Electrical Engineering
Week 10

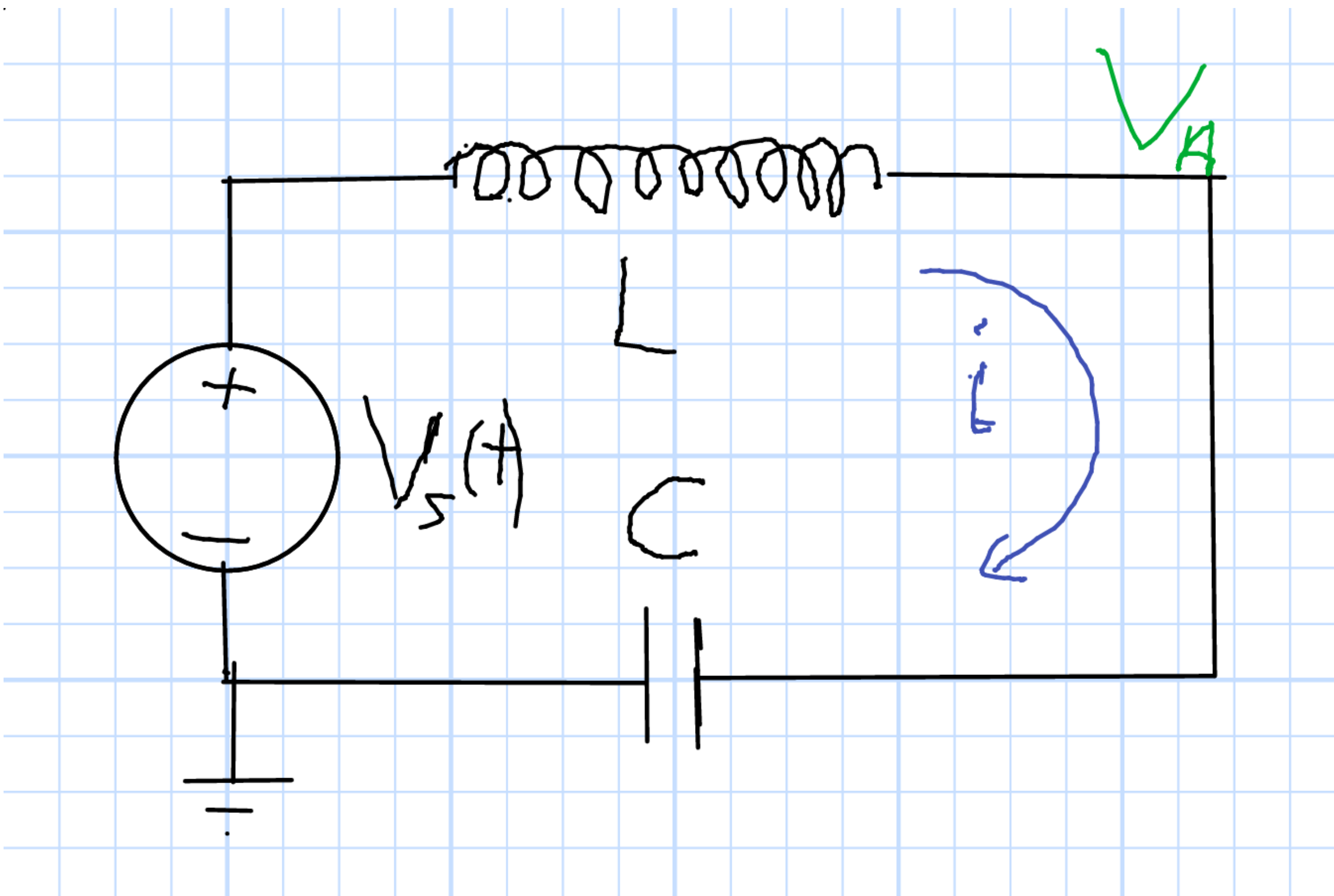
Charles A. DiMarzio
EECE–2210
Northeastern University

