Electrical Engineering
Week 11

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Week 11 Agenda: Circuits and Sine/Cosine Waves

- Complex Arithmetic
- C, L, Reactance and Impedance
- Eli the Ice Man
- Series and Parallel
- Voltage and Current Dividers
- Power, RMS, Peak–to–Peak, Oscilloscopes and Meters
- Resistive and Reactive Power, Power Factor
- One More Op–Amp Circuit

Complex Notation

- Voltage

$$v_x(t) = V_x \cos(\omega t + \phi)$$

- From Euler

$$v_x(t) = \frac{V_x}{2} e^{j(\omega t+\phi)} + \frac{V_x}{2} e^{-j(\omega t+\phi)}$$

$$v_x(t) = \frac{V_x e^{j\phi}}{2} e^{j\omega t} + \frac{V_x e^{-j\phi}}{2} e^{-j\omega t}$$

$$v_x(t) = \frac{\mathbf{V}_x}{2} e^{j\omega t} + \frac{\mathbf{V}_x^*}{2} e^{-j\omega t}$$

- Phasor

$$\mathbf{V}_x = V_x \angle \phi = V_x e^{j\phi}$$

$$v_x(t) = \text{Re}\left[\mathbf{V}_x e^{j\omega t}\right]$$

- Example Cosine

$$v(t) = 5\text{V} \times \cos \omega t$$

$$\mathbf{V} = 5\text{V}$$

- Example Sine

$$v(t) = 5\text{V} \times \sin \omega t$$

$$\mathbf{V} = j5\text{V} = 5\text{V} \angle 90^\circ$$

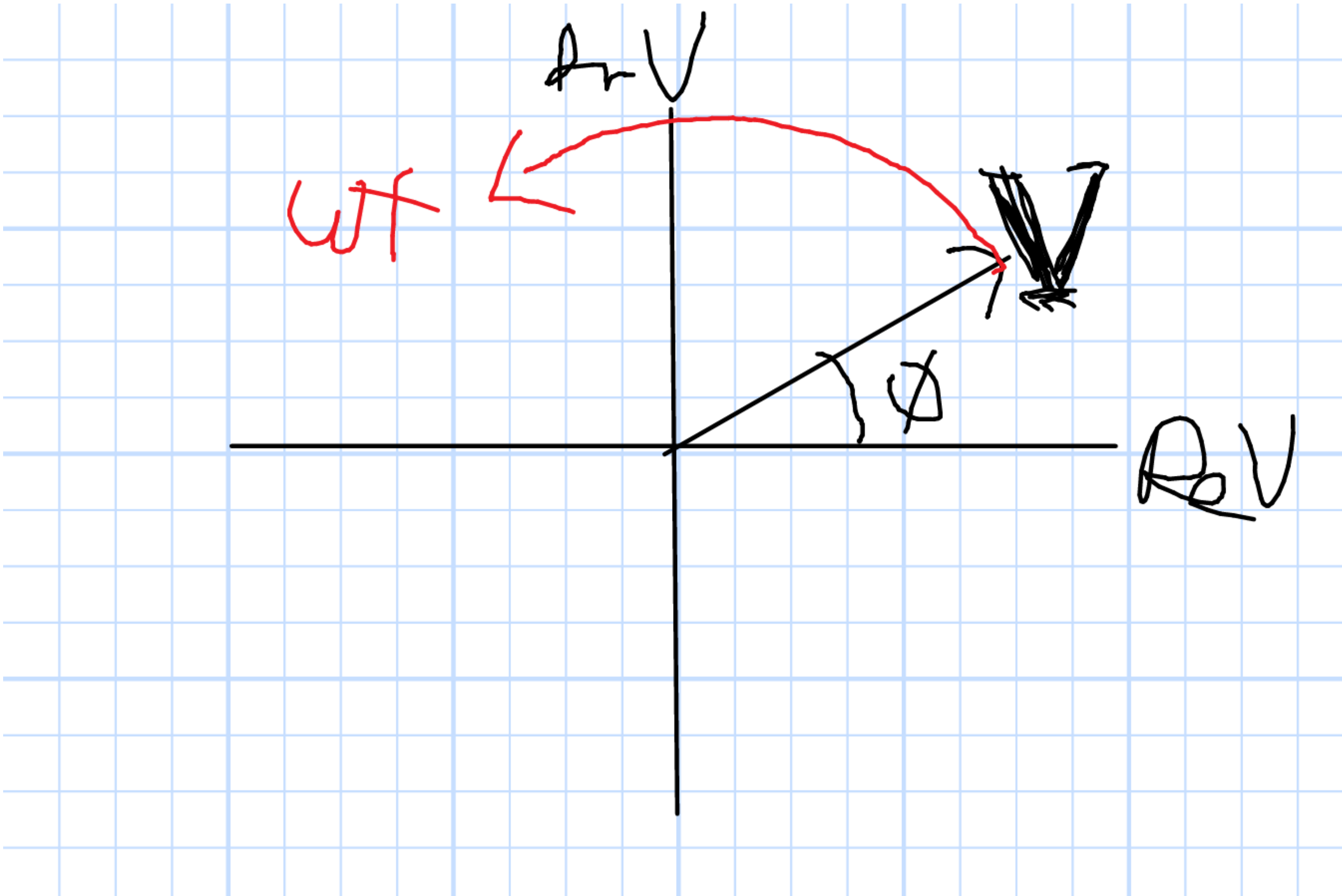
- Example with Phase

$$v(t) = 5\text{V} \times \cos(\omega t + 30^\circ)$$

$$\mathbf{V} = 5\text{V} \angle 30^\circ = 5\text{V} e^{j30^\circ \frac{\pi}{180^\circ}}$$

$$\mathbf{V} = 4.33 + 2.5j$$

Phasor Graphical Representation



Complex Arithmetic

- Addition/Subtraction

$$z_1 = x_1 + jy_1 \qquad z_2 = x_2 + jy_2$$

$$z = z_1 + z_2$$

$$x = x_1 + x_2 \qquad y = y_1 + y_2$$

- Multiplication/Division

$$z = z_1 z_2 = (x_1 + jy_1)(x_2 + jy_2)$$

$$z = x_1 x_2 - y_1 y_2 + jx_1 y_2 + jy_1 x_2$$

- Easier

$$z_1 z_2 : \qquad V\angle\phi = V_1 V_2 \angle(\phi_1 + \phi_2)$$

$$\frac{z_1}{z_2} : \qquad \frac{V_1}{V_2}\angle(\phi_1 - \phi_2)$$

- Conjugate

$$z = x + jy \qquad z^* = x - jy$$

$$(V\angle\phi)^* = V\angle-\phi$$

$$\left(Ve^{\phi}\right)^* = Ve^{-\phi}$$

- Example

$$\frac{5\angle 30^\circ + 7}{2\angle -45^\circ}$$

```
>> moose=...
(5*exp(1j*30*pi/180)+7)...
/2*exp(-1j*45*pi/180)
moose =
   4.8897 - 3.1219i
>>
```

Capacitor (1)

- Capacitor Equation

$$i = C\frac{dv}{dt}$$

- Voltage Equation

$$v = \mathbf{V}e^{j\omega t}$$

- Current

$$i = Cj\omega\mathbf{V}e^{j\omega t}$$

$$i = \mathbf{I}e^{j\omega t} = Cj\omega\mathbf{V}e^{j\omega t}$$

$$\mathbf{I} = j\omega C\mathbf{V}$$

- Impedance

$$Z = \frac{\mathbf{V}}{\mathbf{I}} = \frac{1}{j\omega C}$$

- Example $C = 1\mu$F

$$Z = \frac{1}{j\omega C} \quad \omega = 2\pi f$$

$$V = 10\text{V at } f$$

- Current

$$I = \frac{V}{Z} \quad i = Ie^{j\omega t}$$

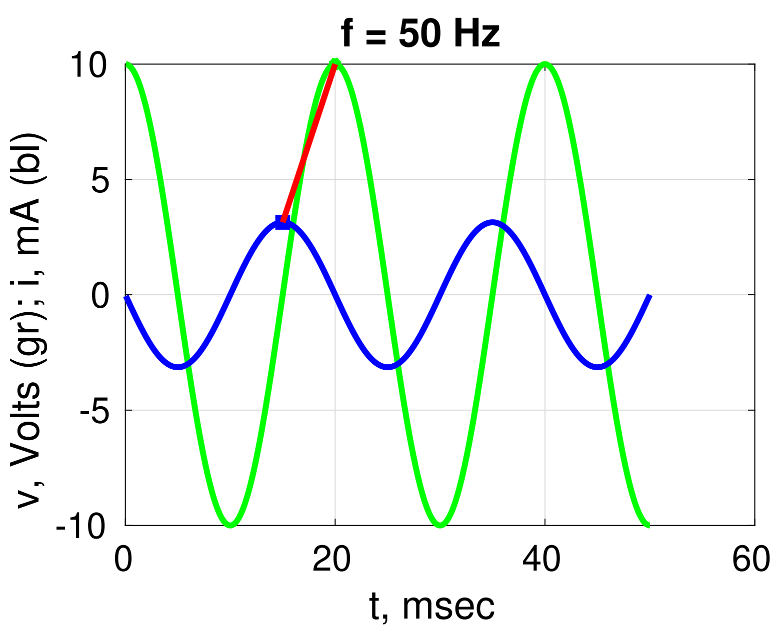
```
>> C=1e-6;V=10;f=50;
>> Z=1/(1j*2*pi*f*C)
Z =
   0.0000e+00 - 3.1831e+03i
>> I=V/Z
I =
   0.0000 + 0.0031i
```

$$I = 3.1\text{mA}\angle 90^\circ$$

Capacitor (2); Current Leads

$C = 1\mu$F $\quad V = 10$V at f

f	Z	$\|I\|$
50Hz	$-j3200\Omega$	3.1mA
500Hz	$-j320\Omega$	31mA
5kHz	$-j32\Omega$	310mA
50kHz	$-j3.2\Omega$	3.1A




```
t=[0.001:0.001:1]*0.05;
C=1e-6;V=10;f=50;omega=2*pi*f;
Z=1/(1j*omega*C);I=V/Z;
vc=real(V*exp(1j*omega*t));
ic=real(I*exp(1j*omega*t));
fv=find(vc==max(vc),1);
fi=find(ic==max(ic),1);
figure;plot(t*1e3,vc,'g',...
  t*1e3,ic*1e3,'b',...
 t(fv)*1e3,vc(fv),'g*',...
 t(fi)*1e3,ic(fi)*1e3,'bs',...
 [t(fv),t(fi)]*1e3,...
   [vc(fv),ic(fi)*1e3],'r');
grid on;xlabel('t, msec');
ylabel('v, Volts (gr); i, mA (bl)')
title('f = 50 Hz');
```

