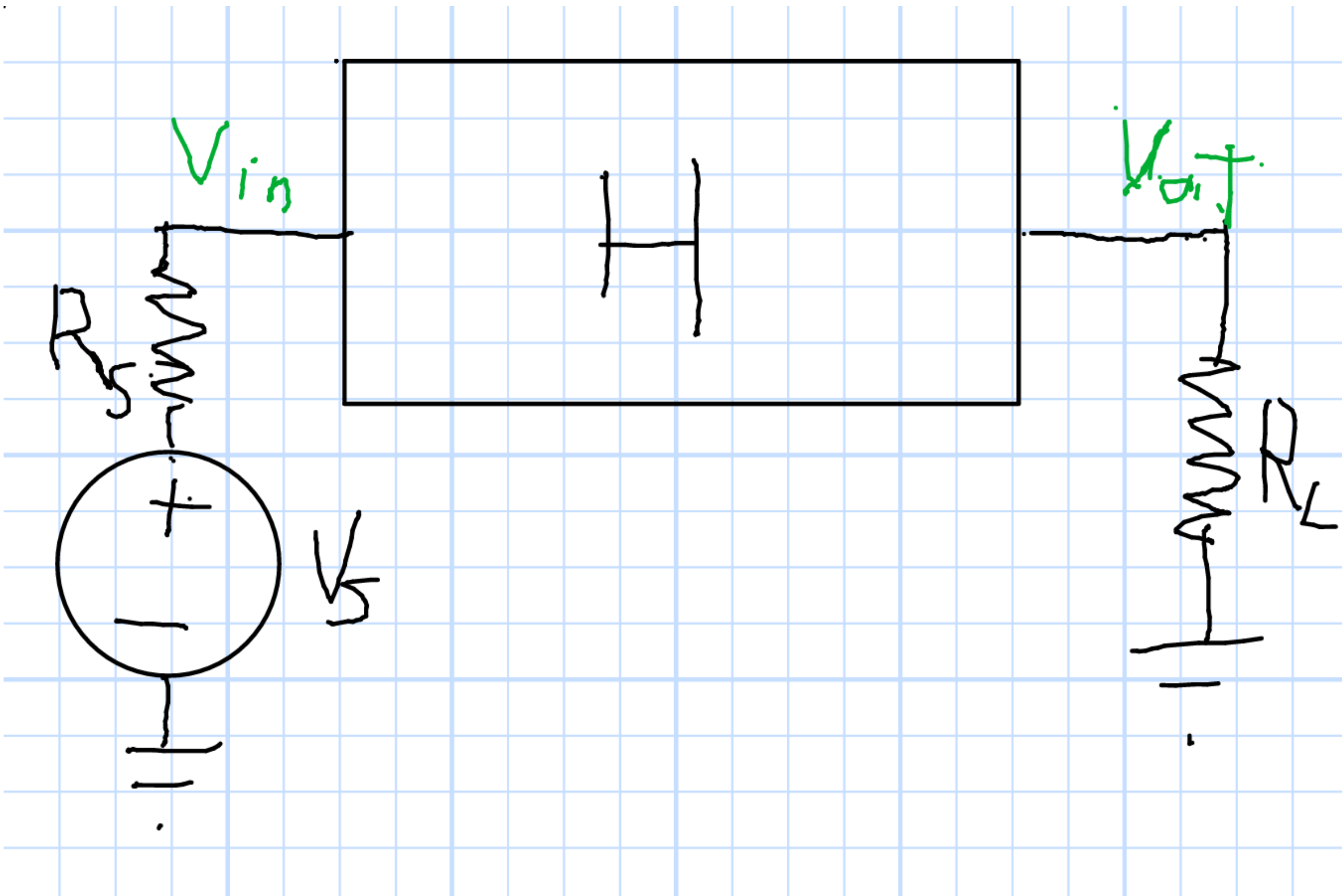
No Even Harmonics, Higher Harmonics Contribute Little

Square Wave Approximation



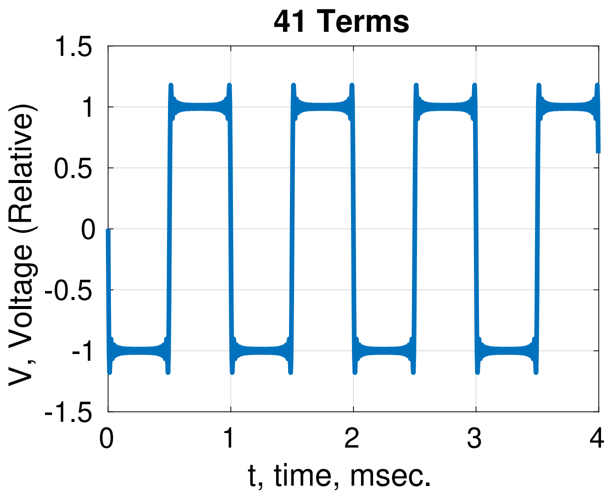
More Harmonics Would Reduce the Ringing (Gibbs Phenomenon)