Oct 2021

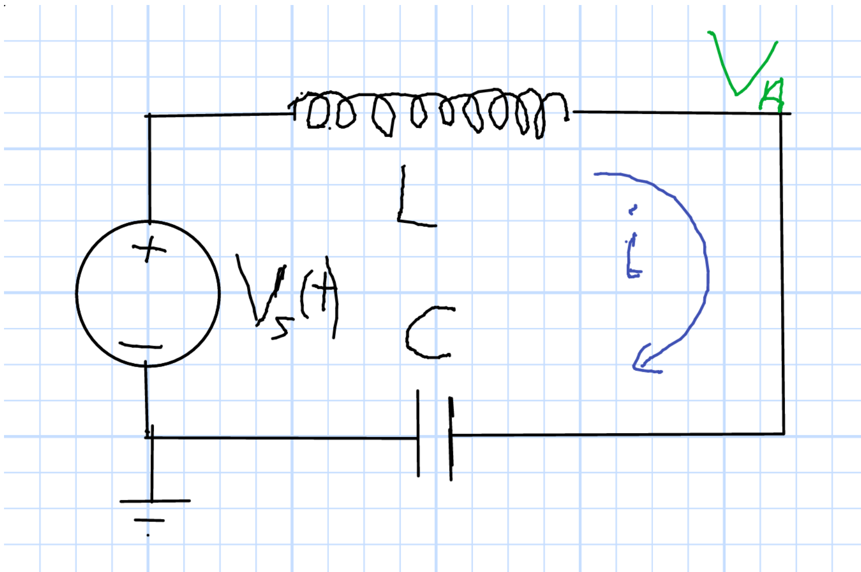
Week 10 Agenda: Second–Order Transients

- Second–Order Equations (LC Circuit)
- Complex Notation
- Series RLC
  - Forced Solution
  - Transient Solution
  - Damping
- Parallel RLC