Inductor: Current Lags

- Inductor Equation

$$v = L\frac{di}{dt}$$

- Current Equation

$$i = \mathbf{I}e^{j\omega t}$$

- Voltage

$$v = Lj\omega\mathbf{I}e^{j\omega t}$$

$$v = \mathbf{V}e^{j\omega t} = Lj\omega\mathbf{I}e^{j\omega t}$$

$$\mathbf{V} = j\omega L\mathbf{I}$$

- Impedance

$$Z = \frac{\mathbf{V}}{\mathbf{I}} = j\omega L$$

- Example $L = 5\text{H}$

$$Z = j\omega L$$

- Current

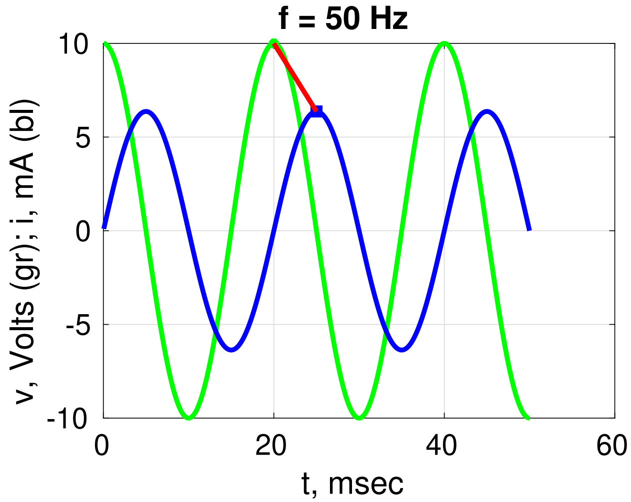
$$I = \frac{V}{Z} \qquad i = Ie^{j\omega t}$$

```
>> L=5;V=10;f=50;omega=2*pi*f;
>> Z=1j*omega*L
Z =
   0.0000e+00 + 1.5708e+03i
>> I=V/Z
I =
   0.0000 - 0.0064i
```

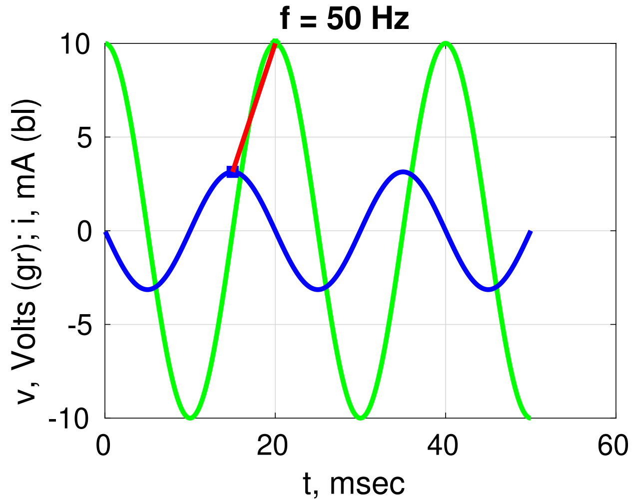
$$Z = 1600\Omega\angle 90^\circ$$

$$I = 6.4\text{mA}\angle{-90^\circ}$$

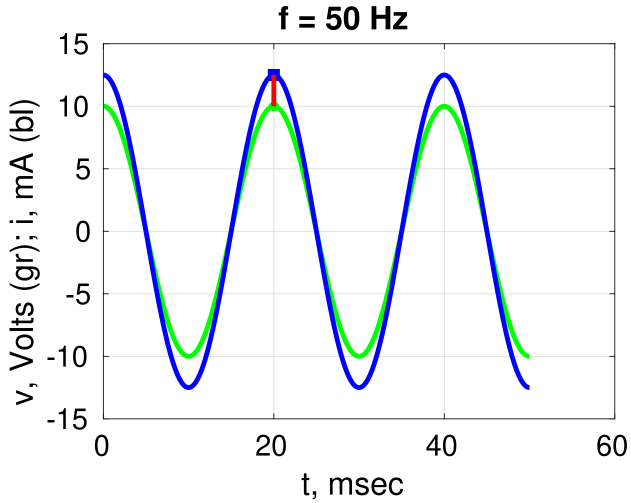
R, L, and C



$L = 500$mH (Peak v before i)



$C = 1\mu$F (Peak v after i)



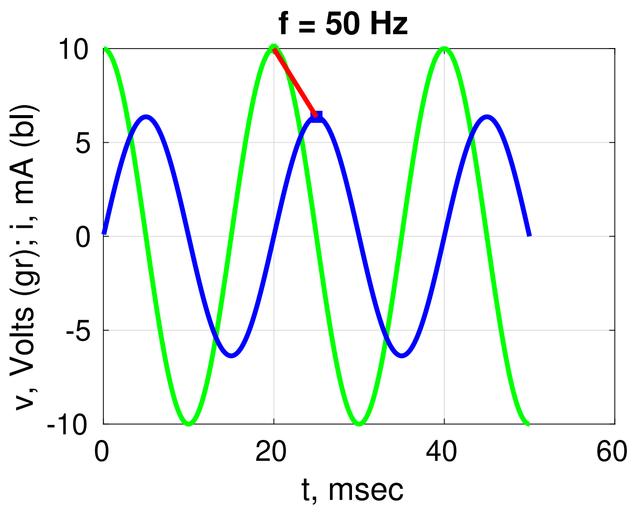
500Ω Resistor

$$Z_C = 3200\Omega\angle{-90^\circ}$$

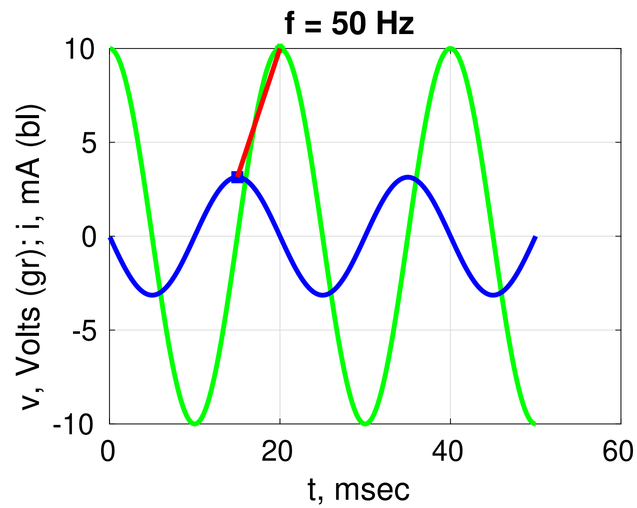
$$Z_L = 1600\Omega\angle{+90^\circ}$$

$$Z_R = 500\Omega\angle 0^\circ$$

Eli the Ice Man



500mH Inductor



1μF Capacitor

- E for Electromotive Force (Voltage)
- Inductor (ELI)

  E (Voltage) Leads I (Current):

  Peak of Voltage (Green) Before Current (Blue)
- Capacitor (ICE)

  I (Current) Leads E (Voltage):

  Peak of Current (Blue) Before Voltage (Green)

Resistance, Reactance, Impedance

- Resistance

$$Z_R = R \qquad \mathbf{V} = \mathbf{I}R$$

- Reactance

$$Z = jX \qquad \mathbf{V} = \mathbf{I}jX$$

- Examples

$$Z_C = X_C = \frac{1}{j\omega C} = \frac{-j}{\omega C} \qquad Z_L = X_L = j\omega L$$

- Impedance

$$Z = R + jX \qquad \mathbf{V} = \mathbf{I}Z$$

AC Circuits & Steady–State Sinusoids

- Voltage Sources, Current Sources, Resistors

$$v = iR$$

- Voltage Sources, Current Sources, Impedances

$$\mathbf{V} = \mathbf{IZ} = \mathbf{I}R + j\mathbf{I}X$$

- Everything We Learned Before Works
  - Series and Parallel Circuits
  - Voltage and Current Dividers
  - Node and Mesh Analysis
  - Even Op–Amps
- Just Remember the Numbers are Complex

$$i = \mathrm{Re}\left(\mathbf{I}e^{j\omega t}\right) \qquad v = \mathrm{Re}\left(\mathbf{V}e^{j\omega t}\right)$$