Transfer Function Concept

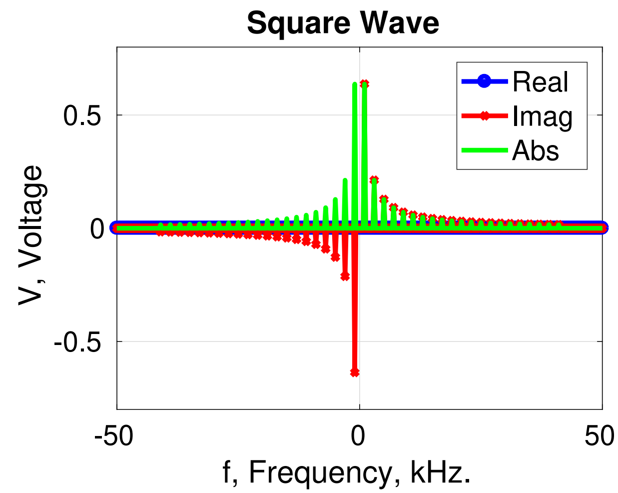


$\mathbf{V}_{out}(\omega) = H(\omega)\,\mathbf{V}_{in}(\omega)$ and No Coupling

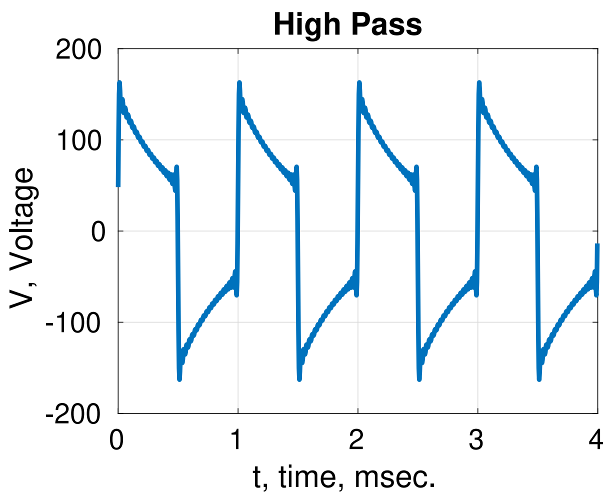
High–Pass Filter/Amplifier



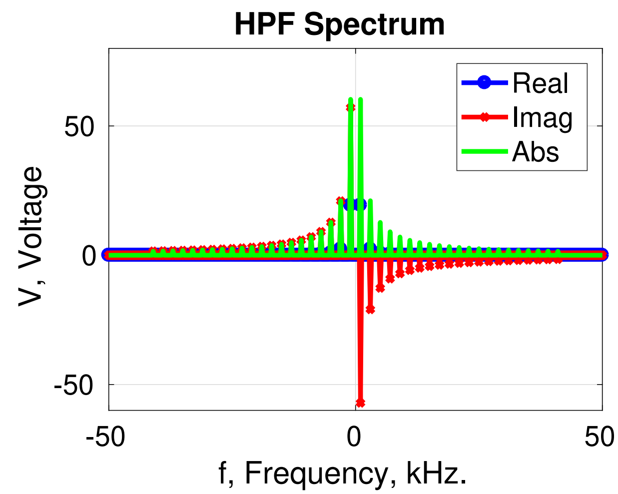
Input



Fourier Series

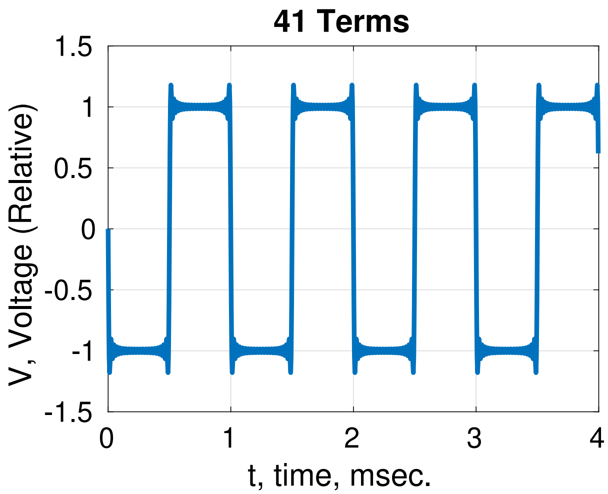


HPF Output

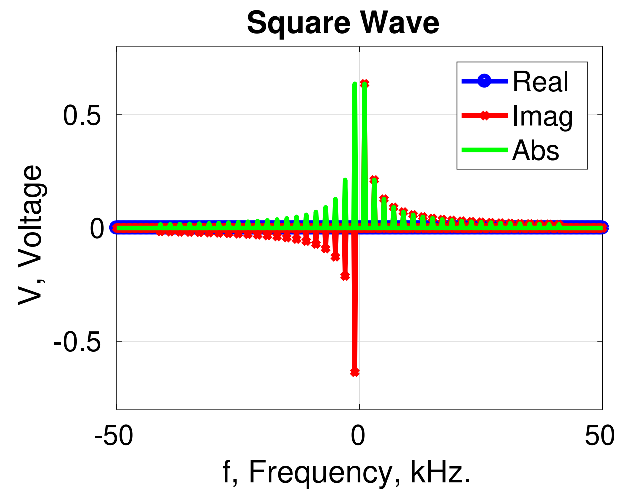


Fourier Series

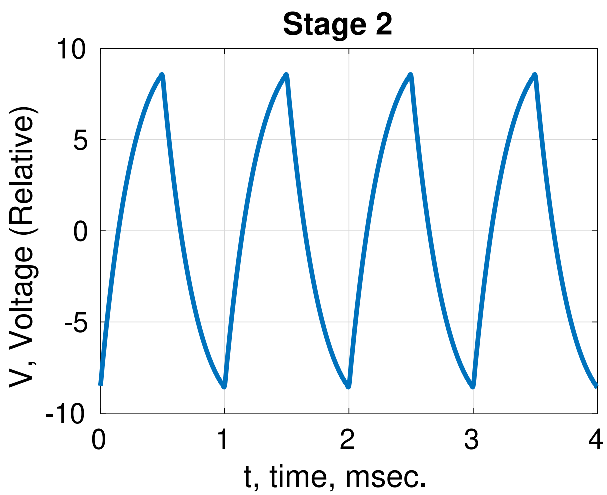
High–Pass and Low Pass



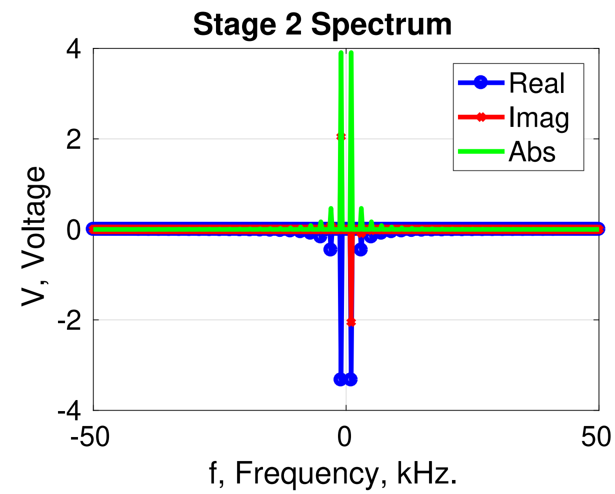
Input



Fourier Series

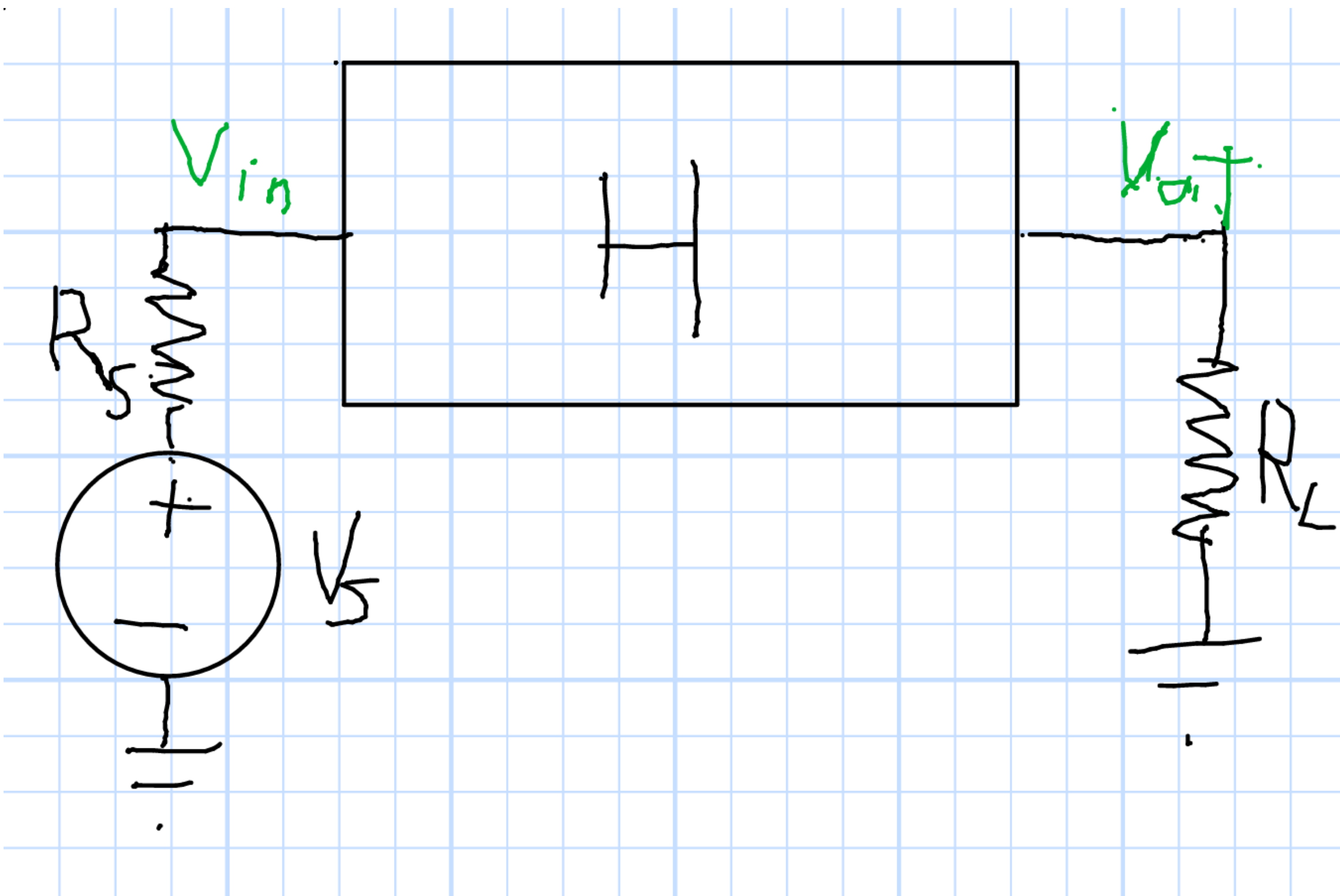


2–Filter Output

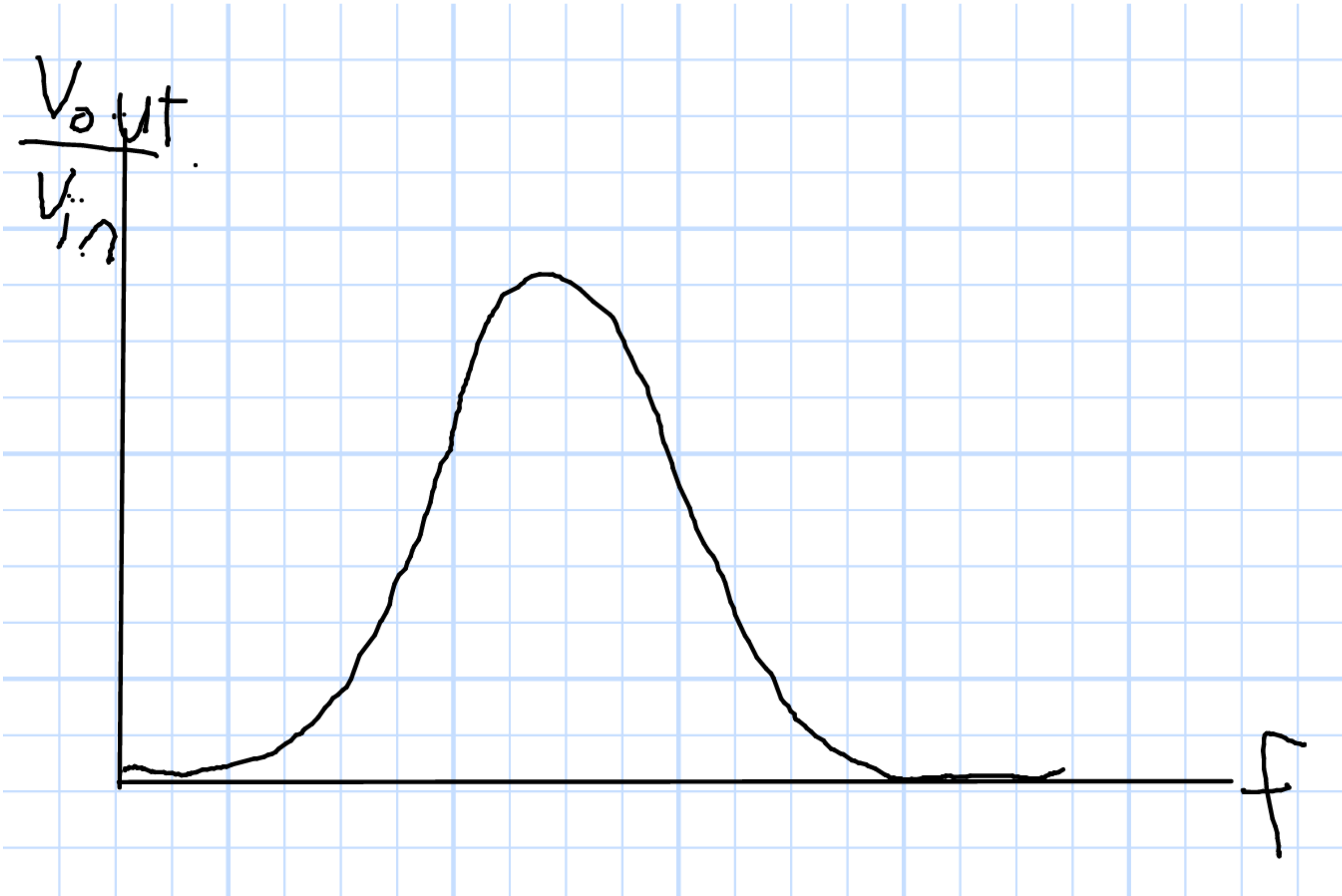


Fourier Series

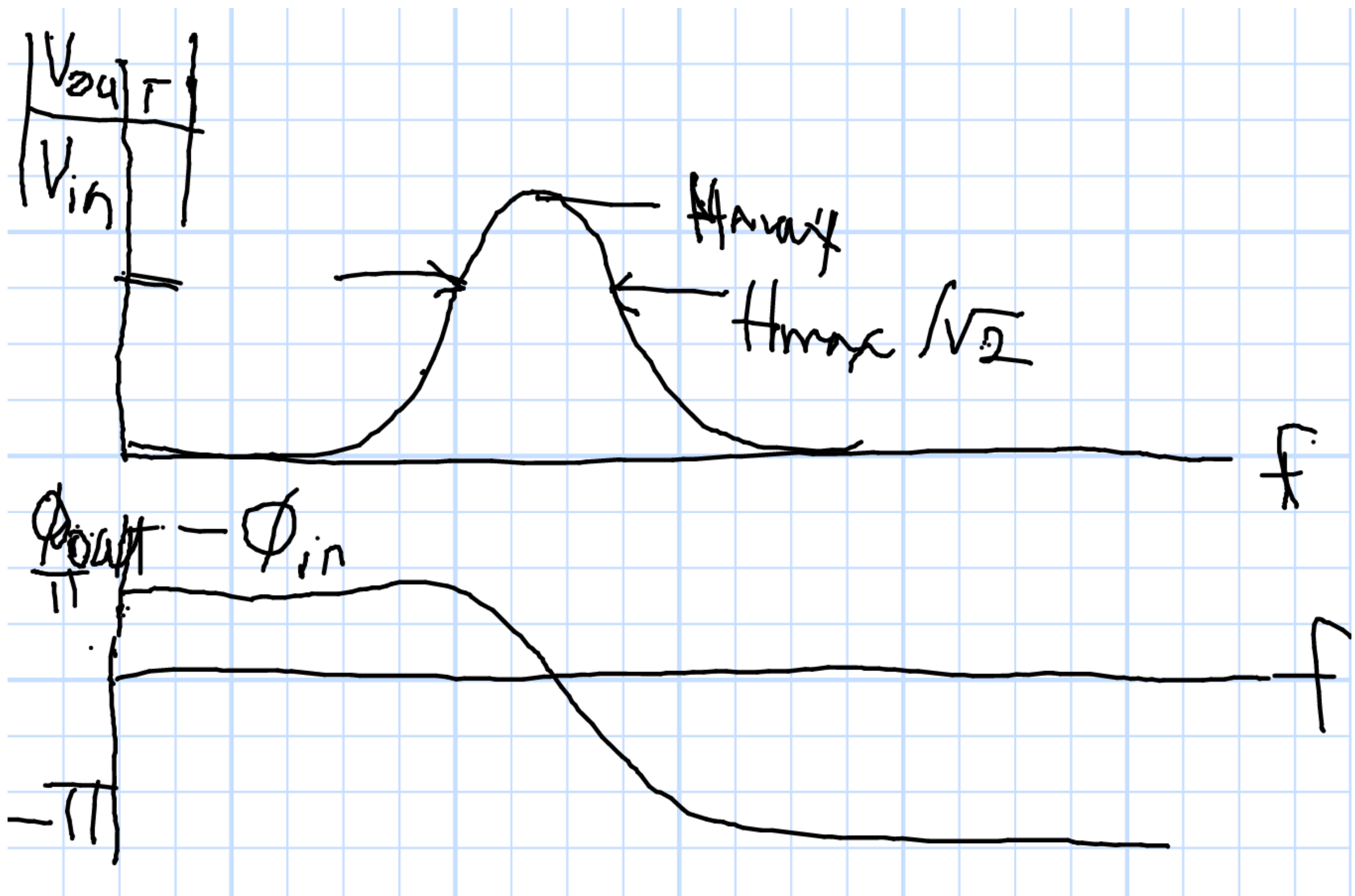
Transfer Function Concept



Transfer Function Concept

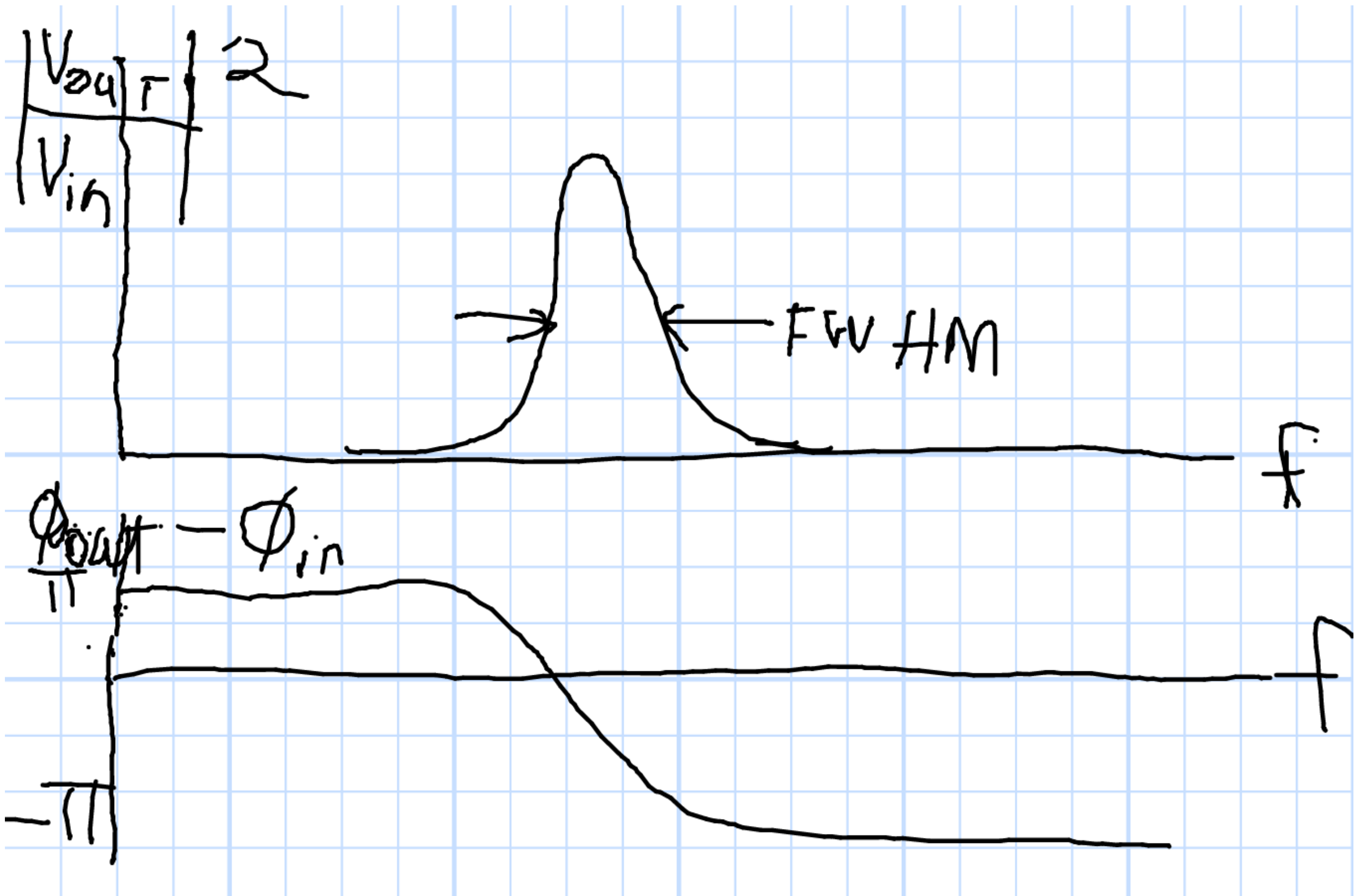


But It's Complex



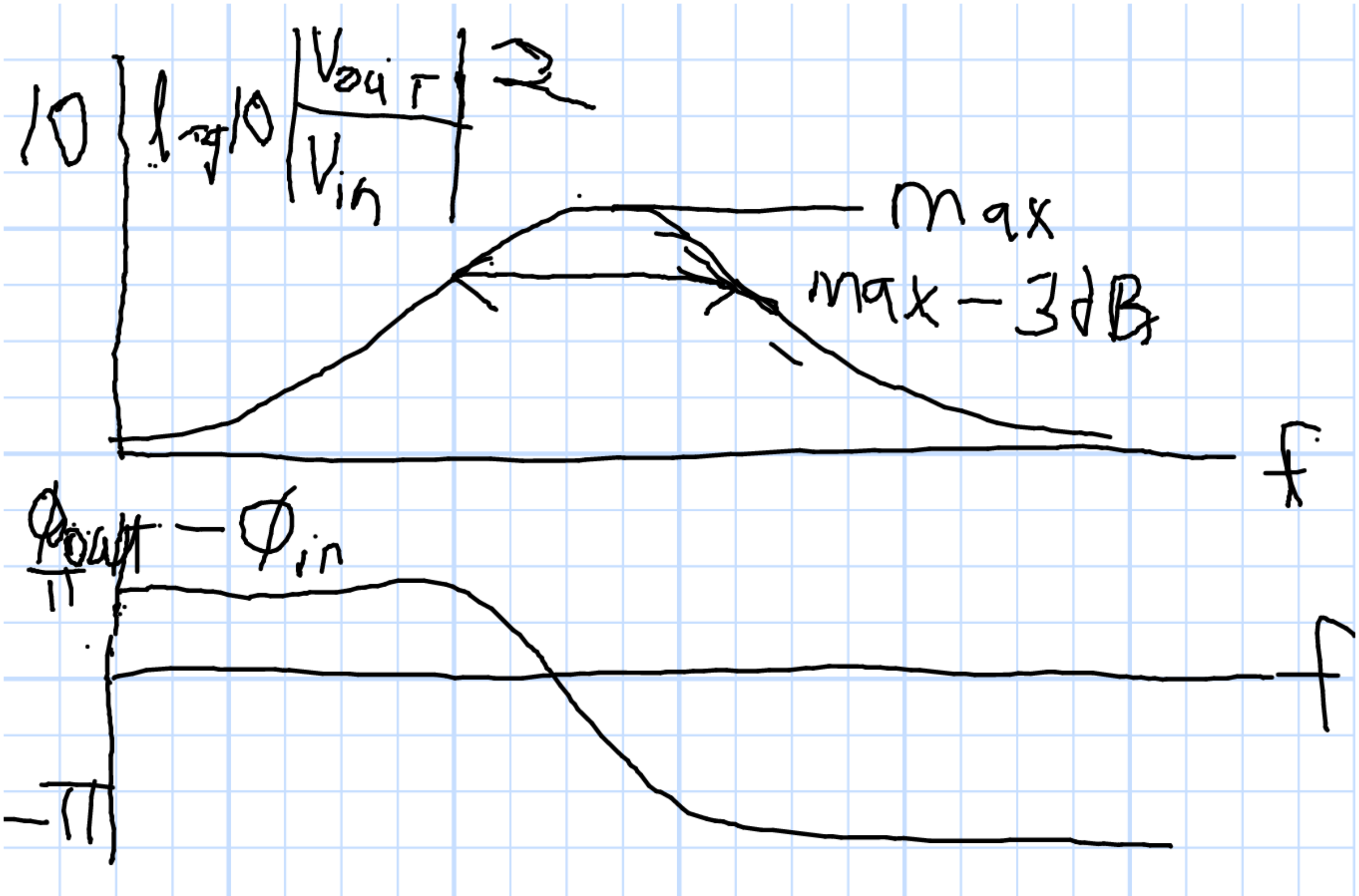
Plot Amplitude and Phase

And It's Nice to Talk About Power



Square the Amplitude.

And Sometimes We Want a Log Scale



Use Decibels

DeciBels, dB

- Always Refers to a Ratio

$$\frac{|V_{out}|}{|V_{in}|} \qquad \text{Or} \qquad \frac{P_{out}}{P_{in}} =? \left|\frac{V_{out}}{V_{in}}\right|^2$$

- Log Base 10