LC Circuit



LC Equations



- KVL

$$v_s(t) = L\frac{di(t)}{dt} + \frac{1}{C}\int i(t)\,dt$$

- Differentiate All Terms

$$\frac{dv_s(t)}{dt} = L\frac{d^2 i(t)}{dt^2} + \frac{1}{C}i(t)$$

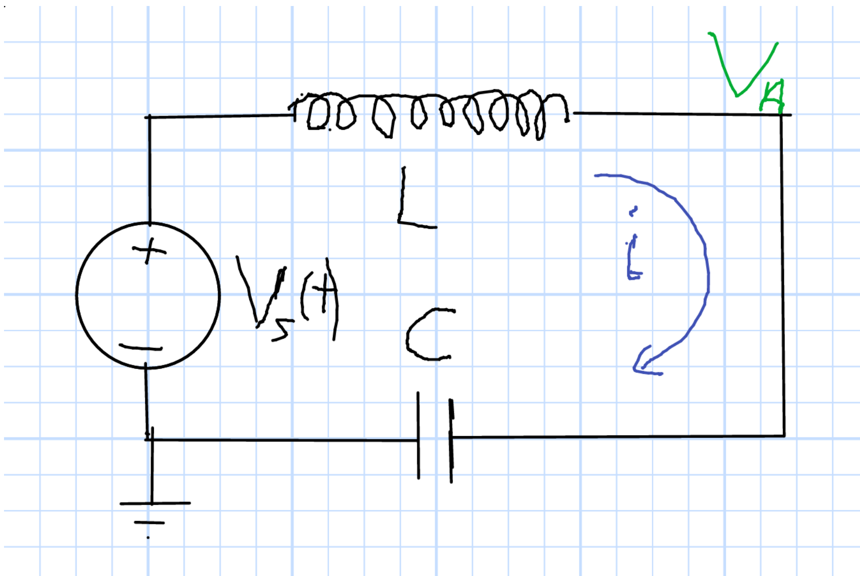
- Propose Solution for i

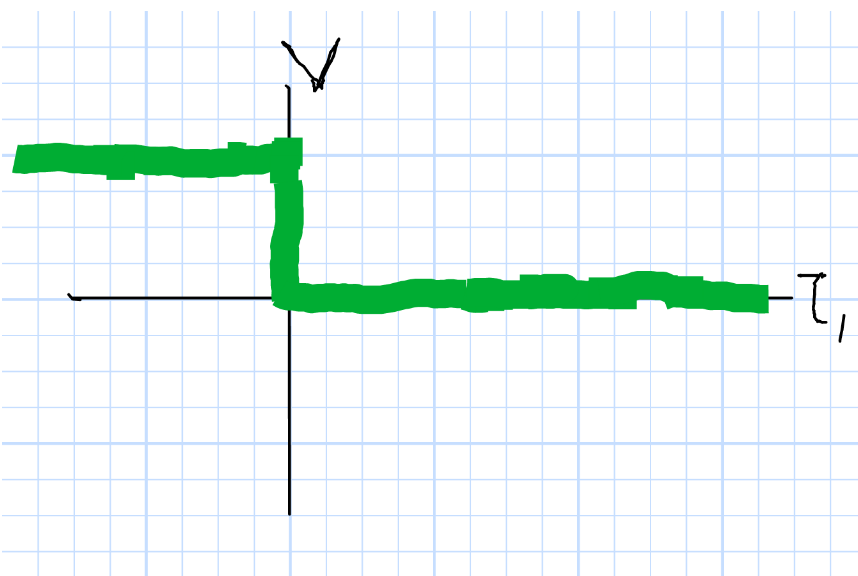
$$i(t) = Ke^{st}$$

- Substitute

$$\frac{dv_s(t)}{dt} = Ls^2 i(t) + \frac{1}{C}i(t)$$

LC Example





- From Previous Page

$$\frac{dv_s(t)}{dt} = Ls^2 i(t) + \frac{1}{C} i(t)$$

- Step Voltage

$$\frac{dv_s(t)}{dt} = -v_s\left(0^-\right)\delta(t)$$

- Solution

$$s^2 = -\frac{1}{\sqrt{LC}} \qquad s = \pm\sqrt{-\frac{1}{LC}}$$

- Express as Frequency

$$s_1 = \sqrt{-1}\omega_0 \qquad s_2 = -\sqrt{-1}\omega_0$$

$$\omega_0 = \frac{1}{\sqrt{LC}}$$

Interlude: Complex Representation

- Current *vs.* Voltage Equations

$$i = C\frac{dV}{dt} \qquad v = L\frac{di}{dt}$$

- Exponential Solutions: First Order Transients

$$\frac{de^{st}}{dt} = se^{st}$$

- Sinusoidal Solutions

$$\frac{d\sin\omega t}{dt} = \omega\cos\omega t \qquad \frac{d\cos\omega t}{dt} = -\omega\sin\omega t$$

- Almost Nice, but Not Quite

Euler's Formula

- Euler's Formula

$$e^{j\theta} = \cos\theta + j\sin\theta \qquad e^{j\omega t} = \cos\omega t + j\sin\omega t$$

- Cosine and Sine

$$\cos\omega t = \frac{e^{j\omega t}}{2} + \frac{e^{-j\omega t}}{2} \qquad \sin\omega t = \frac{e^{j\omega t}}{2j} - \frac{-e^{-j\omega t}}{2j}$$

- Derivatives Are Simple

$$\frac{de^{j\omega t}}{dt} = j\omega e^{j\omega t} \qquad \frac{de^{-j\omega t}}{dt} = -j\omega e^{-j\omega t}$$

$$v = 5\text{V}e^{j\omega t} \qquad i = C\frac{dv}{dt} = j\omega C v$$

Two Problems Instead of One

- Cosine Example: Superposition of Two

$$v_+ = 5\text{V}e^{j\omega t} \qquad i_+ = C\frac{dv}{dt} = j\omega C v$$

$$v_- = 5\text{V}e^{-j\omega t} \qquad i_- = C\frac{dv}{dt} = -j\omega C v$$

- But It's Almost No Extra Work:

A Linear Function of the Complex Conjugate of something is the Complex Conjugate of the Linear Function

$$v = v_+ + v_- = 10\text{V}\cos\omega t$$

$$i = i_+ + i_- = j\omega C v_+ - j\omega C v_-$$

$$= 2\text{Real}\left[j\omega C v_+\right] = -C \times 10\text{V}\sin\omega t$$

i or j: In EE, it's j

- Imaginary Unit

$$\sqrt{-1} = \pm i$$

- But in EE, i is Current
- We use j

$$\sqrt{-1} = \pm j$$

WPJ

- Euler's Formula

$$e^{j\theta} = \cos\theta + j\sin\theta$$

$$e^{j\omega t} = \cos\omega t + j\sin\omega t$$

- Cosine and Sine

$$\cos\omega t = \frac{e^{j\omega t} + e^{-j\omega t}}{2}$$

$$\sin\omega t = \frac{e^{j\omega t} - e^{-j\omega t}}{2j}$$

- Why Exponentials?
  - Compact Notation
  - Easy Math

Solving Problems

Real Quantity	Positive Freq.		Negative Freq.	Phasor
$A\cos\omega t$	$\frac{A}{2}e^{j\omega t}$	$+$	$\frac{A}{2}e^{-j\omega t}$	$A\angle 0$
$A\sin\omega t$	$\frac{-jA}{2}e^{j\omega t}$	$+$	$\frac{jA}{2}e^{-j\omega t}$	$A\angle 90$
$A\cos(\omega t+\phi)$	$\frac{Ae^{j\phi}}{2}e^{j\omega t}$	$+$	$\frac{Ae^{-j\phi}}{2}e^{-j\omega t}$	$A\angle\phi$
$A\cos\phi\cos\omega t - \ldots$ $A\sin\phi\sin\omega t =$				
$a\cos\omega t + b\sin\omega t$	$\frac{a+jb}{2}e^{j\omega t}$	$+$	$\frac{a-jb}{2}e^{-j\omega t}$	$a+jb$