Series and Parallel

- Resistors in Series

$$R = R_1 + R_2$$

- Resistors in Parallel

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$$

- Inductors in Series

$$j\omega L = j\omega L_1 + j\omega L_2$$

- Inductors in Parallel

$$\frac{1}{j\omega L} = \frac{1}{j\omega L_1} + \frac{1}{j\omega L_2}$$

- Capacitors in Series

$$\frac{1}{j\omega C} = \frac{1}{j\omega C_1} + \frac{1}{j\omega C_2}$$

- Capacitors in Parallel

$$\frac{1}{1/j\omega C} = \frac{1}{1/j\omega C_1} + \frac{1}{1/j\omega C_2}$$

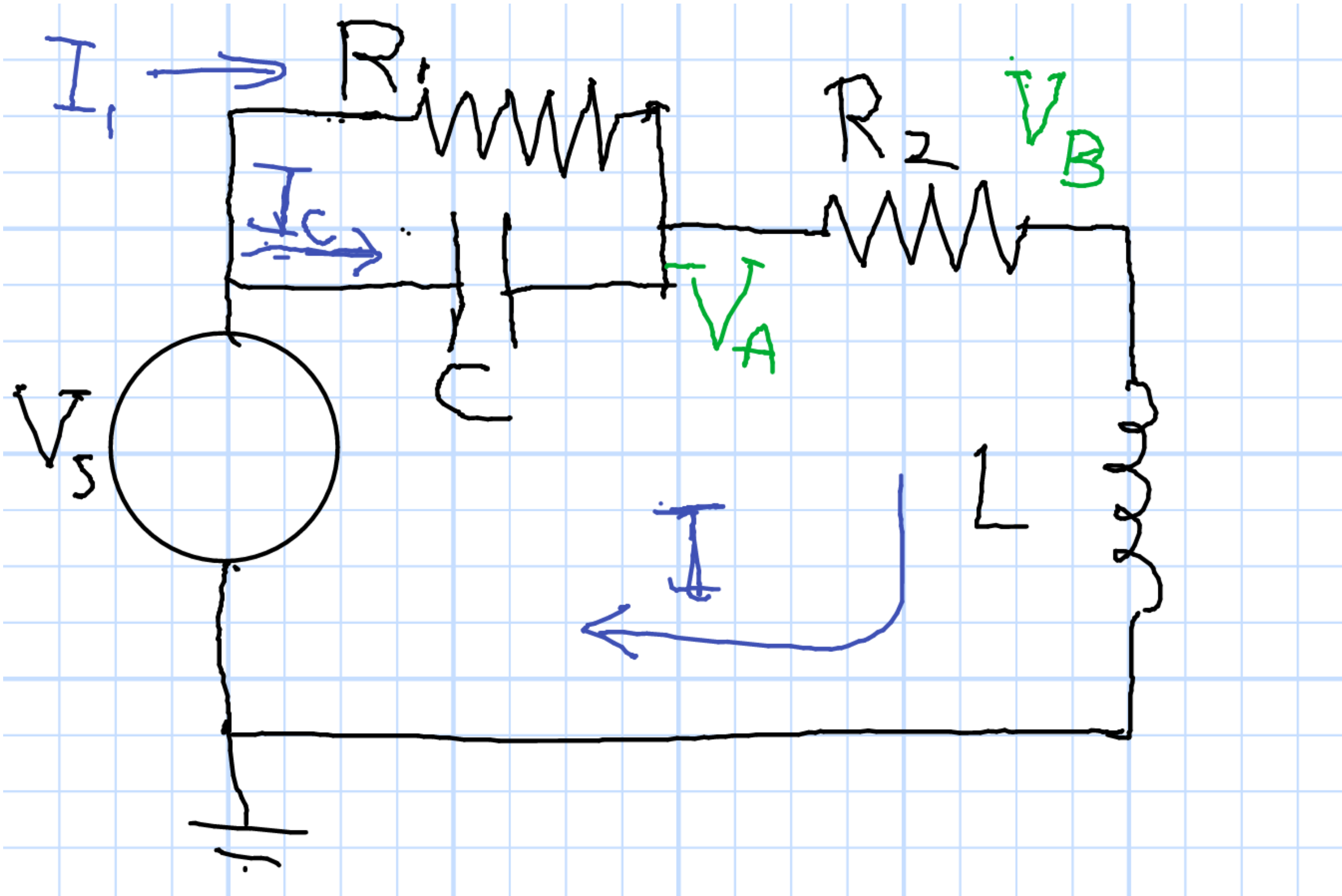
- Impedances in Series

$$Z = Z_1 + Z_2$$

- Impedances in Parallel

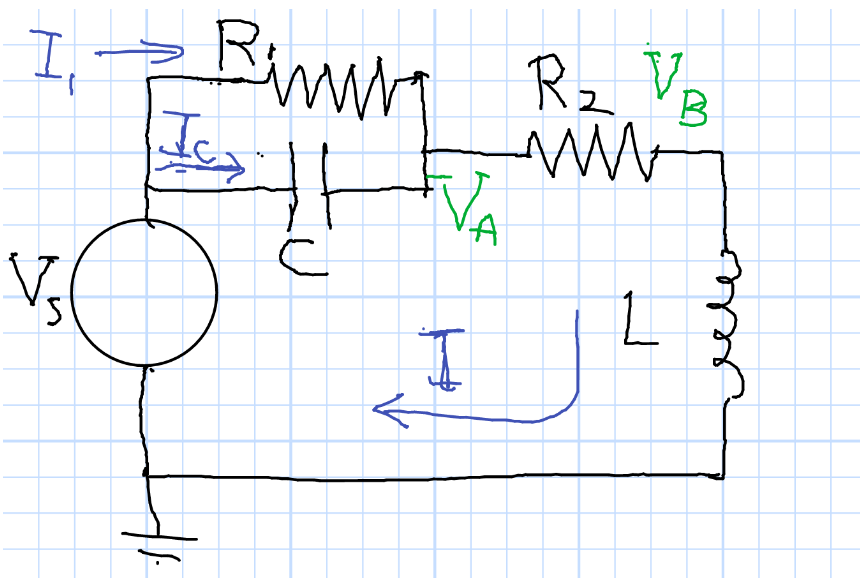
$$\frac{1}{Z} = \frac{1}{Z_1} + \frac{1}{Z_2}$$

Voltage Divider



$R_1 = 1\text{k}\Omega$, $R_2 = 2\text{k}\Omega$, $C = 470\text{pF}$, $L = 3\text{mH}$, $V = 20\text{V}$, $f = 100\text{kHz}$

Overall Impedance



$R_1 = 1\text{k}\Omega \qquad R_2 = 2\text{k}\Omega$

$C = 470\text{pF} \qquad L = 3\text{mH}$

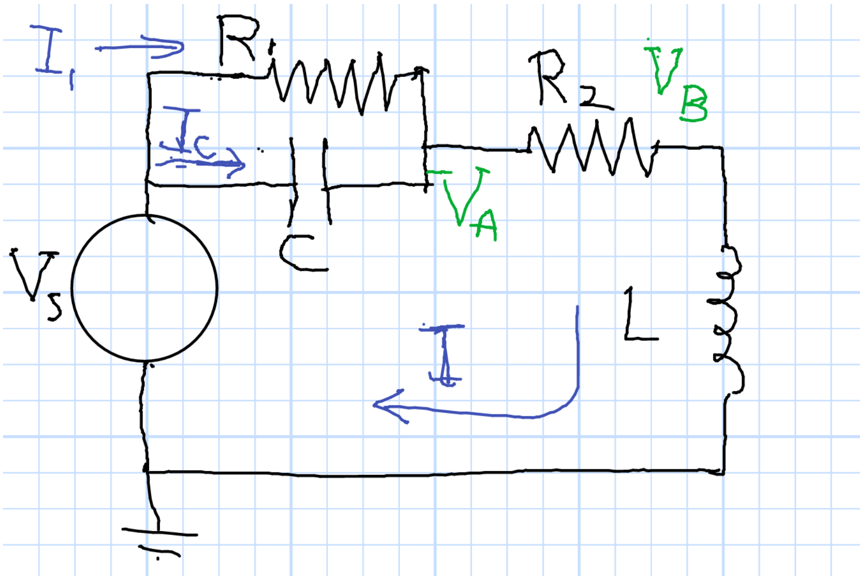
$V = 20\text{V at } f = 100\text{kHz}$

$$Z = \left(R_1 \parallel \frac{1}{j\omega C}\right) + R_2 + j\omega L$$

```
ZC=1/(1j*omega*C);
ZL=1j*omega*L;
Z1C=1/(1/R1+1/ZC);
>> Z=Z1C+R2+ZL
Z =
   2.9198e+03 + 1.6133e+03i
>> abs(Z)
ans =
   3.3359e+03
>> angle(Z)*180/pi
ans =
   28.9229
```

$$Z = 3.3\text{k}\Omega\angle 29^\circ$$

Current



$R_1 = 1\text{k}\Omega$ $R_2 = 2\text{k}\Omega$

$C = 470\text{pF}$ $L = 3\text{mH}$

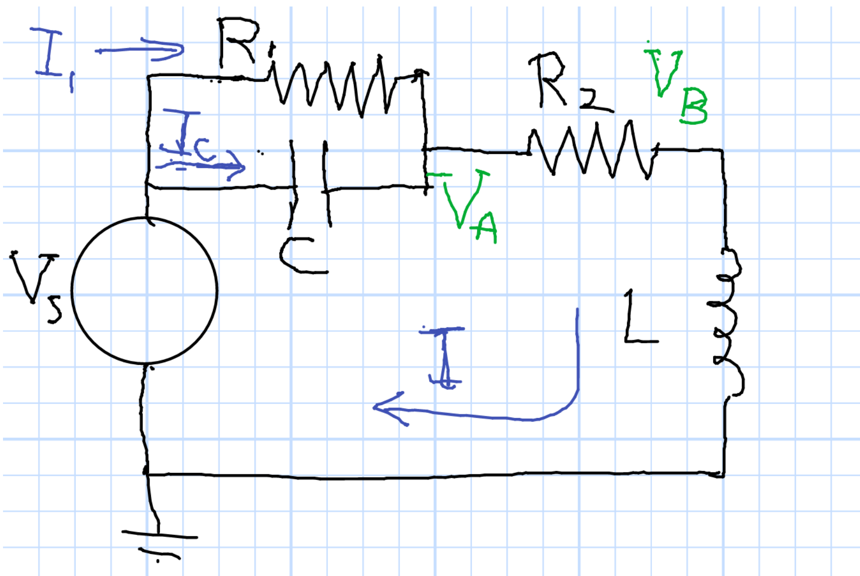
$V = 20\text{V}$ at $f = 100\text{kHz}$

$$I = \frac{V}{Z} = \frac{20\text{V}}{Z}$$

```
>> I=V/Z
I =
   0.0052 - 0.0029i
>> abs(I)
ans =
    0.0060
>> angle(I)*180/pi
ans =
  -28.9229
```

$$I = 6\text{mA}\angle{-29^\circ}$$

Current Divider



$R_1 = 1\text{k}\Omega$ $R_2 = 2\text{k}\Omega$

$C = 470\text{pF}$ $L = 3\text{mH}$

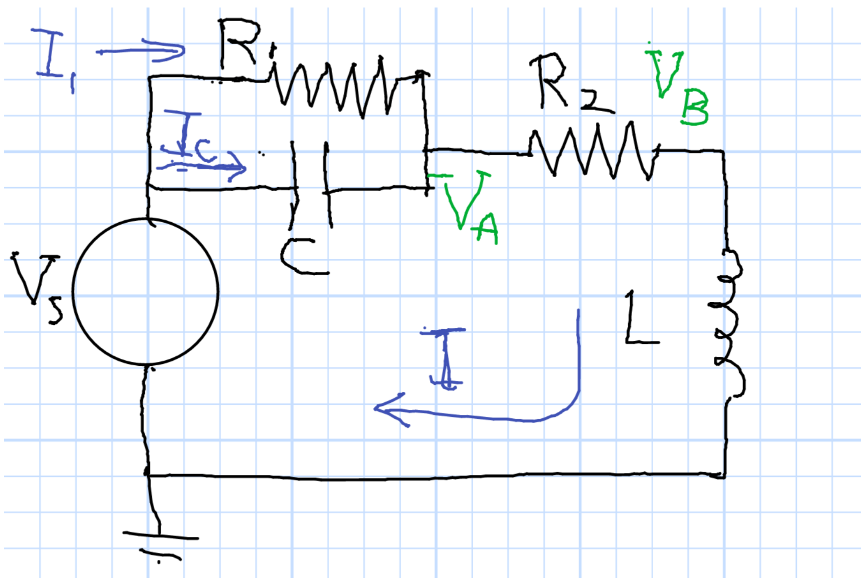
$V = 20\text{V at } f = 100\text{kHz}$

$$I_C = I\frac{R_1}{R_1 + Z_C}$$

$$I_1 = I\frac{Z_C}{R_1 + Z_C}$$

```
IC=I*R1/(R1+ZC)
IC =
   0.0012 + 0.0012i
I1=I*ZC/(R1+ZC)
I1 =
   0.0040 - 0.0041i
Icheck=IC+I1, I
Icheck =
   0.0052 - 0.0029i
I =
   0.0052 - 0.0029i
```