$$\log_{10}\frac{P_{out}}{P_{in}}$$

- But That's too Coarse; Multiply by 10

$$R_{dB} = 10\log_{10}\frac{P_{out}}{P_{in}} = 20\log_{10}\frac{|V_{out}|}{|V_{in}|} \qquad \frac{|V_{out}|}{|V_{in}|} = 10^{R_{dB}/20}$$

- lower case d, Capital B (Named for Alexander Graham Bell)

Some Ratios in dB

dB	Power Ratio	Voltage Ratio
-40	0.0001	0.01
-30	0.001	$\sqrt{0.001} \approx 0.032$
-20	0.01	0.1
-10	0.1	$\sqrt{0.1} \approx 0.32$
-6	0.25	0.5
-3	0.5	$\sqrt{0.5} \approx 0.71$
0	1	1
3	2	$\sqrt{2} \approx 1.4$
6	4	2
10	10	$\sqrt{10} \approx 3.2$
20	100	10
30	1000	$\sqrt{1000} \approx 32$
40	10000	100

The cool thing is that they add.

Some Special References

- Electronics and Optics dBm

$$10\log_{10}\frac{P}{1\text{mW}}$$

- Sometimes Used for Voltage, Assuming 50 Ohms

$$10\log_{10}\frac{|V|^2}{50\Omega\times 1\text{mW}}$$

- Example: $1\text{V}_{RMS}\rightarrow 13\text{dBm}$

- Radar Meteorology dBZ (One 1mm raindrop per m^3)

- Acoustics dBu and more

Comment on Digitization

- Eight–Bit Digitizer
  - Smallest Step = 1, Max Error = 0.5
  - Largest Value $2^8 - 1 = 255$
  - Ratio

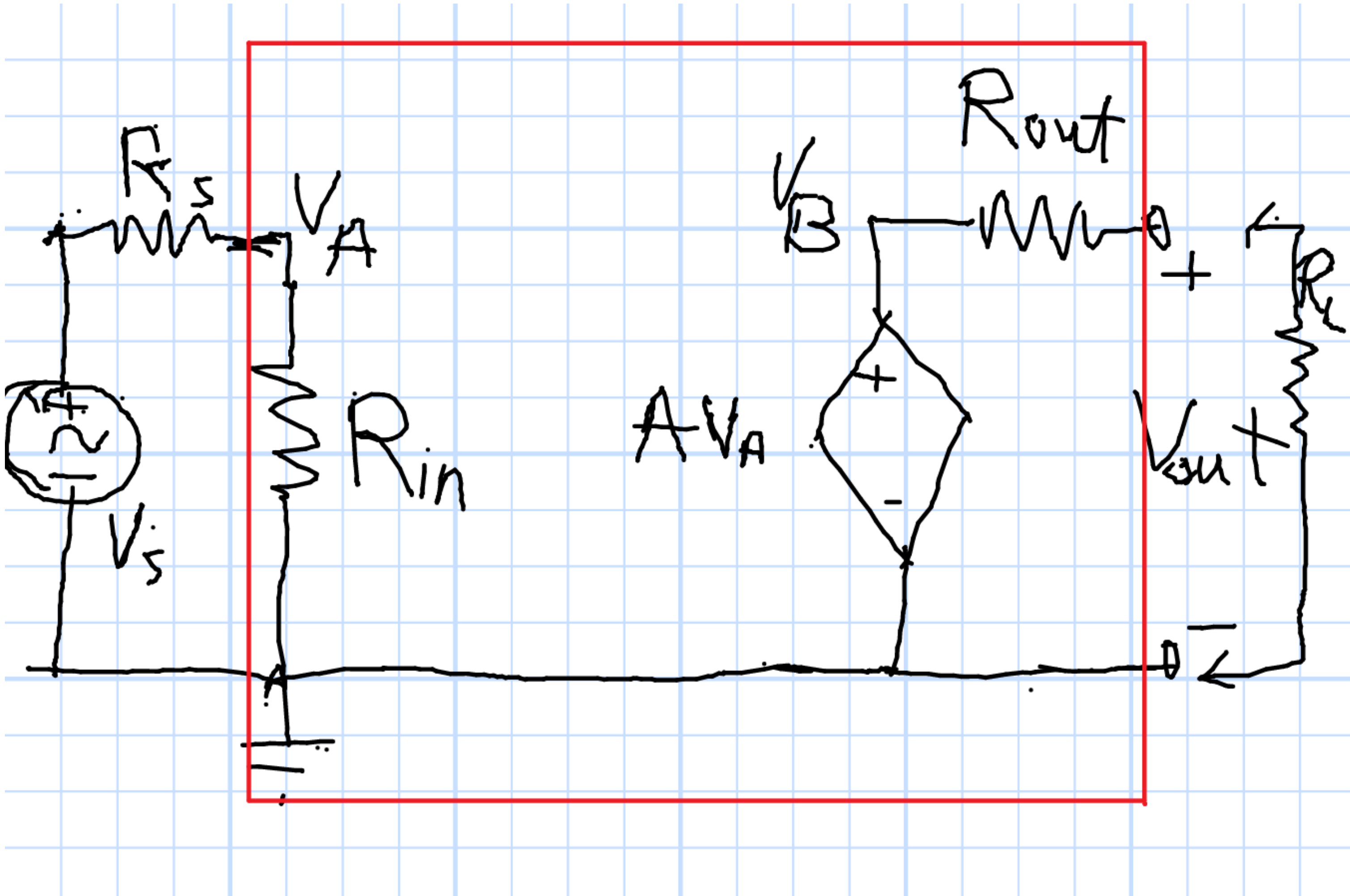
$$20\log_{10}\frac{255}{1} = 48\text{dB}$$