Problem Solution

Unknown $= x + jy$

Pos. Freq. $+$ C.C.

$$2\mathrm{Re}\left[\frac{x+jy}{2}e^{j\omega t}\right]$$

$$\mathrm{Re}\left[(x+jy)\,e^{j\omega t}\right]$$

Complex Notation (1)

- Phasor A

$$Ae^{j\omega t} = (a + jb)\, e^{j\omega t}$$

- Include Negative Frequency Part

$$\frac{A}{2}e^{j\omega t} + \frac{A^*}{2}e^{-j\omega t} = \left(\frac{a}{2} + j\frac{b}{2}\right) e^{j\omega t} + \left(\frac{a}{2} - j\frac{b}{2}\right) e^{-j\omega t} =$$

$$\frac{a}{2}\left(e^{j\omega t} + e^{-j\omega t}\right) + j\frac{b}{2}\left(e^{j\omega t} - e^{-j\omega t}\right)$$

- From Euler's Formula

$$\cos\omega t = \frac{e^{j\omega t} + e^{-j\omega t}}{2} \qquad \sin\omega t = \frac{e^{j\omega t} - e^{-j\omega t}}{2j}$$

- Result

$$\frac{A}{2}e^{j\omega t} + \frac{A^*}{2}e^{-j\omega t} = a\cos\omega t + b\sin\omega t$$

Complex Notation (2)

- Do all the math with $Ae^{j\omega t}$
- Add the negative freqency part at the end
- Don't worry about trigonometric identities ever again
- Phase and Amplitude are in the Phasor

$$|a+jb| = \sqrt{a^2+b^2} = |A| \qquad \phi = \angle A = \arctan\frac{b}{a}$$

- This only works for linear systems
- Some people use A, and some use $A/2$; be careful.

Steady–State Sinusoids (Later)

Resistors

$v = iR$ $\qquad$ $\mathbf{V} = \mathbf{I}R$

Capacitors

$i = C\frac{dv}{dt}$ $\qquad$ $\mathbf{I} = j\omega C\mathbf{V}$ $\qquad$ $\mathbf{V} = \frac{1}{j\omega C}\mathbf{I}$

Inductors

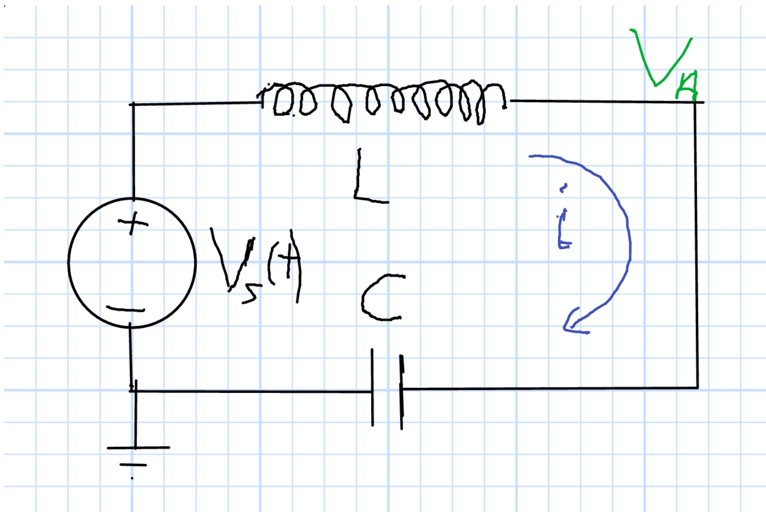
$v = L\frac{di}{dt}$ $\qquad$ $\mathbf{V} = j\omega L\mathbf{I}$

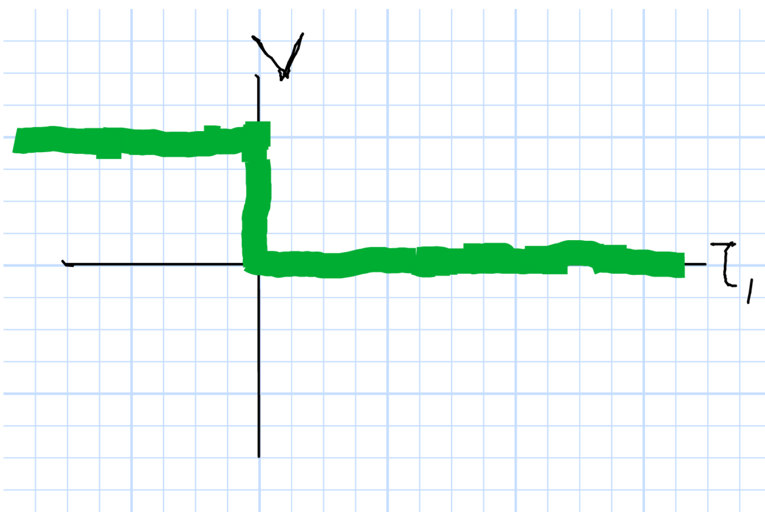
Easy? . . .