Voltage Divider



$R_1 = 1\text{k}\Omega$ $\quad$ $R_2 = 2\text{k}\Omega$

$C = 470\text{pF}$ $\quad$ $L = 3\text{mH}$

$V = 20\text{V at } f = 100\text{kHz}$

$$V_A = V\frac{R_2 + j\omega L}{Z}$$

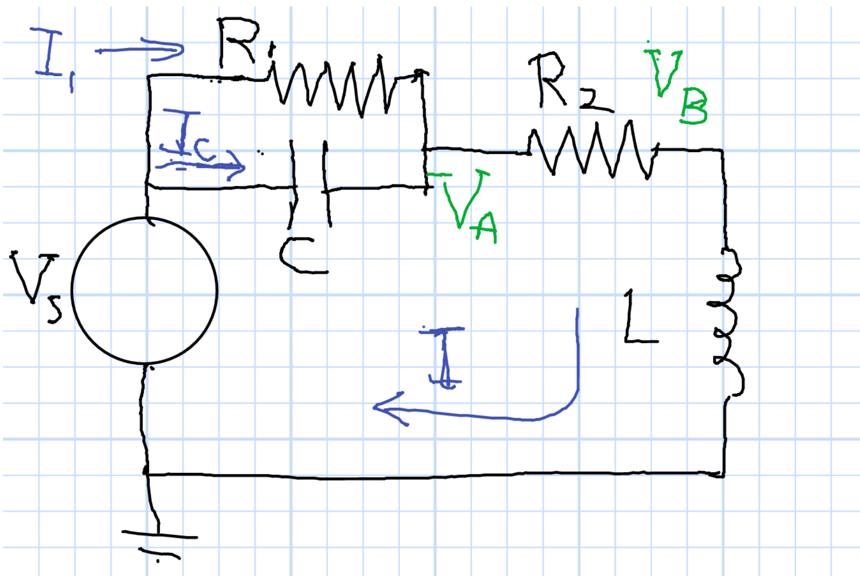
```
>> VA=V*(R2+ZL)/Z
VA =
  15.9609 + 4.0924i
>> abs(VA),angle(VA)*180/pi
ans =
   16.4772
ans =
   14.3809
```

$$V_A = 16.4\text{V}\angle 14^\circ$$

$$V_B = V\frac{j\omega L}{Z}$$

$$= 11.3\text{V}\angle 61^\circ$$

Power



$R_1 = 1\text{k}\Omega$ $\quad R_2 = 2\text{k}\Omega$

$C = 470\text{pF}$ $\quad L = 3\text{mH}$

$V = 20\text{V}$ at $f = 100\text{kHz}$

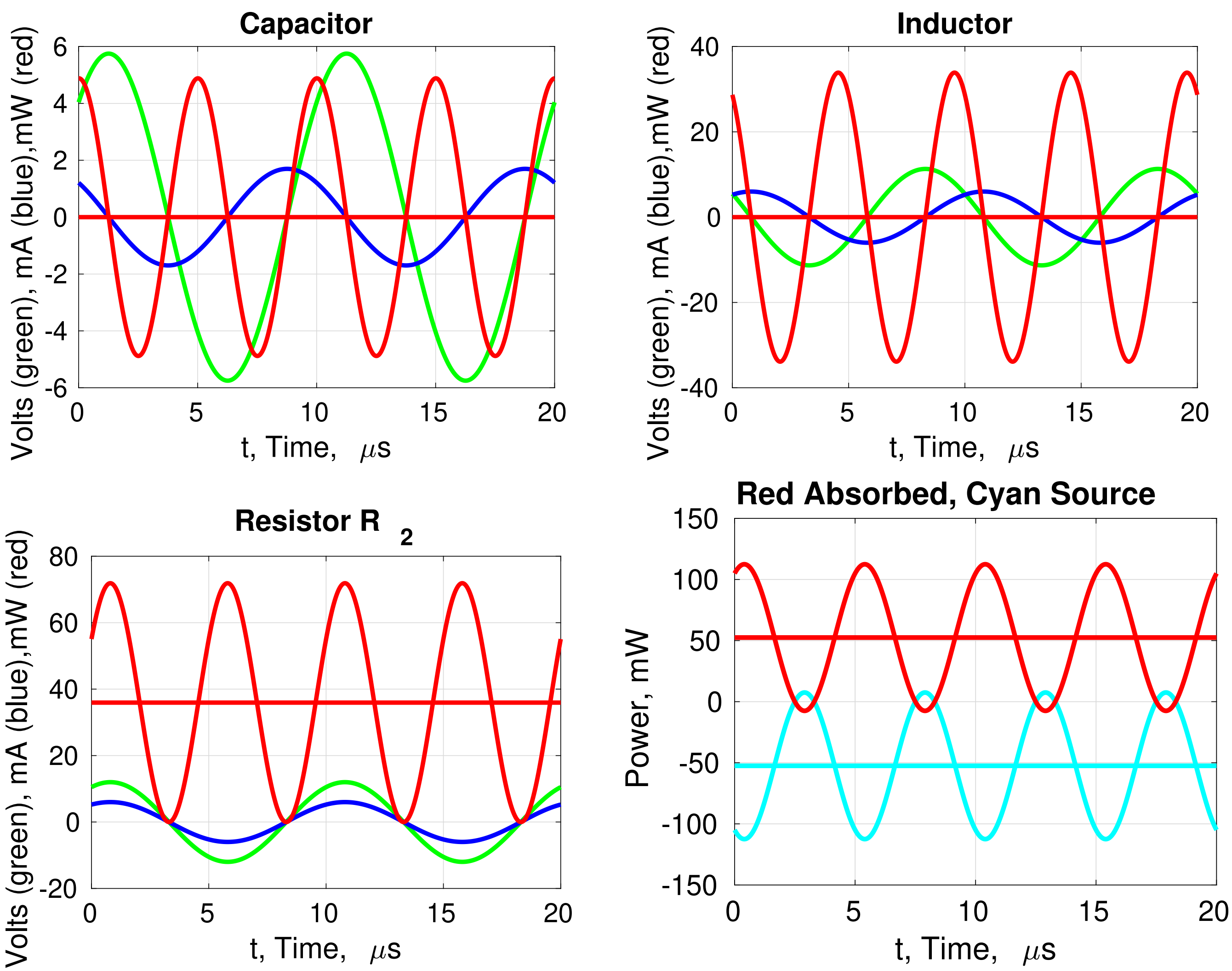
Capacitor: $V_C = V_S - V_A$

$$v_c = \text{Re}\, V_C e^{j\omega t} \qquad i_c = \text{Re}\, I_C e^{j\omega t}$$

$$p_c = v_c i_c$$

```
>> t=[0:0.001:1]*2/f;
>> vc=real(VC*exp(1j*omega*t));
ic=real(IC*exp(1j*omega*t));
pc=vc.*ic;
figure;plot(t*1e6,vc,'g',...
   t*1e6,ic*1e3,'b',...
   t*1e6,pc*1e3,'r',...
   t([1,end])*1e6,...
    real(VC*conj(IC))/2*[1,1]*1e3,...
    'r');
grid on;
```

Power Summary



Note That Source Absorbs Power at Times.

Resistive and Reactive Power

- Power Equations

$$v = \text{Re}\left(Ve^{j\omega t}\right) \quad i = \text{Re}\left(Ie^{j\omega t}\right) \qquad p = vi$$

- Complex Notation

$$v = \frac{V}{2}e^{j\omega t} + \frac{V^*}{2}e^{-j\omega t} \qquad i = \frac{I}{2}e^{j\omega t} + \frac{I^*}{2}e^{-j\omega t}$$

$$p = \frac{VI^*}{4} + \frac{V^*I}{4} + \frac{VI}{4}e^{j2\omega t} + \frac{V^*I^*}{4}e^{-j2\omega t}$$

$$p = \text{Re}\left(\frac{VI^*}{2}\right) + \frac{|VI|}{2}\cos\left(2\omega t + \phi\right)$$

$$\phi = \text{Angle}\left(\frac{VI}{2}\right)$$

Average Power

- Previous Page

$$p = \text{Re}\left(\frac{VI^*}{2}\right) + \frac{|VI|}{2}\cos\left(2\omega t + \phi\right)$$

$$\phi = \text{Angle}\left(\frac{VI}{2}\right)$$

- Time Average (over whole number of cycles)

$$\overline{p} = \text{Re}\left(\frac{VI^*}{2}\right) = \frac{|VI|}{2}\cos\phi$$

- RMS Voltage and Current

$$\overline{p} = \frac{|V|}{\sqrt{2}}\frac{|I|}{\sqrt{2}}\cos\phi = V_{RMS}I_{RMS}\cos\phi$$

Resistive and Reactive Power

- Previous Page

$$p = \text{Re}\left(\frac{VI^*}{2}\right) + \frac{|VI|}{2}\cos\left(2\omega t + \phi\right) \qquad \overline{p} = \text{Re}\left(\frac{VI^*}{2}\right)$$