- In General, for n–bit Digitizer

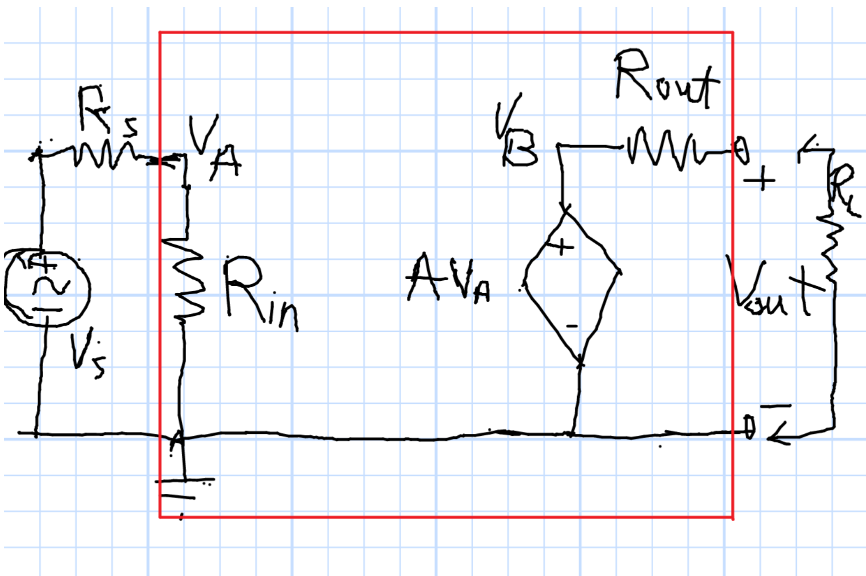
$$20\log_{10}\frac{2^n - 1}{1} \approx 20\log_{10} 2^n = 20n\log_{10} 2$$

Dynamic Range in dB $\approx 6n$

Generic Amplifier



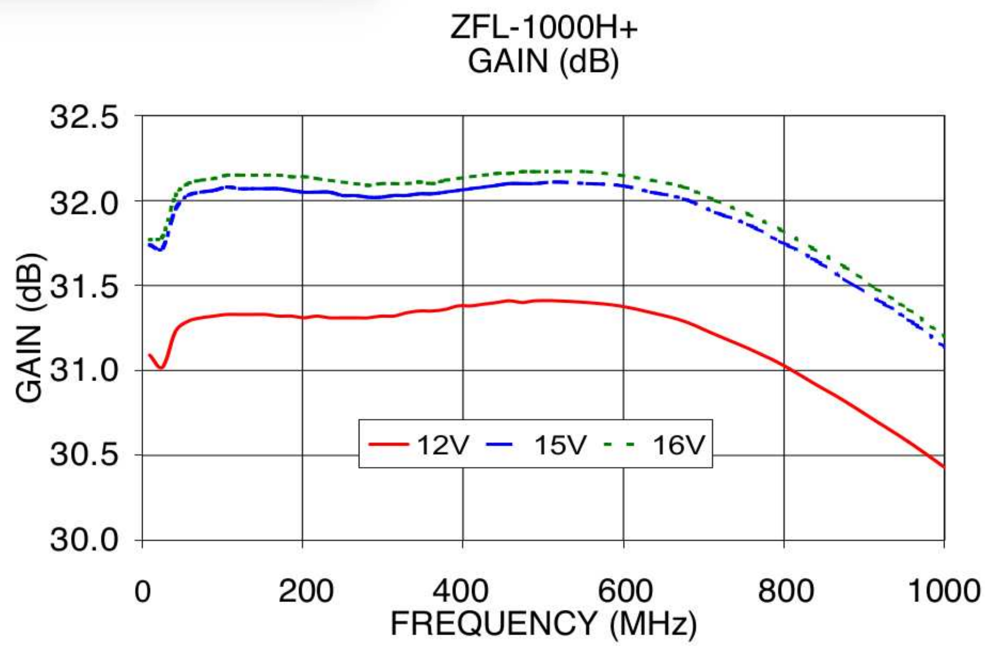
Amplifier Specifications



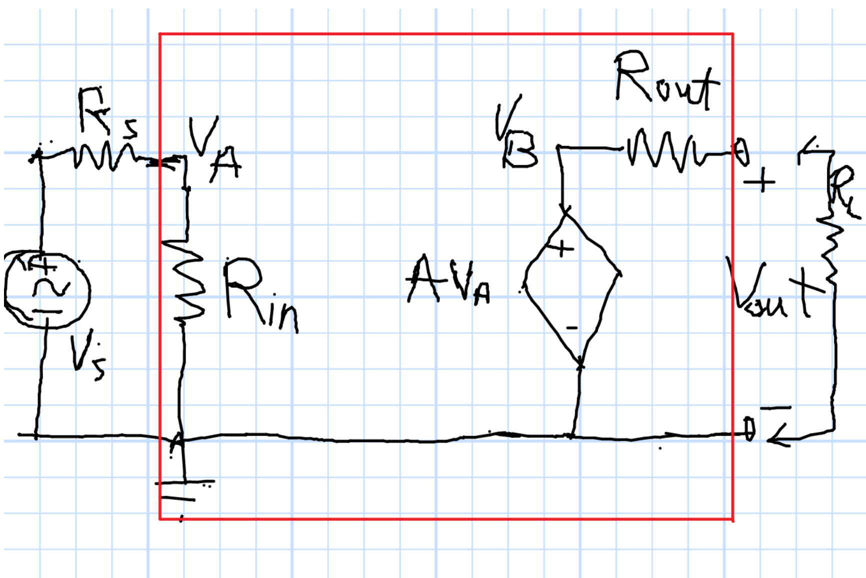
- Gain & Bandwidth Examples
  - 30dB $\pm$ 1dB from 30Hz to 16kHz
  - 60dB at 70MHz and $<$ 30dB at 80MHz
- Impedances
  - High R_{in}, Low R_{out} for Maximum Voltage Gain
  - Matched for Maximum Power Gain
  - Often 50 Ohms
- Maximum Outputs
- Noise at Input
- Other

Amplifier Example

30dB Gain, 50Ω in and out, SMA Connectors
+15V Supply, Max Output 20dBm
10 to 1000MHz, Power +15V, from Minicircuits.



Amplifier Measurements



- Example: 50Ω Amplifier
- Input Measured Correctly

$$\mathbf{V}_A = \frac{\mathbf{V}_S}{2}$$

- Source without Load

$$\mathbf{V}_A = V_S$$

- Output Measured Correctly

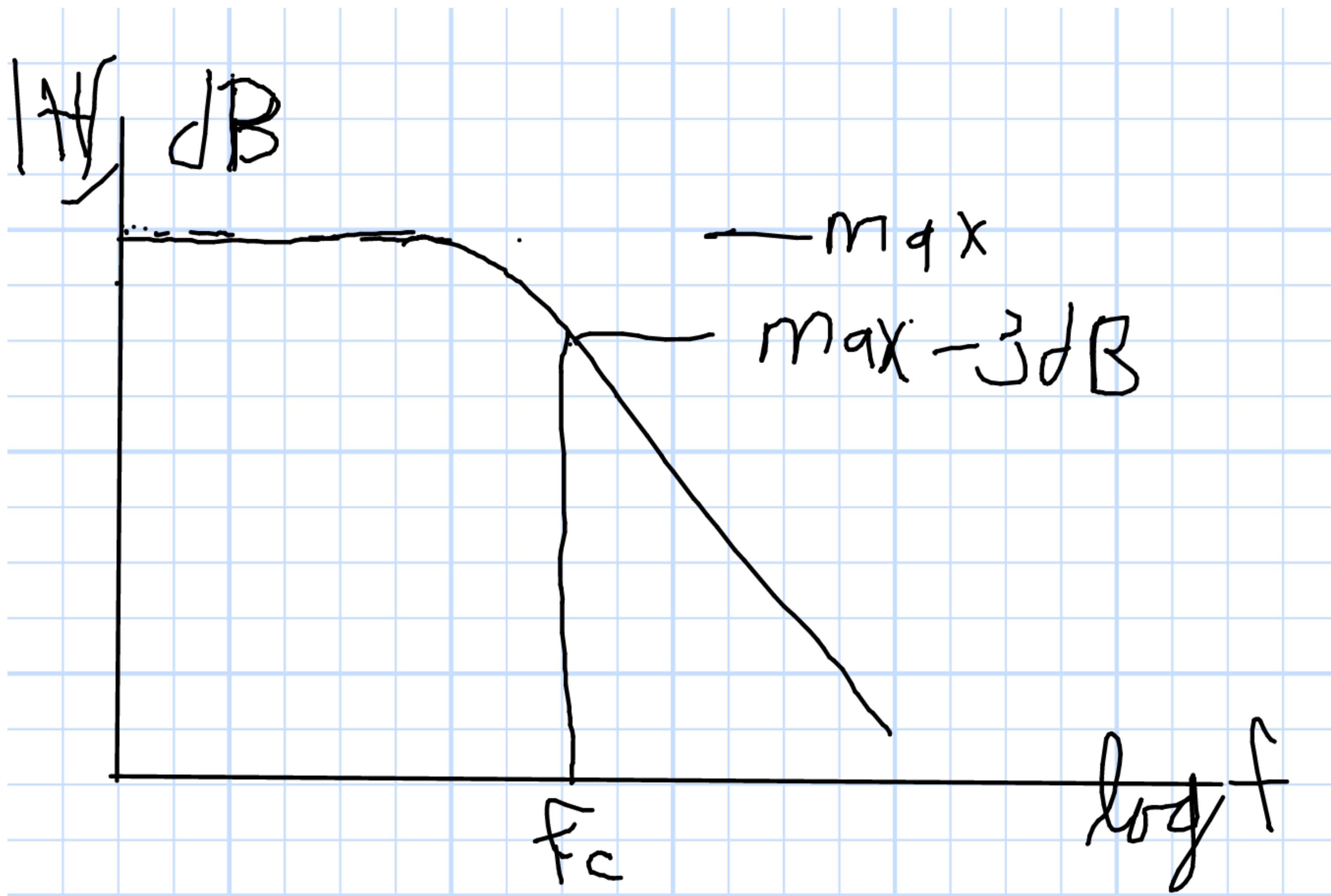
$$\mathbf{V}_{OUT} = \frac{\mathbf{V}_B}{2}$$