Easy!

Back to the LC Example

Exponential Solutions with Complex Exponents





- Remember

$$i(t) = Ke^{st}$$

$$s_1 = \sqrt{-1}\omega_0 \qquad s_2 = -\sqrt{-1}\omega_0$$

$$\omega_0 = \frac{1}{\sqrt{LC}}$$

- Solution

$$i(t) = Ke^{j\omega_0 t} + K^* e^{-j\omega_0 t}$$

- Voltage if $v_s = 0$

$$[v_s(t) - v_A(t)] = L\frac{di(t)}{dt}$$

$$v_A(t) = -jK\omega_0 e^{j\omega_0 t} + jK^*\omega_0 e^{-j\omega_0 t}$$

LC Solution

- From Previous Page

$$v_A(t) = -jK\omega_0 e^{j\omega_0 t} + jK^*\omega_0 e^{-j\omega_0 t}$$

- 2 Unknowns, Real and Imaginary Parts of K

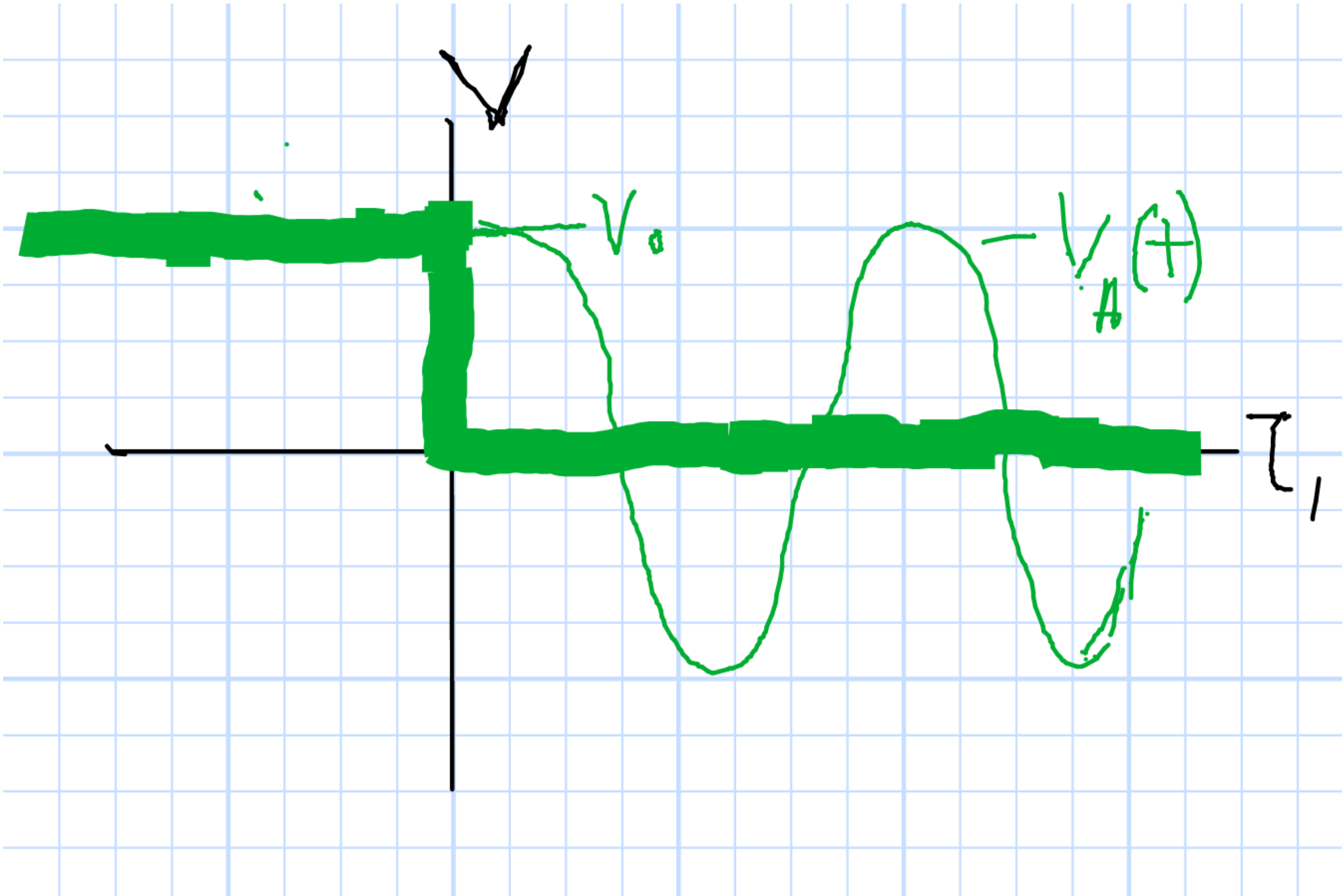
- v Continuous Across Capacitor (V_A)