- Complex Power

$$\mathbf{S} = \frac{\mathbf{VI}^*}{2}$$

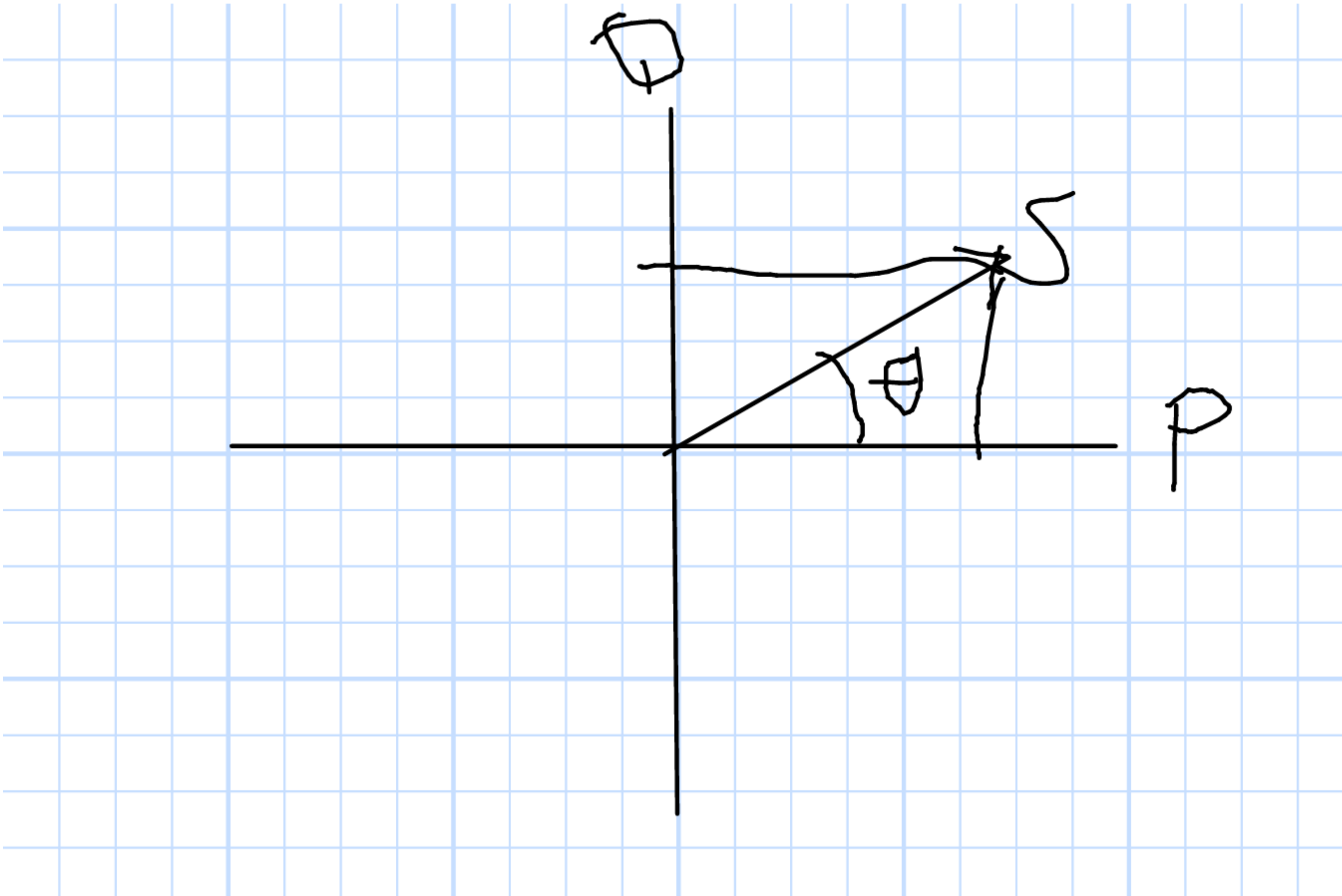
- Resistive Power

$$P = \text{Re}\,\mathbf{S}$$

- Reactive Power

$$Q = \text{Im}\,\mathbf{S}$$

Power Factor



$\mathbf{S} = P + jQ$. Power Factor: $\cos\theta$

P Watts, S Volt-Amps = VA, Q, Volt-Amps-Reactive = VAR

Generators

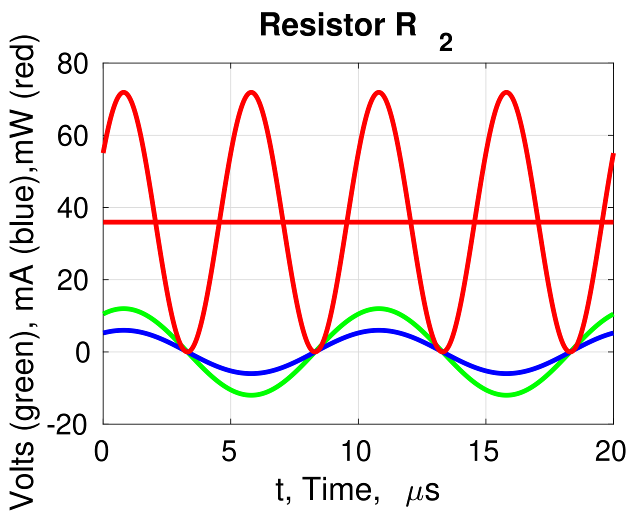
150 kVA (exmod.uk)

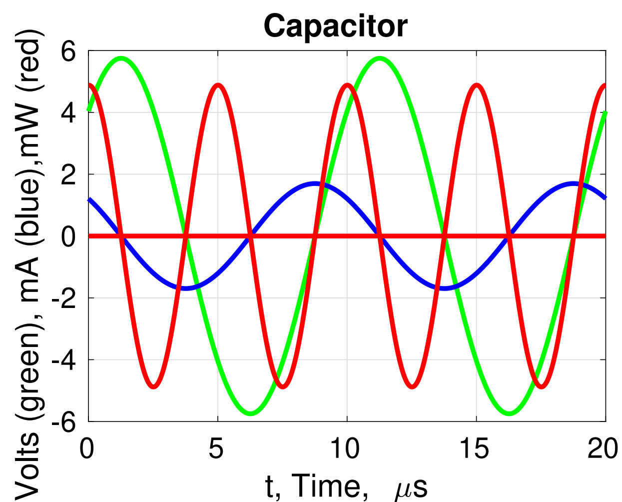
600 MVA (nuclearstreet.com)

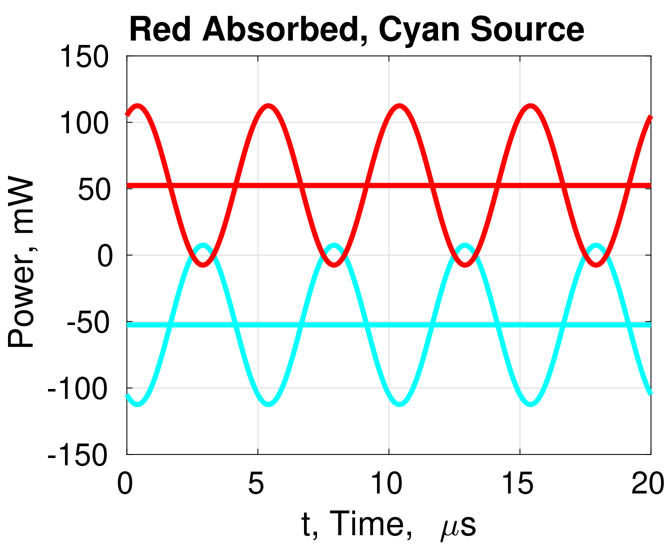
1 kVA indiamart.com

1177MVA (transformer-technology.com)

Power Summary







- Resistor; $\phi = 0$

$$\bar{p} = v_{rms} i_{rms} = \frac{v_{rms}^2}{R}$$

- Cap/Inductor; $\phi = \pm 90^\circ$

$$\bar{p} = 0$$

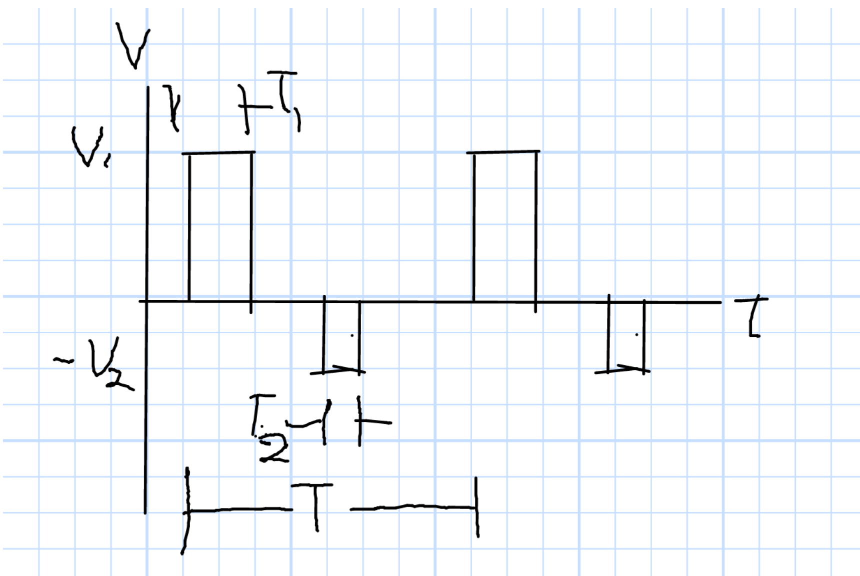
- General Sinusiods

$$\bar{p} = v_{rms} i_{rms} \cos\phi$$

RMS More Generally

$$p = \frac{v^2}{R} \qquad \bar{p} = \frac{v_{rms}^2}{R} \qquad \bar{p} = \frac{w(t:t+T)}{T}$$