- Output without Load

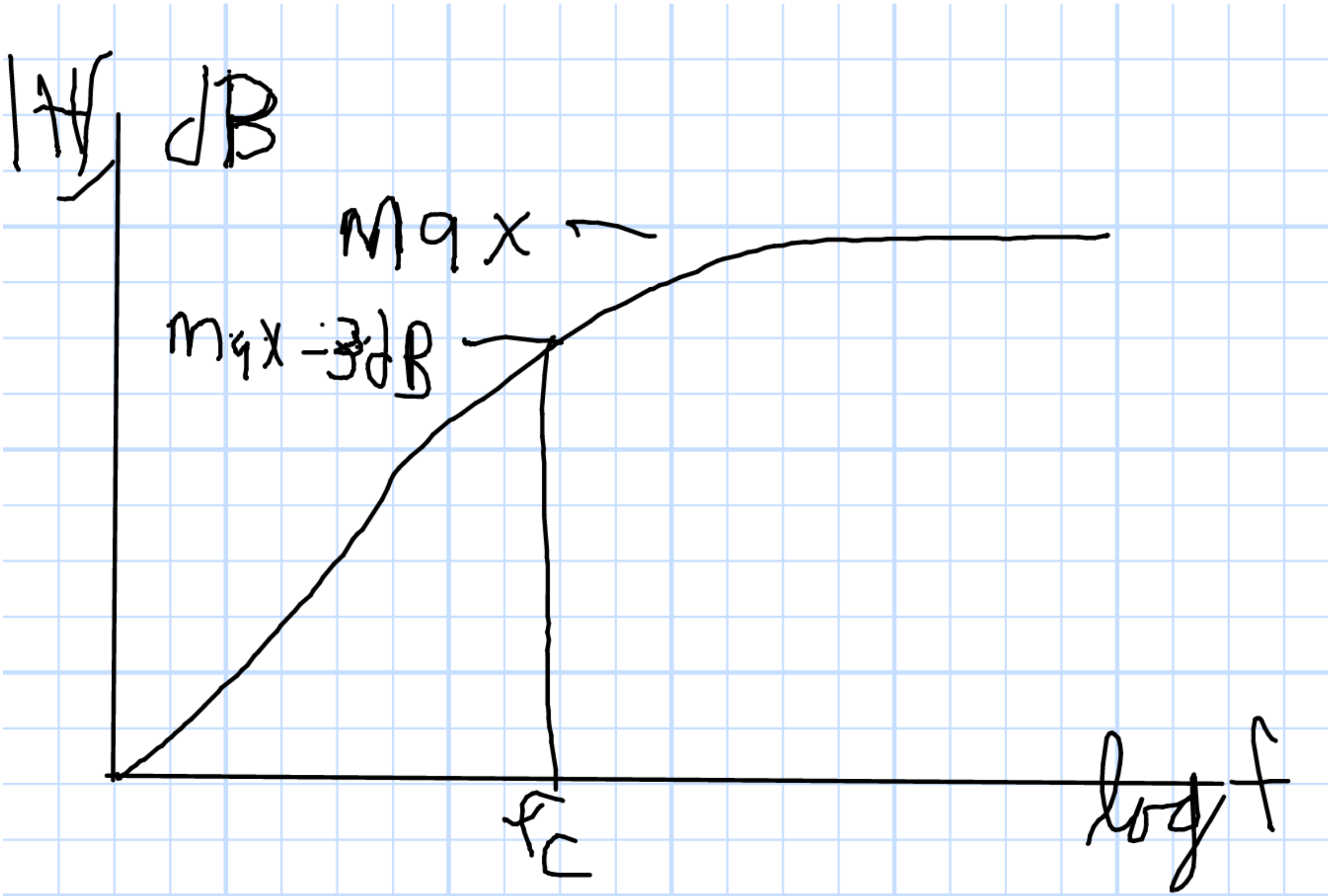
$$\mathbf{V}_{OUT} = \mathbf{V}_B$$

- Watch P–P *vs.* RMS

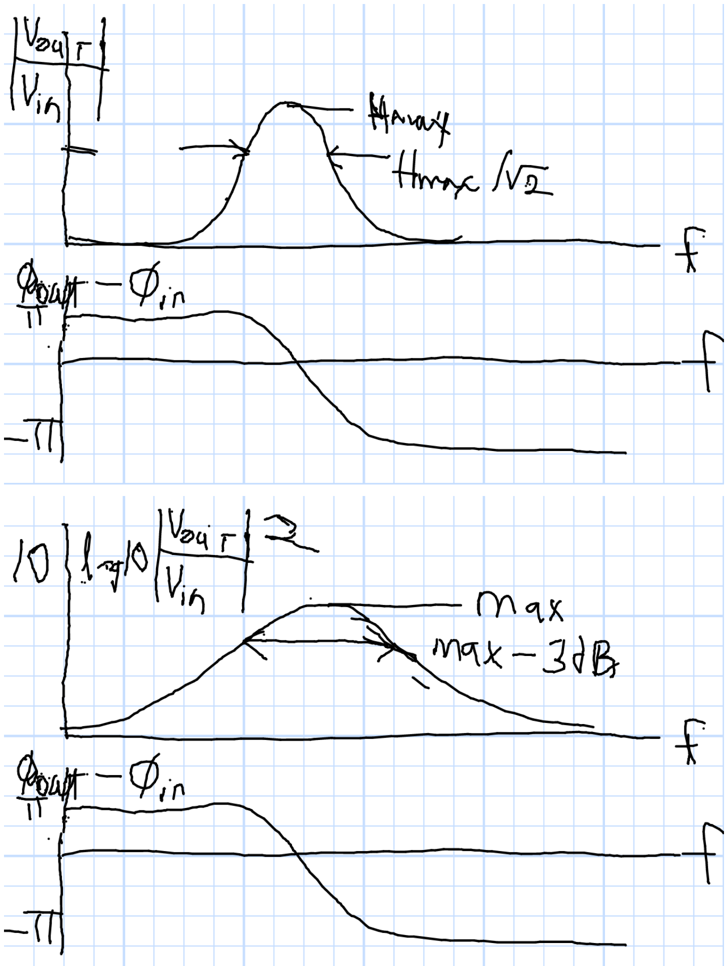
Low–Pass Filter

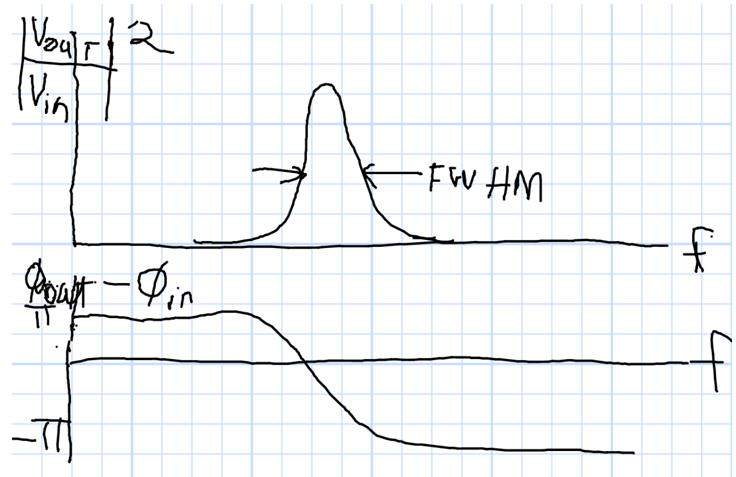


High–Pass Filter



Half–Power Point





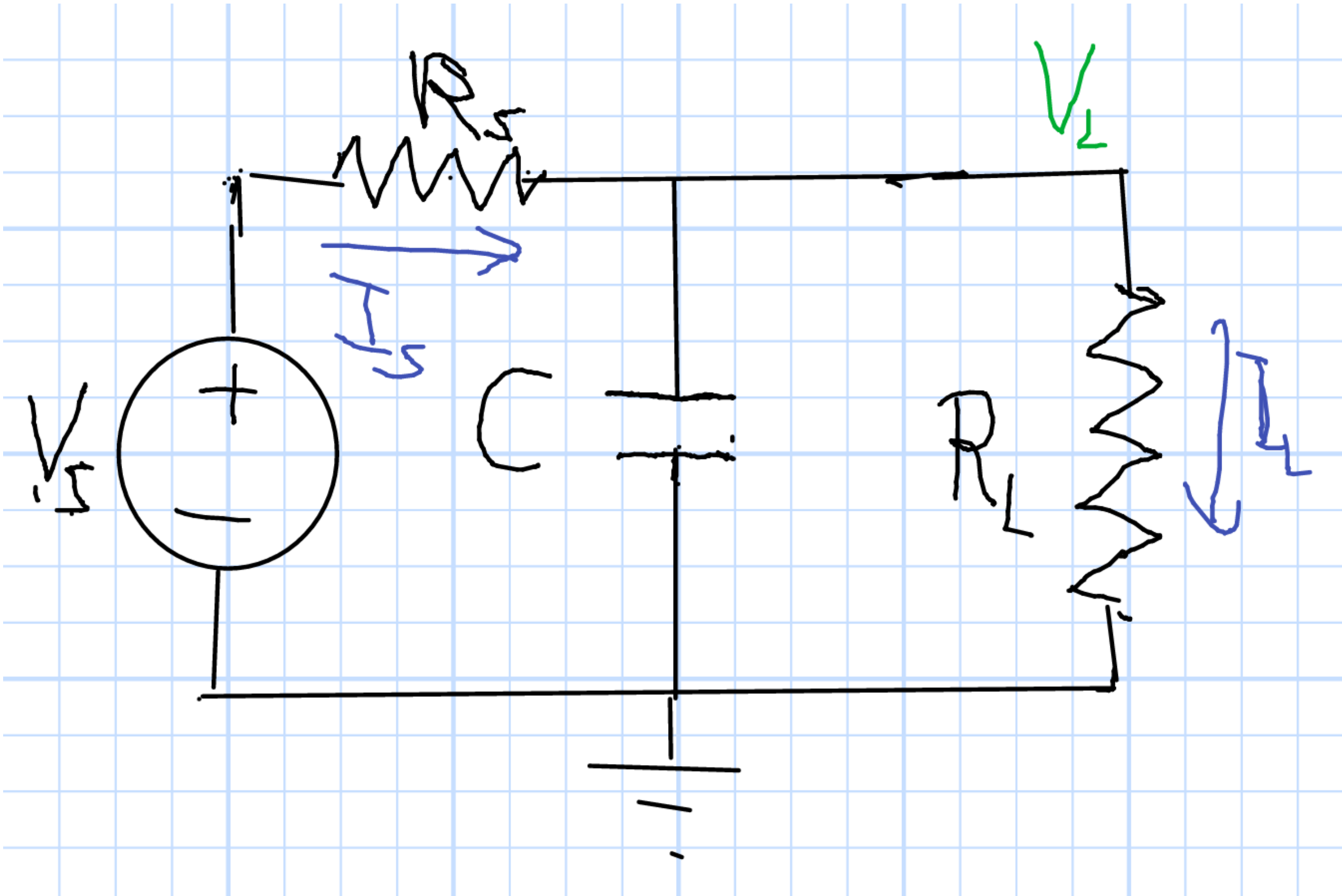
- Power Gain is Voltage Gain Squared

$$A_P = |A_V|^2 = \left|\frac{V_{out}}{V_{in}}\right|^2$$

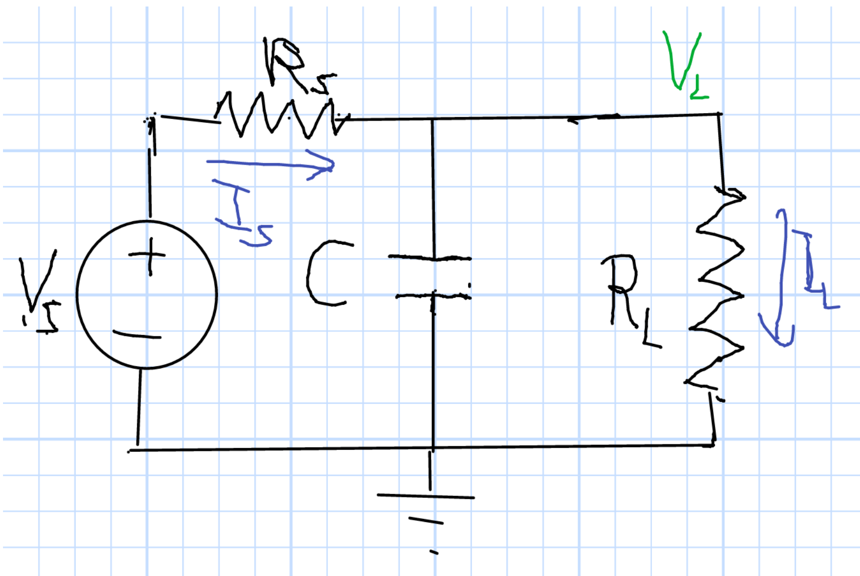
- in deciBels

$$10 \log_{10} A_P = 20 \log_{10} |A_V|$$

Example Circuit



Solution



$R_S = 1\text{k}\Omega$ $R_L = 500\Omega$

$C = 68\text{nF}$

- Voltage Divider

$$\frac{V_L}{V_S} = H = \frac{R_L \parallel \frac{1}{j\omega C}}{R_S + \left(R_L \parallel \frac{1}{j\omega C}\right)}$$