- i Continuous Through Inductor (and Capacitor)

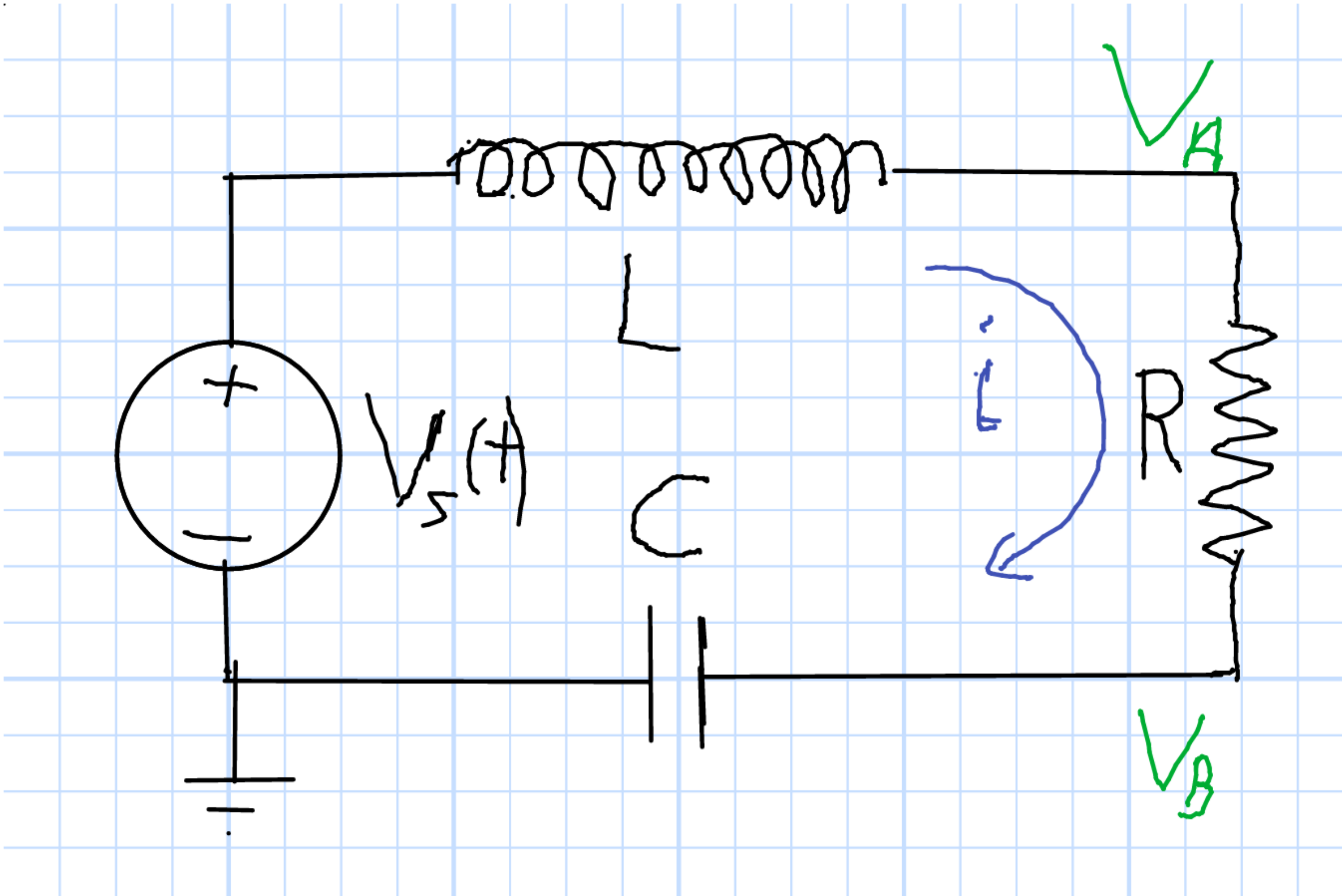
$$v_A\left(0^+\right) = v_s\left(0^-\right) \qquad \frac{dv_A}{dt}\,|\,0^+ = 0$$