$$\bar{p} = \frac{1}{T}\int_t^{t+T} \frac{v^2}{R} dt$$



$$\bar{p} = \frac{1}{RT}\left(v_1^2 t_1 - v_2^2 t_2\right)$$

$$\bar{p} = \frac{v_{rms}^2}{R}$$

$$v_{rms}^2 = \frac{v_1^2 t_1 - v_2^2 t_2}{T}$$

Voltage Readings

- Equations

$$v = \text{Re}\left(Ve^{j\omega t}\right) = 20\text{V}\cos\omega t$$

$$v_{max} = |V| \qquad v_{min} = -|V|$$

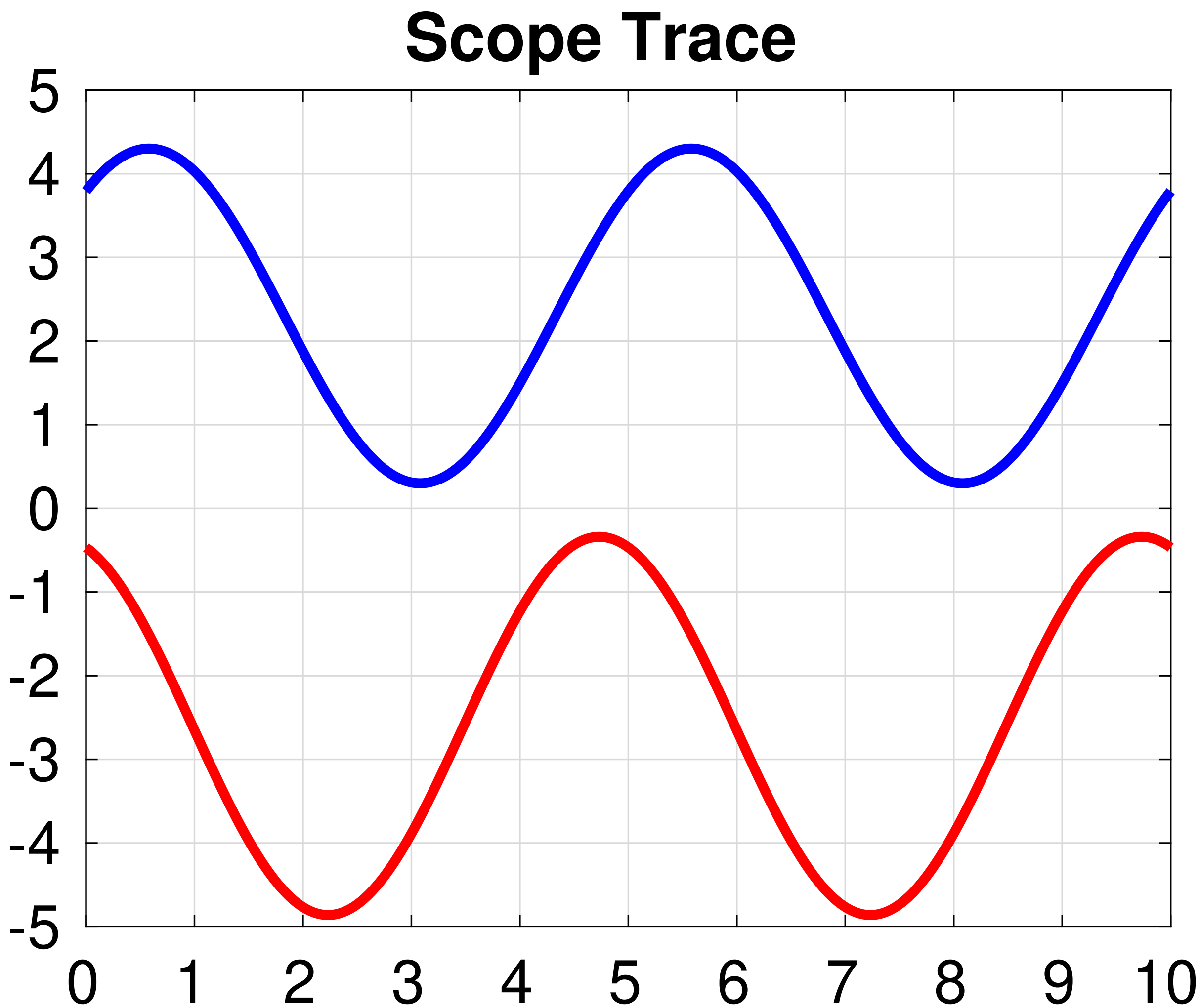
- Volt Meter Measures RMS

$$v_{rms} = \frac{|V|}{\sqrt{2}} = 14\text{V}$$

- Oscilloscope Measures Peak–to–Peak

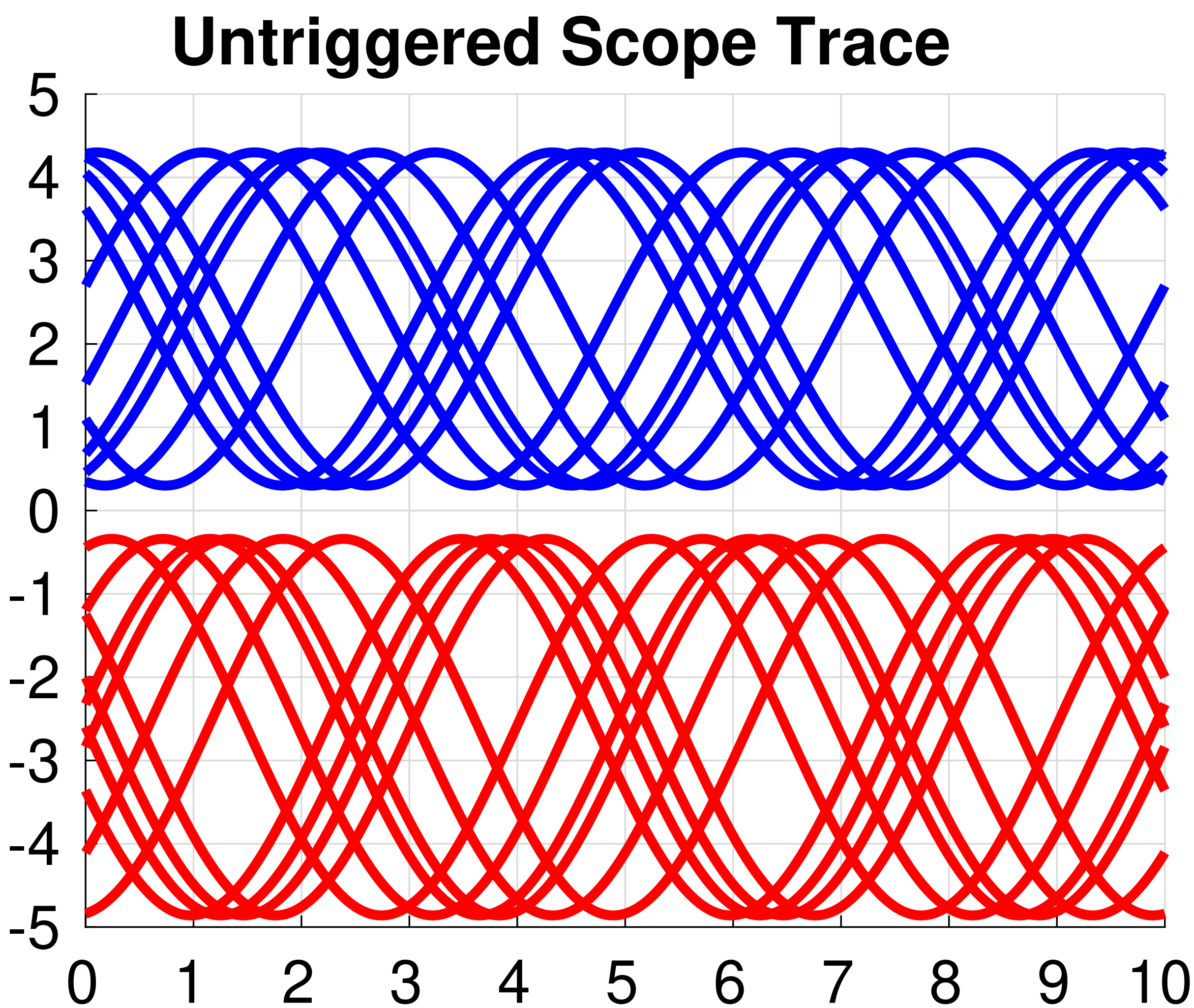