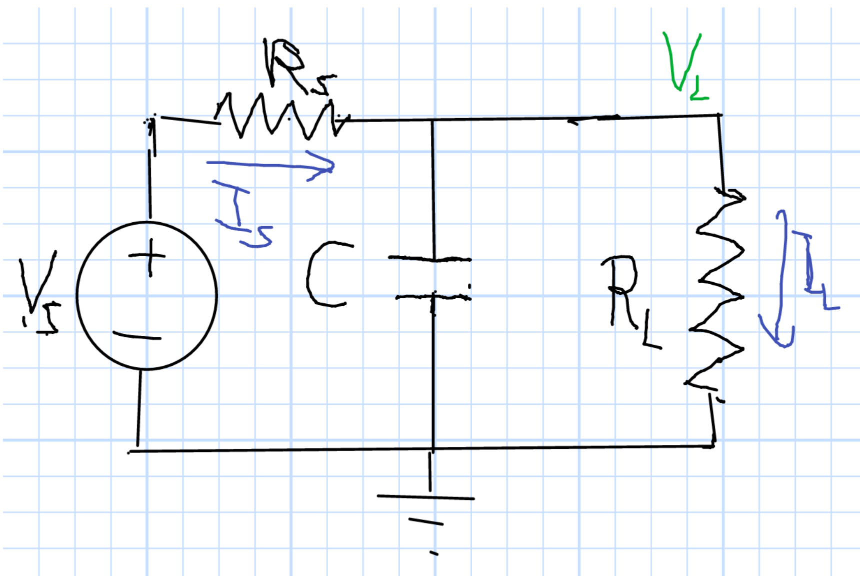
- Simplify

$$H = \frac{1}{\frac{R_S}{R_L \| \frac{1}{j\omega C}} + 1}$$

$$= \frac{1}{\frac{R_S\left(R_L + \frac{1}{j\omega C}\right)}{\frac{R_L}{j\omega C}} + 1}$$

$$= \frac{R_L}{R_S R_L j\omega C + R_S + R_L}$$

Transfer Function Values



$R_S = 1\text{k}\Omega \qquad R_L = 500\Omega$

$C = 68\text{nF}$

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$$H = \frac{R_L}{R_S R_L j\omega C + R_S + R_L}$$

- Simplify a Bit More

$$H = \frac{1/R_S}{j\omega C + \frac{1}{R_S \| R_L}}$$

- Max at $f = 0$ (Cap Open)

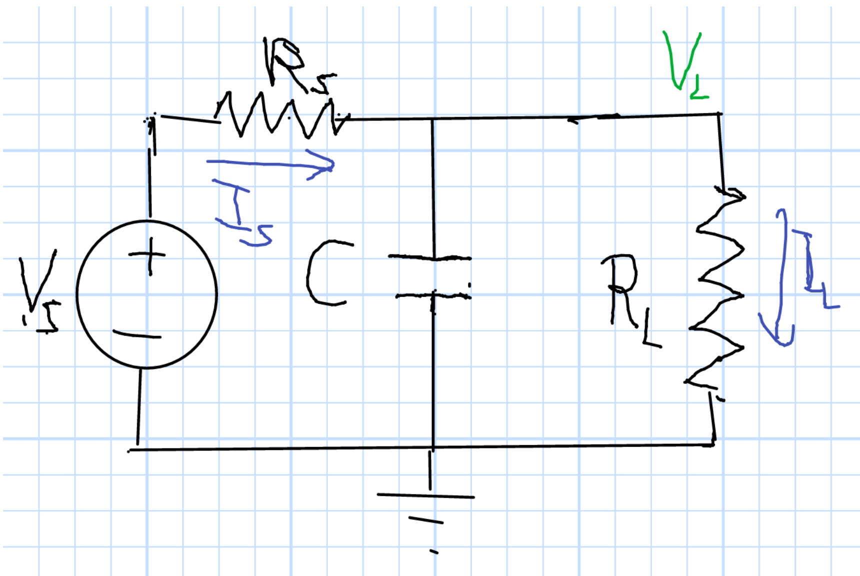
$$\frac{R_S \parallel R_L}{R_S} = \frac{R_L}{R_S + R_L} = \frac{1}{3}$$

- Cutoff

$$\frac{1}{\omega C} = R_S \parallel R_L$$

$$f_C = \frac{1}{2\pi \left(R_S \parallel R_L\right) C} = \frac{1}{2\pi\tau}$$

Transfer Function



$$R_S = 1\text{k}\Omega \qquad R_L = 500\Omega$$

$$C = 68\text{nF}$$

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$$H_{max} = 1/3$$

$$\frac{1}{\omega C} = R_S \parallel R_L$$

$$f_C = \frac{1}{2\pi \left(R_S \parallel R_L\right) C} \approx 7\text{kHz}$$

- Cutoff Significance

$$H = \frac{H_{max}}{\sqrt{2}} \times 1\angle{-45^\circ}$$

- Half–Power Point
- Half Phase Shift
- Equation

$$H = \frac{H_{max}}{1 + jf/f_c}$$

Approximations

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$$H = \frac{H_{max}}{1 + jf/f_c}$$