$$v_A(t+) = v_s\left(0^-\right)\cos\omega_0 t$$

LC Result



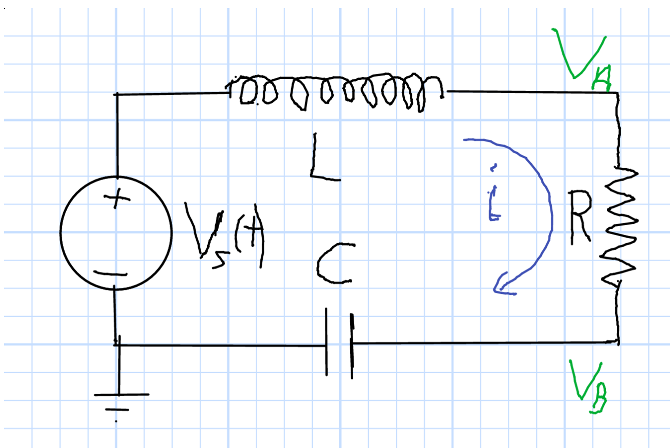
RLC Series Circuit



RLC Series Circuit Analysis

- KVL

$$L\frac{di(t)}{dt} + Ri(t) + \frac{1}{C}\int i(t)\,dt = v_s(t)$$



- Undamped Resonant Frequency $\omega_0 = \frac{1}{\sqrt{LC}}$
- Damping Coefficient $\alpha = \frac{R}{2L}$

- Differentiate

$$L\frac{d^2i(t)}{dt^2} + R\frac{di(t)}{dt} + \frac{1}{C}i(t) = \frac{dv_s(t)}{dt}$$

- Divide by L

$$\frac{d^2i(t)}{dt^2} + \frac{R}{L}\frac{di(t)}{dt} + \frac{1}{LC}i(t) = \frac{1}{L}\frac{dv_s(t)}{dt}$$

- Use Definitions, ω_0 and α

$$\frac{d^2i(t)}{dt^2} + 2\alpha\frac{di(t)}{dt} + \omega_0^2 i(t) = \frac{1}{L}\frac{dv_s(t)}{dt}$$

General Second–Order DE

- From Previous Page

$$\frac{d^2 i(t)}{dt^2} + 2\alpha \frac{di(t)}{dt} + \omega_0^2 i(t) = \frac{1}{L}\frac{dv_s(t)}{dt}$$

- Define Forcing Function, $f = \frac{dv_s}{dt}\frac{1}{L}$

$$\frac{d^2 i(t)}{dt^2} + 2\alpha \frac{di(t)}{dt} + \omega_0^2 i(t) = f(t)$$

- Particular (Forced) and Complementary Solutions
- DC: Forced Solution is Steady–State Solution, i_p
- Transient Solution, i_c for

$$\frac{d^2 i(t)}{dt^2} + 2\alpha \frac{di(t)}{dt} + \omega_0^2 i(t) = 0$$

Damping

- Transient Equation From Previous Page

$$\frac{d^2 i(t)}{dt^2} + 2\alpha \frac{di(t)}{dt} + \omega_0^2 i(t) = 0 \qquad i(t) = ke^{st}$$

- Characteristic Equation

$$s^2 i + 2\alpha s i + \omega_0^2 i = 0 \qquad s^2 + 2\alpha s + \omega_0^2 = 0$$

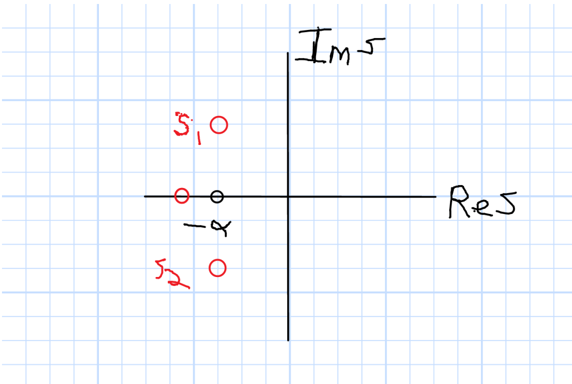
- Solutions

$$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2} \qquad s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2}$$