$$v_{pp} = 2|V| = 40\text{V}$$

Reading an Oscilloscope



Ch1: V 10V/cm Ch2: V_B, 5V/cm Time base 20μs

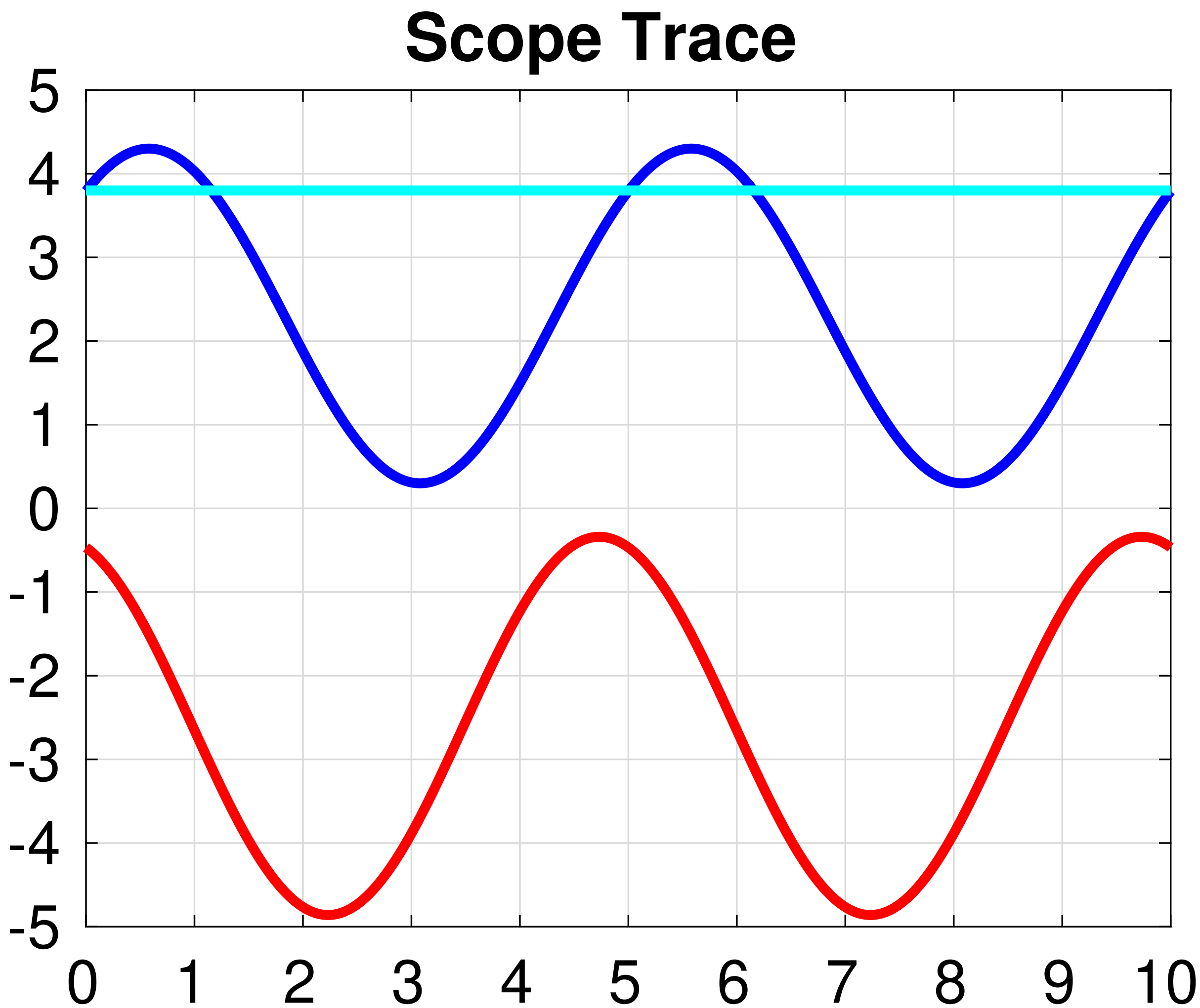
Untriggered Oscilloscope



Ch1: V 10V/cm Ch2: V_B, 5V/cm Time base 20μs

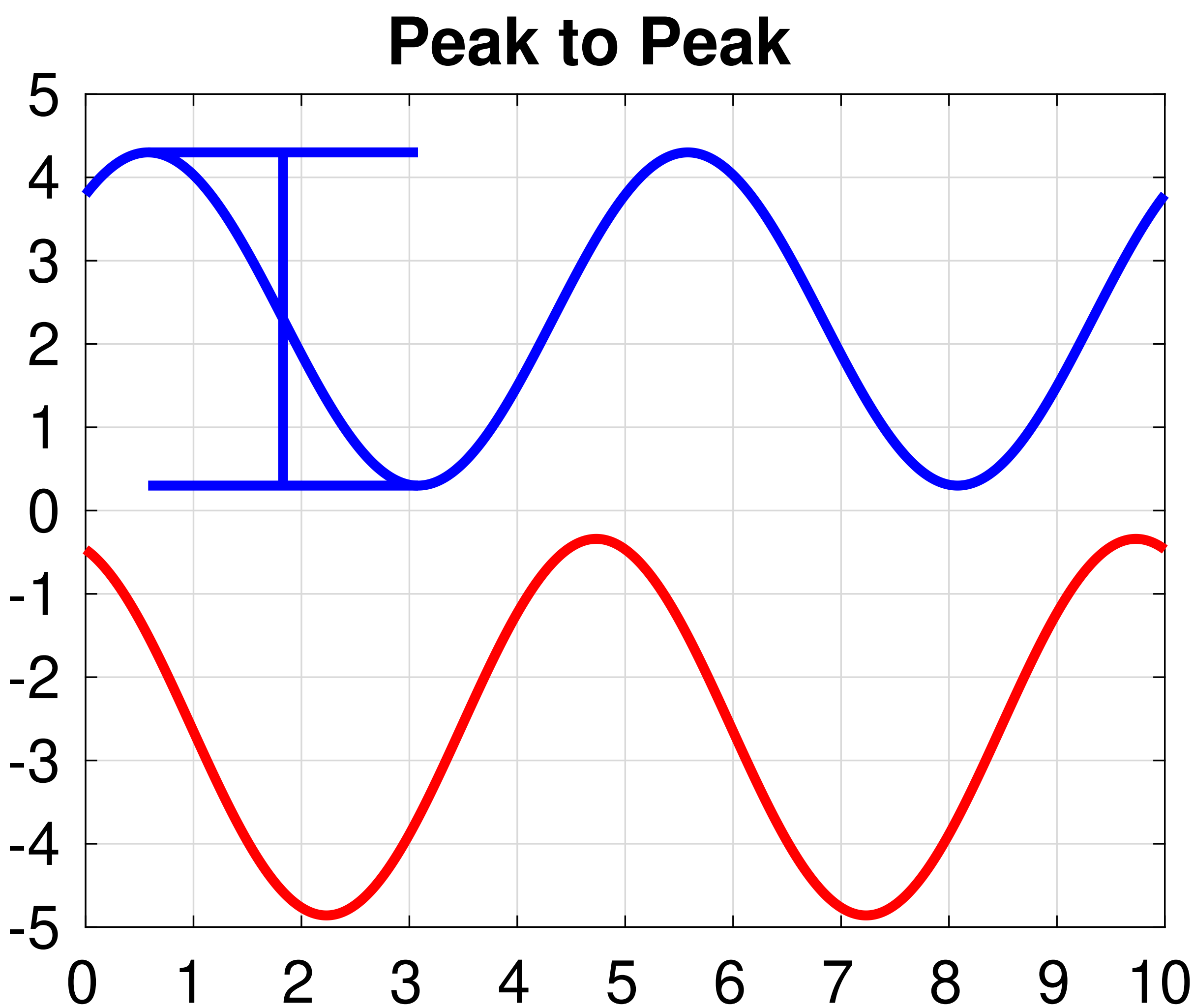
Trigger Mode: Auto

Channel–1 Trigger



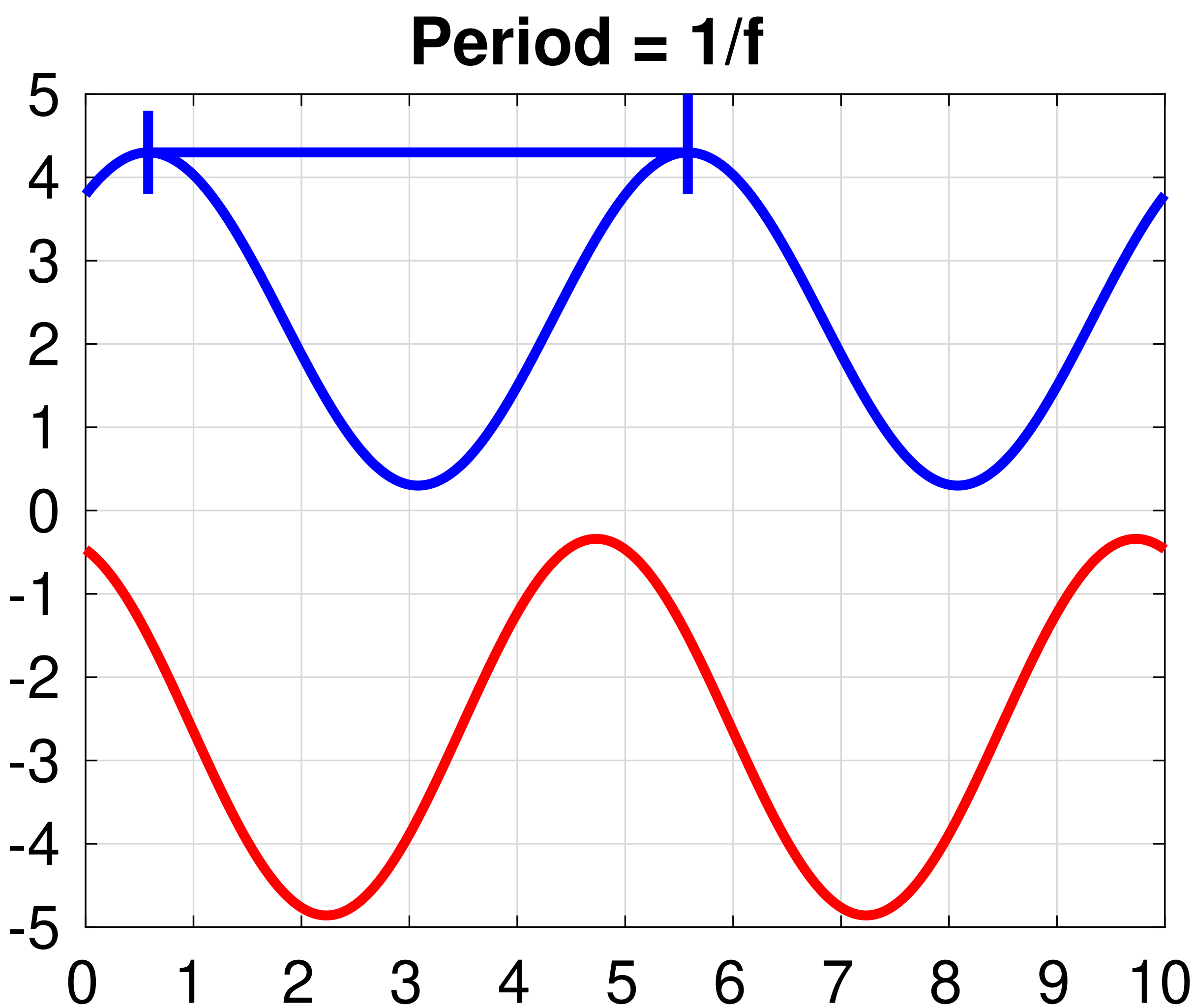
Ch1: V 10V/cm Ch2: V_B, 5V/cm Time base 20μs
Trigger Ch1, 15V, +slope

Reading the Voltage



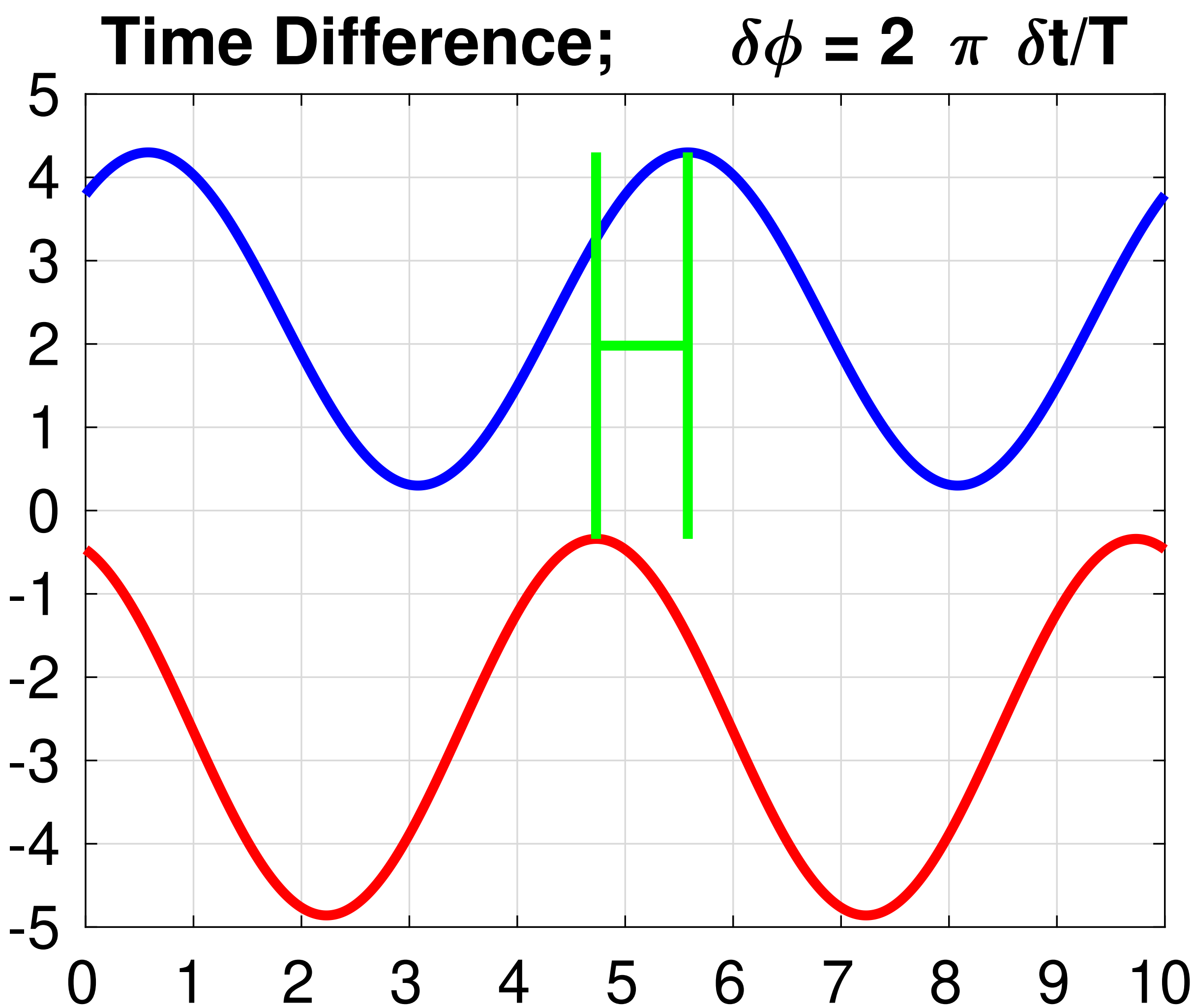
Ch1: V, 40V$_{pp}$/(10V/cm) Ch2: V_B, 22.6V$_{pp}$/(5V/cm)