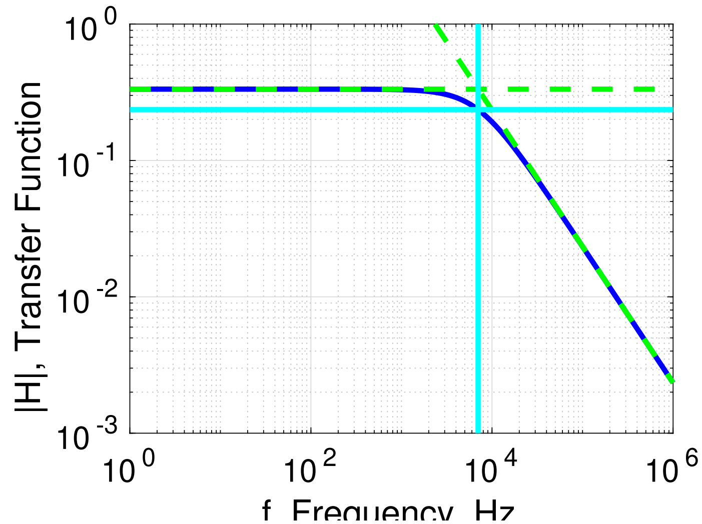
- Low Frequency

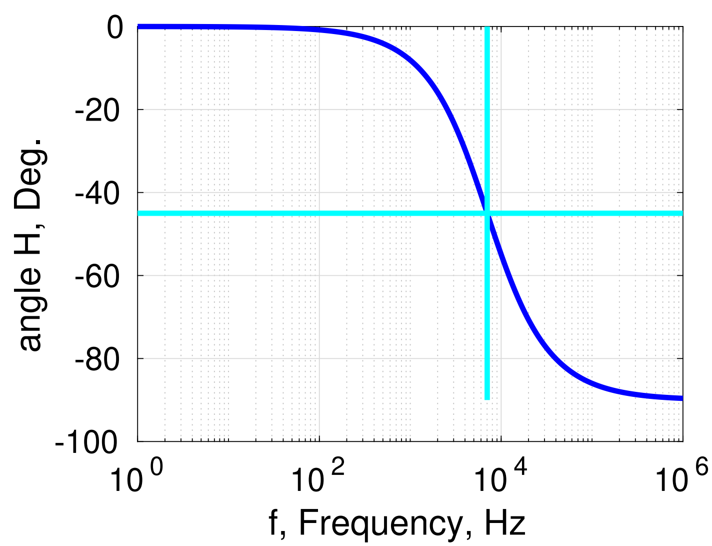
$$H \approx H_{max}$$

- High Frequency

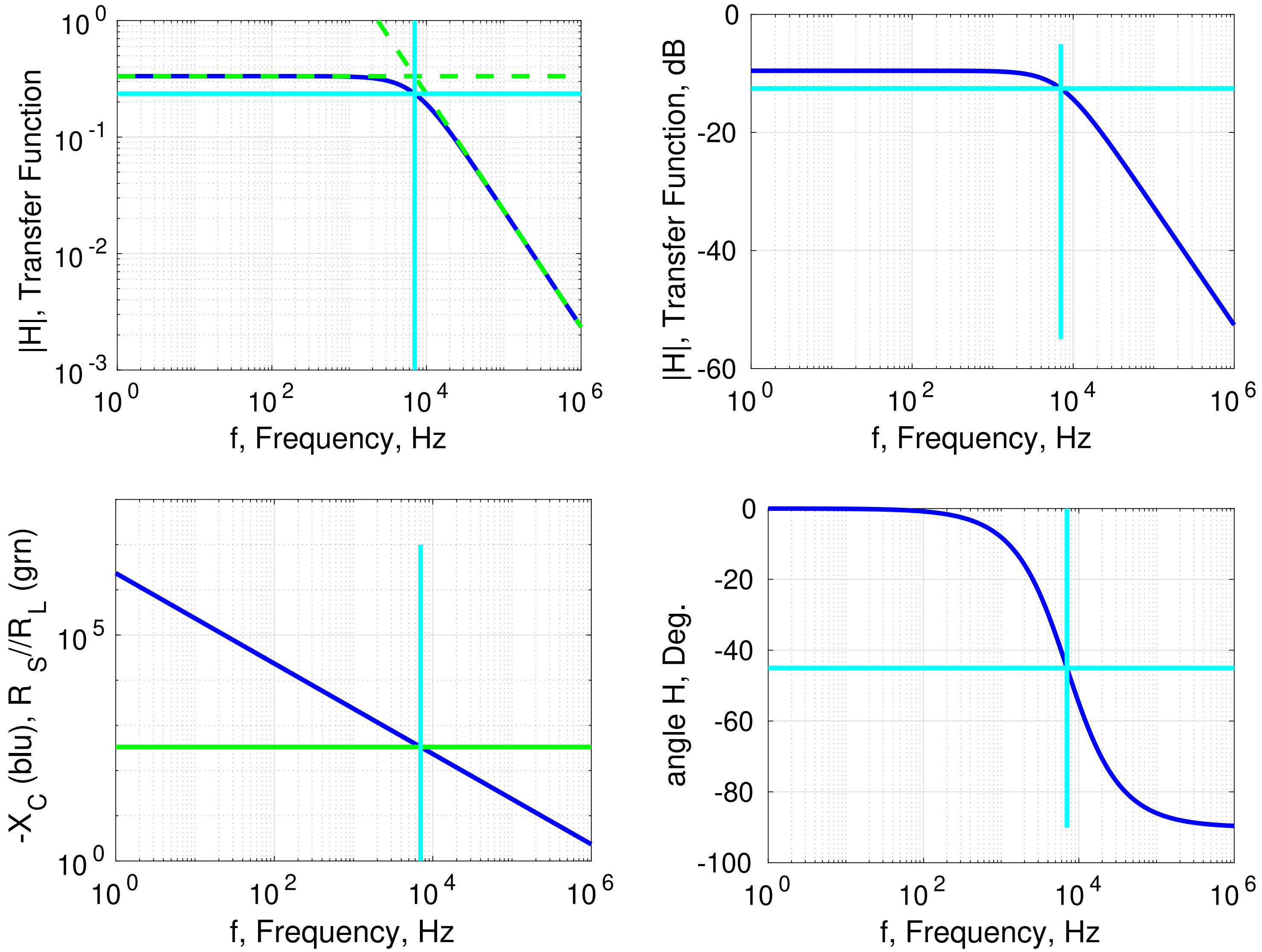
$$|H| = \frac{|H|_{max} f_c}{f}$$

- 20 dB per Decade
- 6 dB per Octave



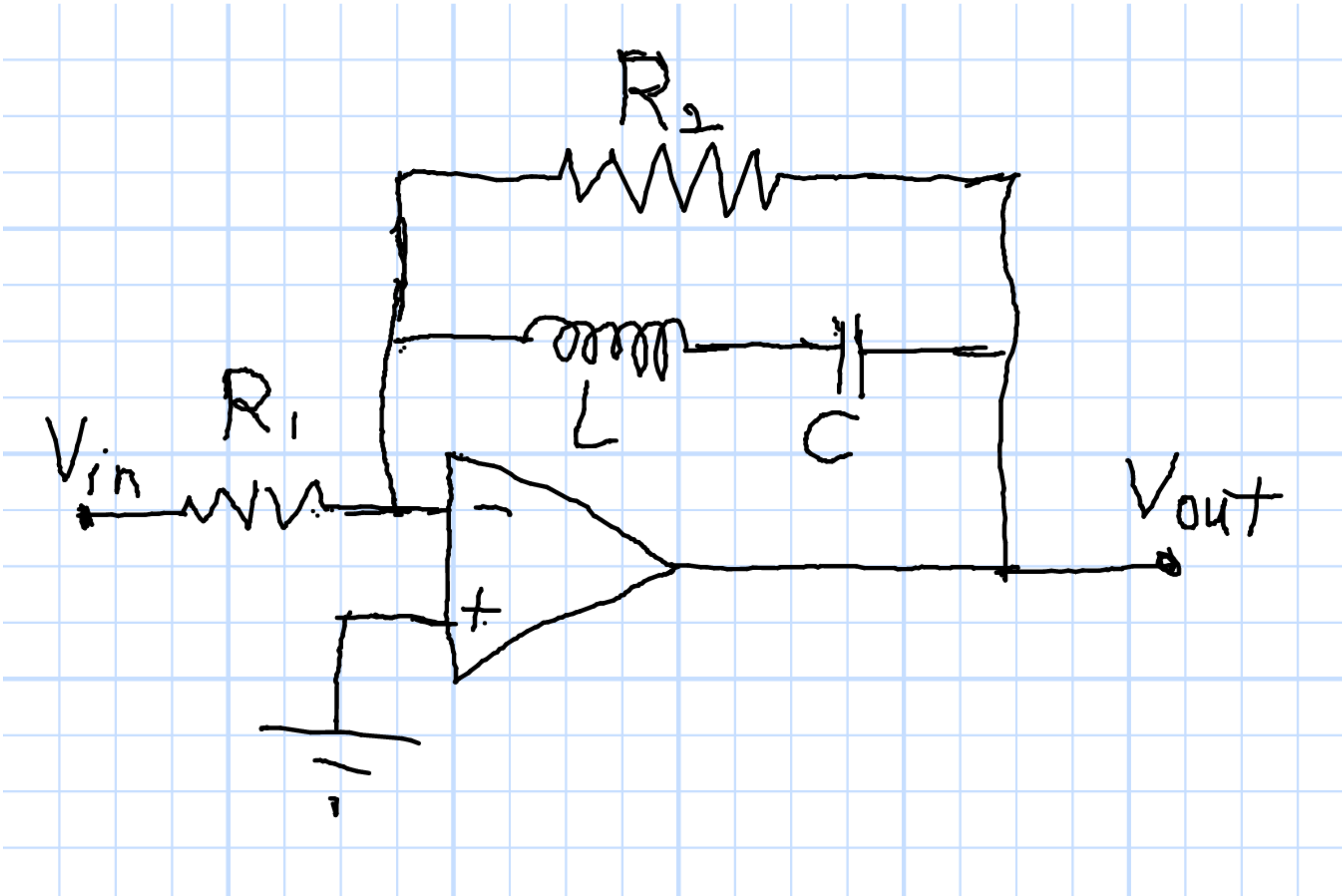


Plots



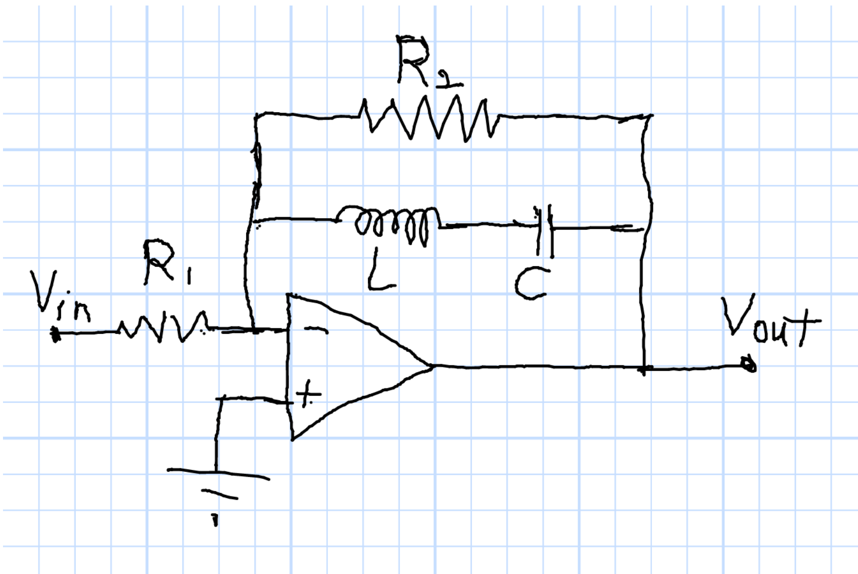
|H|, Transfer Function
10^0
10^-1
10^-2
10^-3
10^0
10^2
10^4
10^6
f, Frequency, Hz
|H|, Transfer Function, dB
0
-20
-40
-60
-X_C (blu), R_S//R_L (grn)
10^5
angle H, Deg.
-80
-100

Example Circuit: Active Notch Filter



$A_v = \frac{-Z_2}{Z_1}$; $f \approx 0$, $Z_C \approx \infty$ or $f \to \infty$, $Z_L \approx \infty$; $A_v = \frac{-R_2}{R_1}$

Notch Filter Solution



Design Frequency $f_0 = 70\text{MHz}$, $C = 2.3\text{nF}$, $L = 1/\omega_0^2/C$
$R_1 = 50\Omega$, $R_2 = 50\Omega$

Non–Ideal Components
$R_{CP} = 1\text{G}\Omega$, R_{LS} Various

- Ideal Solution

$$Z_1 = R_1$$

$$Z_2 = R_2 \parallel \left(j\omega L + \frac{1}{j\omega C} \right)$$

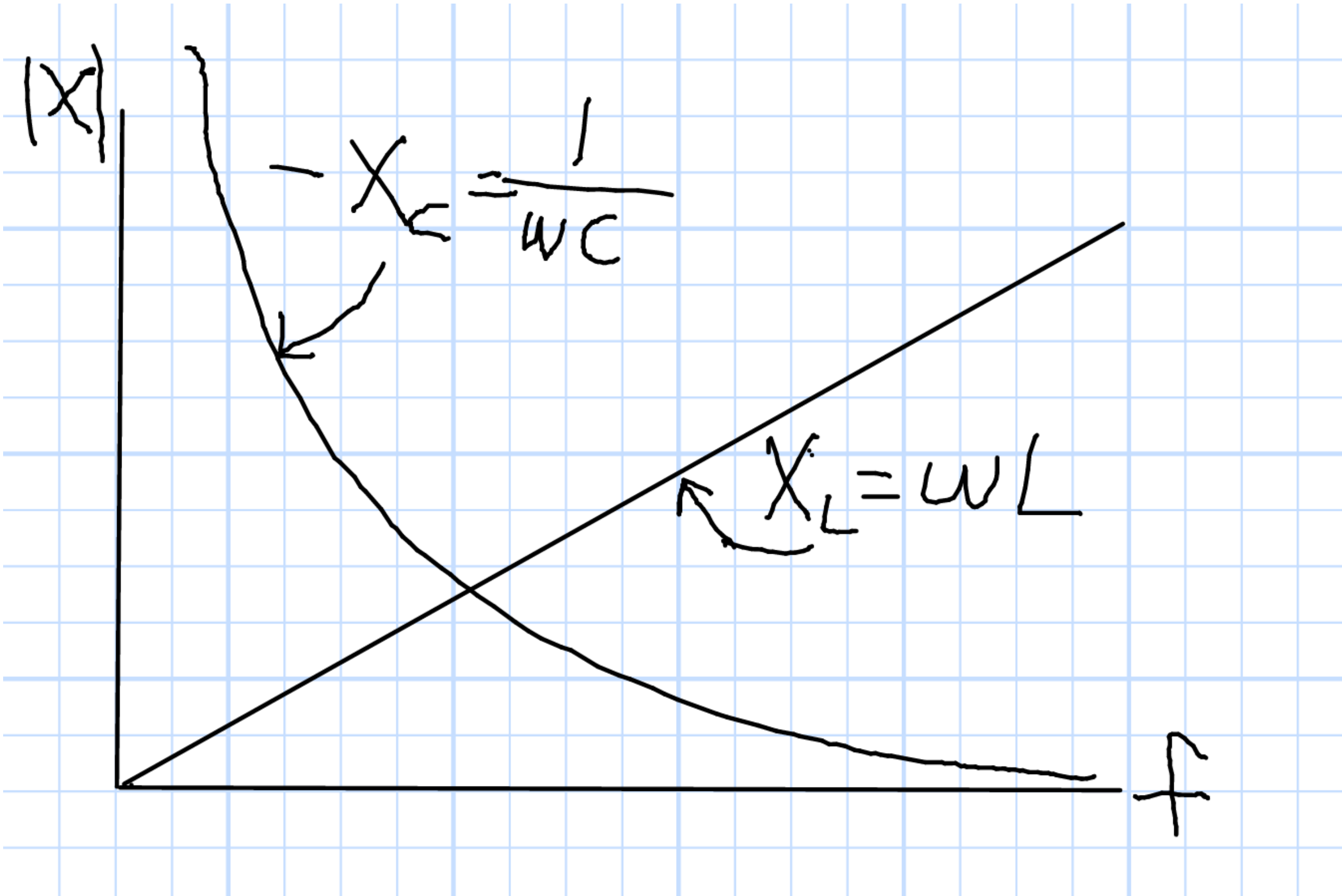
$$A_V = -\frac{Z_2}{Z_1}$$

$$Max\,|A_V| = \frac{R_2}{R_1}$$

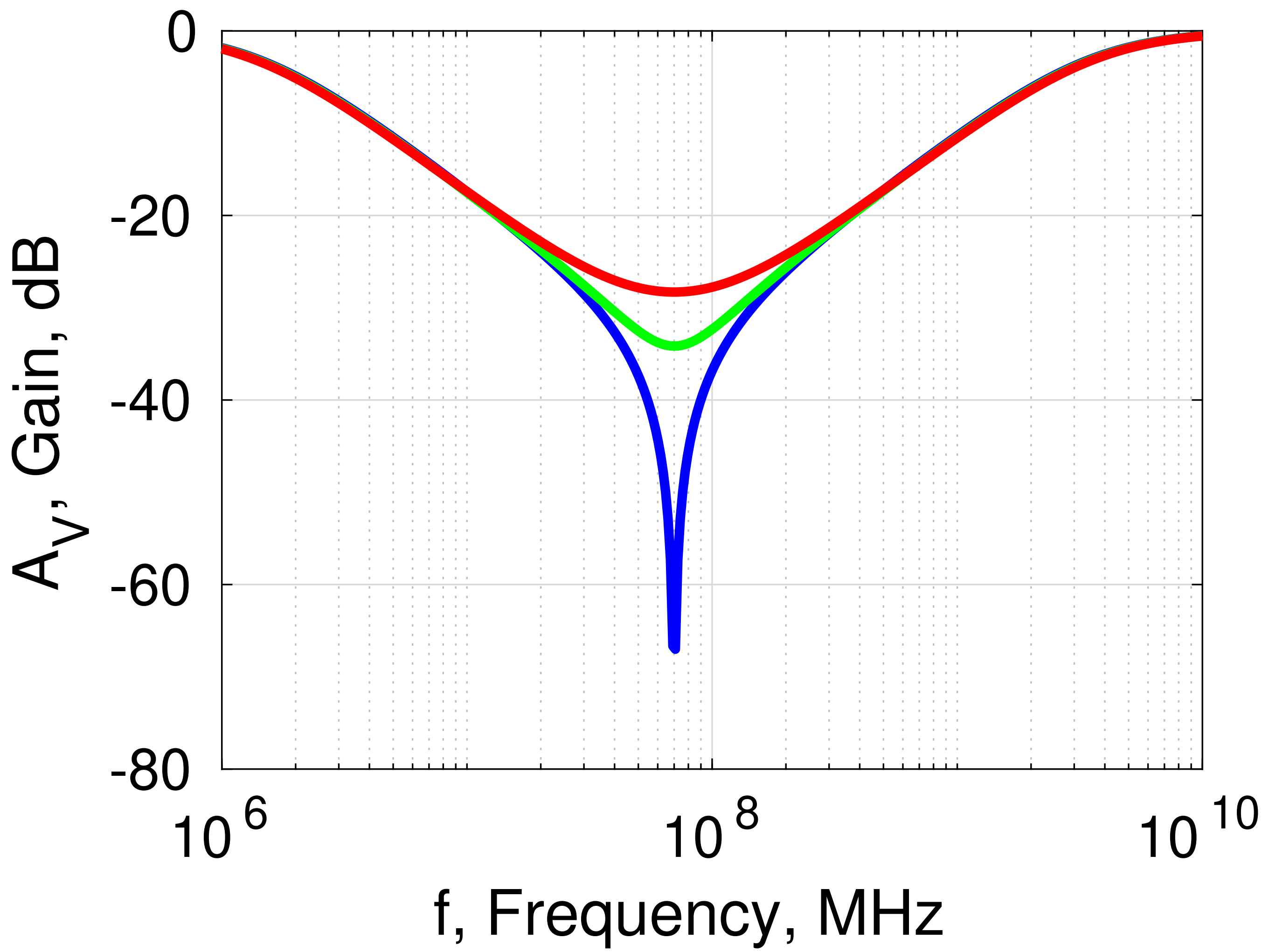
- Actual with Series R_{LS} and Parallel R_{CP}

$$Z_2 = R_2 \parallel \left[j\omega L + R_{LS} + \left(R_{CP} \parallel \frac{1}{j\omega C} \right) \right]$$

Reactance Tradeoff

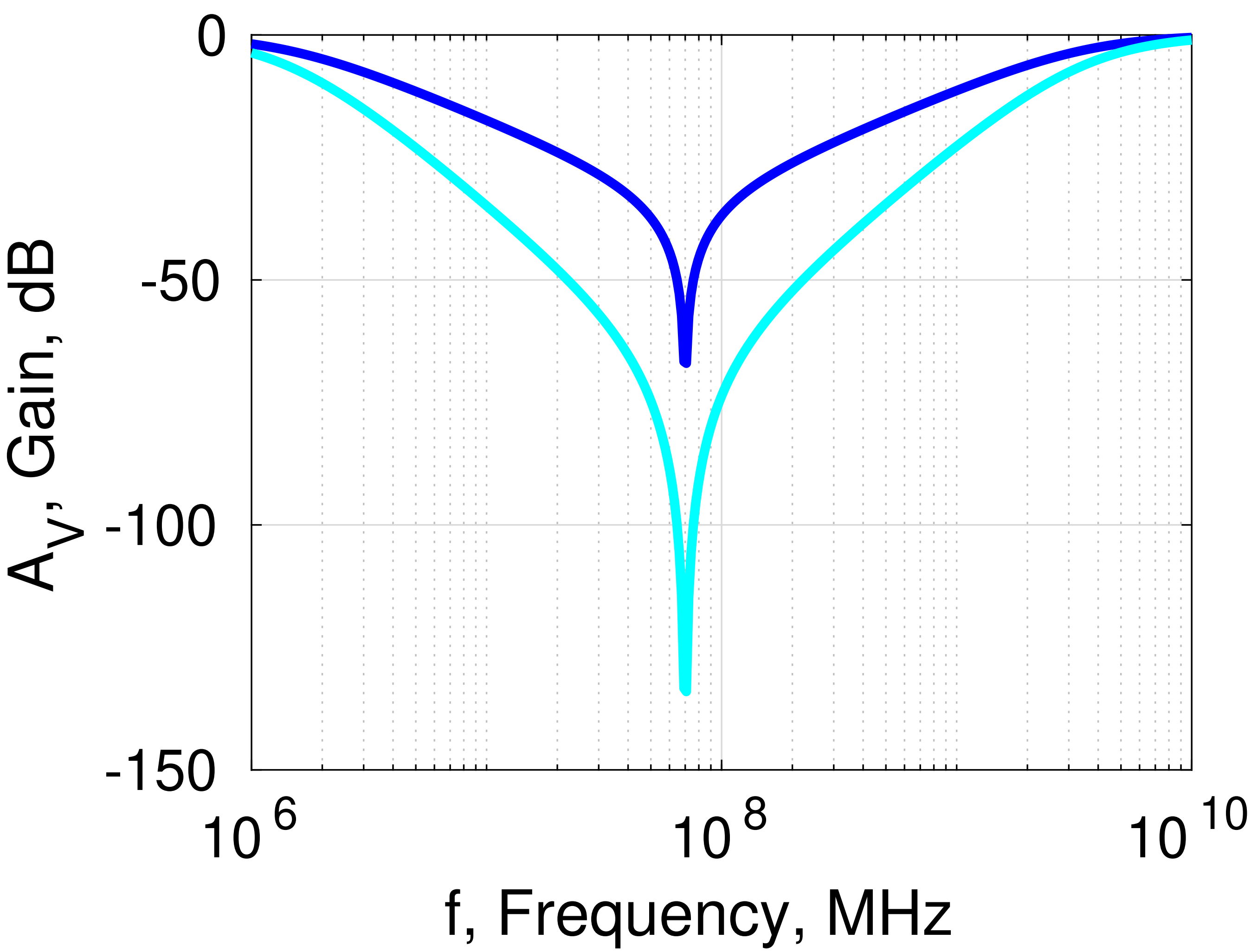


Notch Transfer Function



$R_{LS} = 0$ (Blue), 1Ω (Green), 2Ω (Red)

Cascade Transfer Function

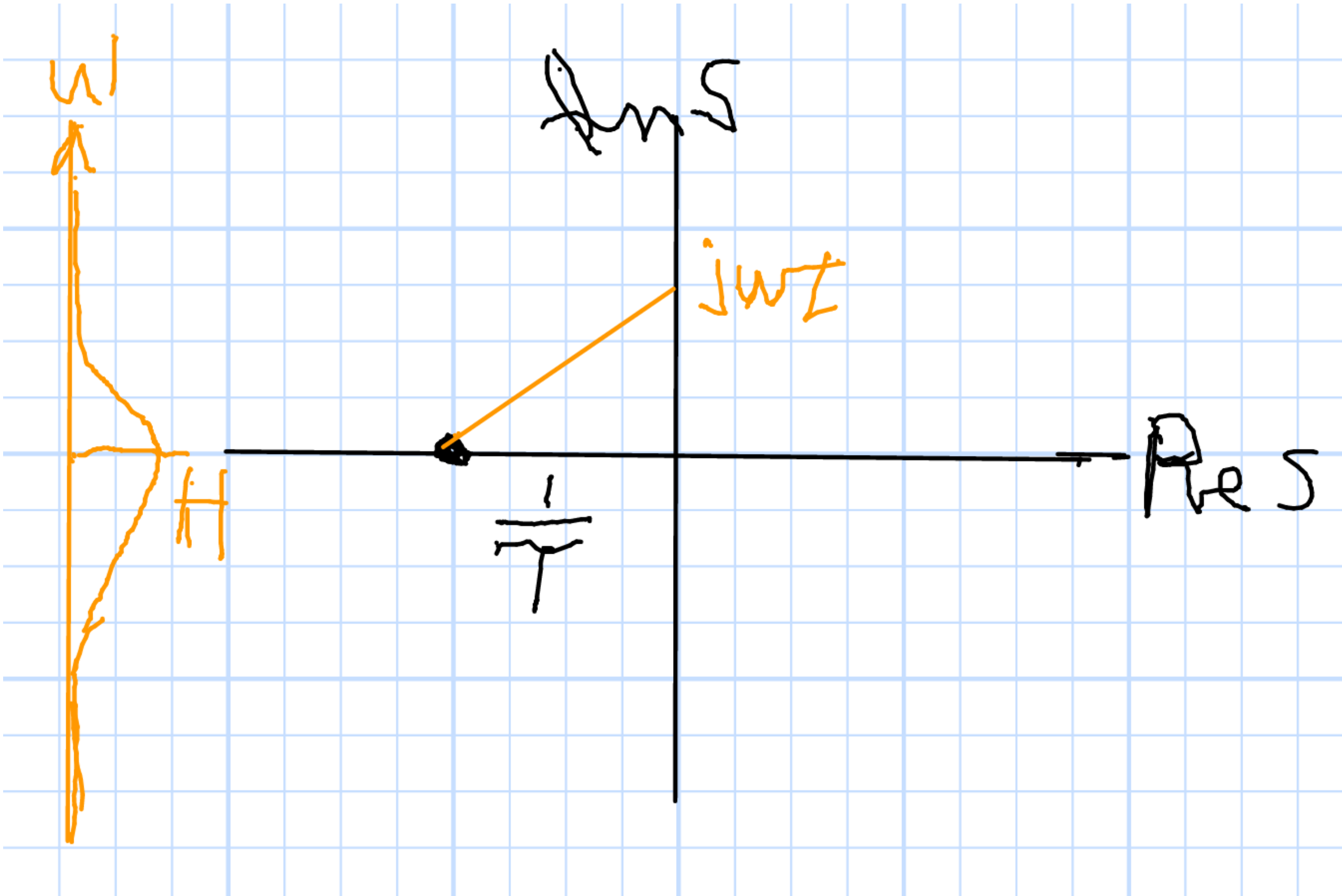


$R_{LS} = 0$ (Blue), 2 Cascaded Notches (Cyan)

Cascaded Circuits

- General Rule: $A = A_1 \times A_2 \ldots$ (Add dB)
- Gain $|A| > 1$ (0dB): Gain Increases with Cascaded Circuits
- Loss $|A| < 1$ (0dB): Gain Decreases with Cascaded Circuits
- Slopes Increase with Cascaded Circuits
- Bandwidths Decrease

First–Order Circuit



Second–Order Circuit