Reading the Period



$T = 1/f$; 10μs/2μs/cm

Reading the Phase Difference



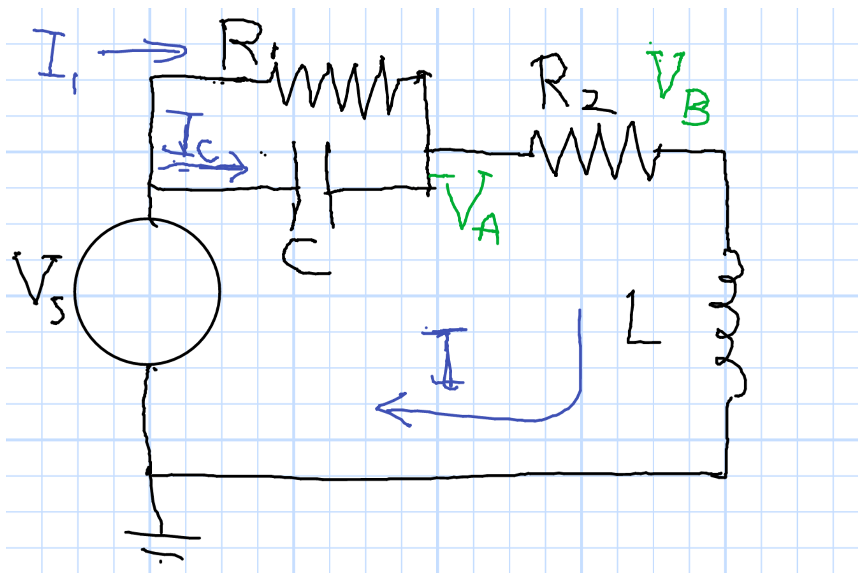
$\delta t = T\frac{\delta\phi}{2\pi} = T\frac{\delta\phi}{360^\circ}$; 1.7$\mu$s/2$\mu$s/cm

Trigger Ch1, 15V, +slope

Questions

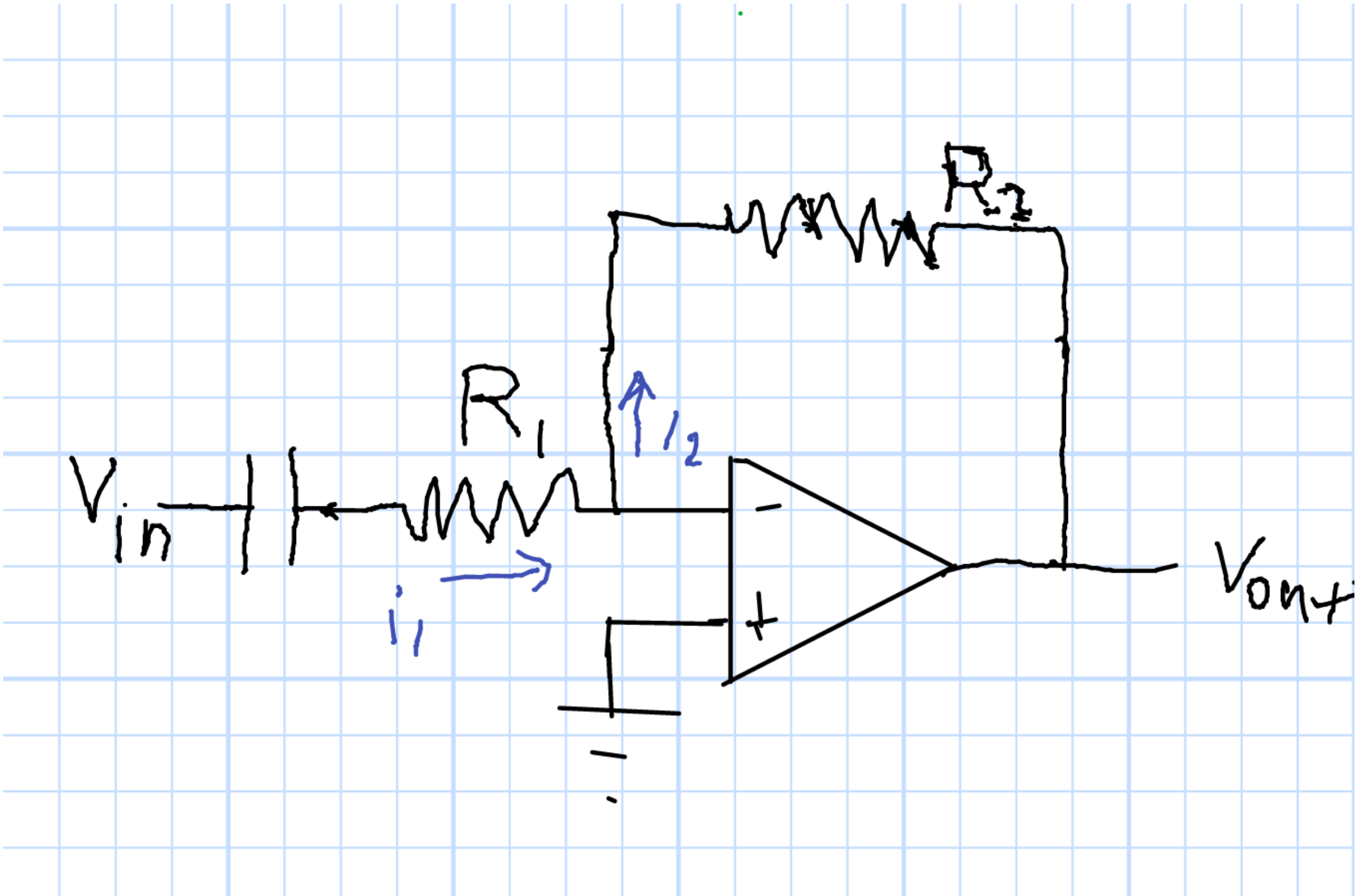
How much does all of this change in the current circuit if the input to the oscilloscope is 1MΩ?

How about if it is 50Ω?

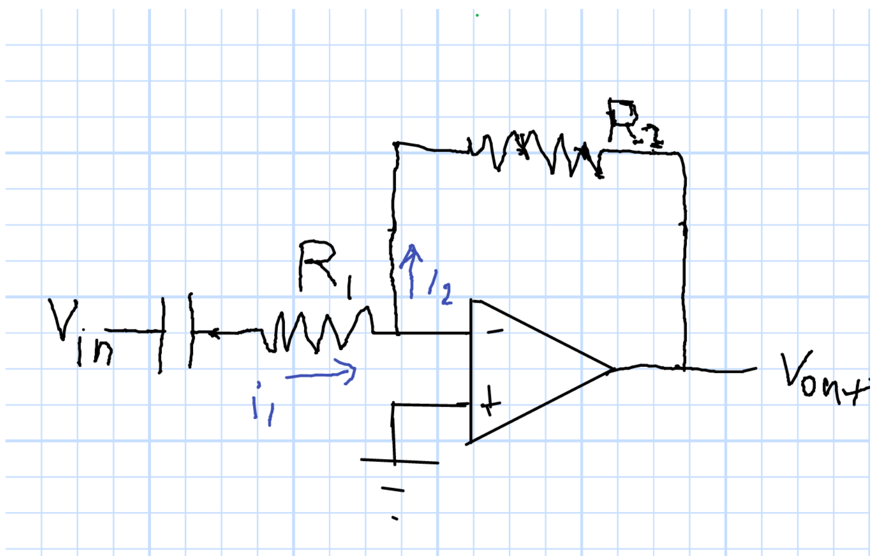


$R_1 = 1\text{k}\Omega$, $R_2 = 2\text{k}\Omega$ $Z_C = -j3.4\text{k}\Omega$, $Z_L = j1.9\text{k}\Omega$,
$V = 20\text{V}$, $f = 100\text{kHz}$

AC–Coupled Op–Amp Circuit



AC–Coupled Amplifier Solution



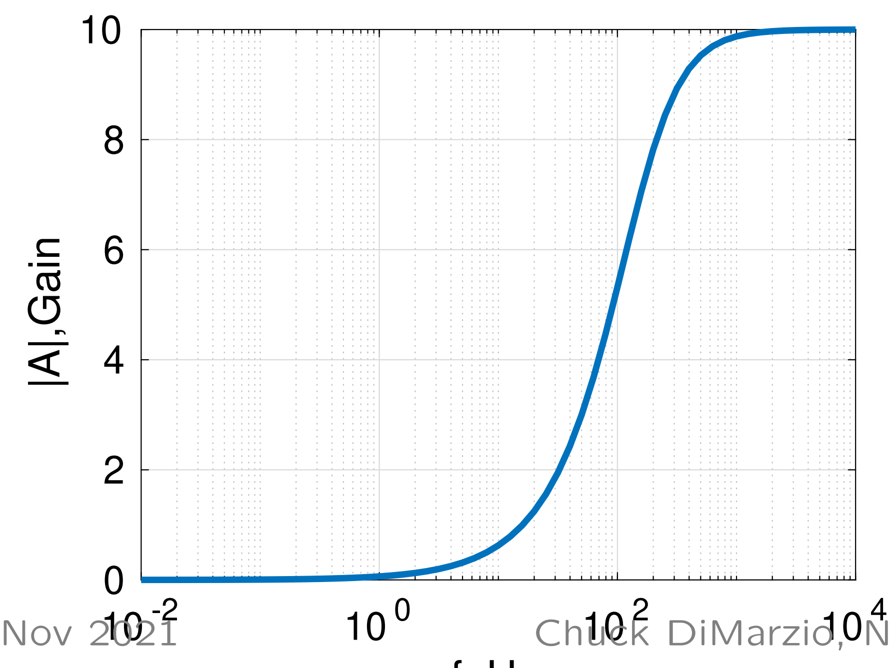
$$A_v = -\frac{R_2}{R_1 + \frac{1}{j\omega C}}$$

$R_1 = 1\text{k}\Omega \qquad R_2 = 10\text{k}\Omega$

$C = 1\mu\text{F} \qquad X_C = \frac{1}{j2\pi f C}$

$X_C = -160\text{k}\Omega$ at 1Hz

$X_C = -160\Omega$ at 1kHz




```
>> f=10.^[-2:0.1:4];
>> R1=1000;R2=10e3;C=1e-6;
>> A=-R2./(R1+1./(1j*2*pi*f*C));
>> figure;semilogx(f,abs(A));
grid on;
>> xlabel('f, Hz');
ylabel('|A|,Gain');
```