- Define Damping Ratio

"zeta" $\qquad \zeta = \frac{\alpha}{\omega_0}$

Damping in the Complex Plane



Underdamped

$\zeta < 1$

$\alpha < \omega_0$

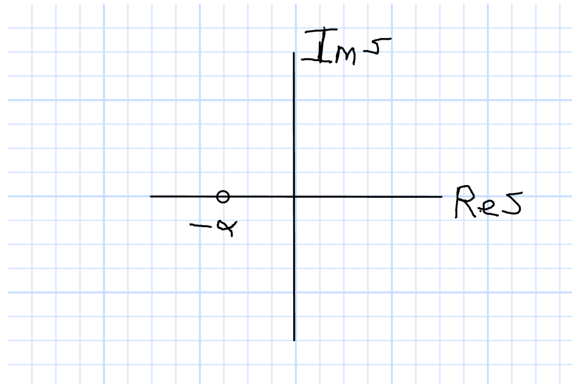
$\Re s < 0$

$\Im s$ Symmetric

Solutions:

Damped Sinusoids

Frequency Shifted



Critically Damped

$\zeta = 1$

$\alpha = \omega_0$

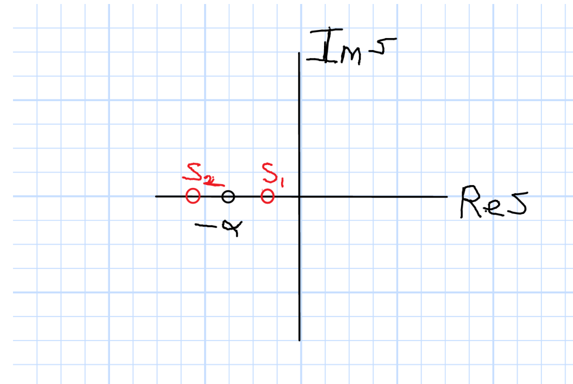
$\Re s < 0$

$\Im s = 0$

Degenerate

Exponential

$+e^{-t/\tau}$



Overdamped

$\zeta > 1$

$\alpha > \omega_0$

$\Re s < 0$

$\Im s = 0$

Exponentials

Underdamped (1)

- Solutions for s

$$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2} \qquad s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2} \qquad \alpha < \omega_0$$

- Solution Contains $\sqrt{\text{Negative}}$

$$s_1 = -\alpha + j\sqrt{\omega_0^2 - \alpha^2} \qquad s_2 = -\alpha - j\sqrt{\omega_0^2 - \alpha^2}$$

- Natural Frequency

$$\omega_n = \sqrt{\omega_0^2 - \alpha^2}$$

$$s_1 = -\alpha + j\omega_n \qquad s_2 = -\alpha - j\omega_n$$

- Solution

$$i\left(t\right) = Ke^{-\alpha + j\omega_n t} + K^* e^{-\alpha - j\omega_n t}$$

Underdamped (2)

- Previous Page (K Complex)

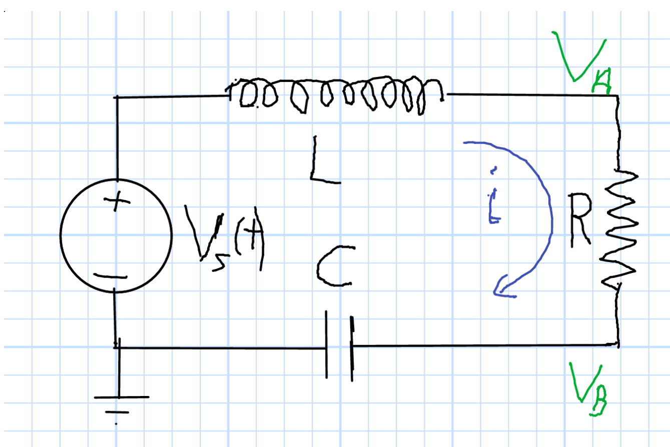
$$i\left(t\right) = Ke^{-\alpha+j\omega_n t} + K^* e^{-\alpha-j\omega_n t}$$

- Alternative Form (K_1,K_2 Real)

$$i\left(t\right) = K_1 e^{-\alpha t}\cos\omega_n t + K_2 e^{-\alpha t}\sin\omega_n t$$

- Add Particular Solution

$$i\left(t\right) = K_1 e^{-\alpha t}\cos\omega_n t + K_2 e^{-\alpha t}\sin\omega_n t + 0$$



- Initial Conditions $i\left(0+\right) = 0$

$$i\left(t\right) = K_2 e^{-\alpha t}\sin\omega_n t$$

$$K_2 = \frac{v_S\left(0^+\right)}{\omega_n L}$$

Example

- Let's Make $\omega_0 = 2\pi \times 5\text{kHz}$

$$\omega_0 = \frac{1}{\sqrt{LC}} \qquad L = \frac{1}{\sqrt{\omega_0^2 C}}$$

$$C = 1\mu\text{F} \quad \rightarrow \quad L = 1\text{mH}$$

- Choose Underdamped $\zeta = 0.1$

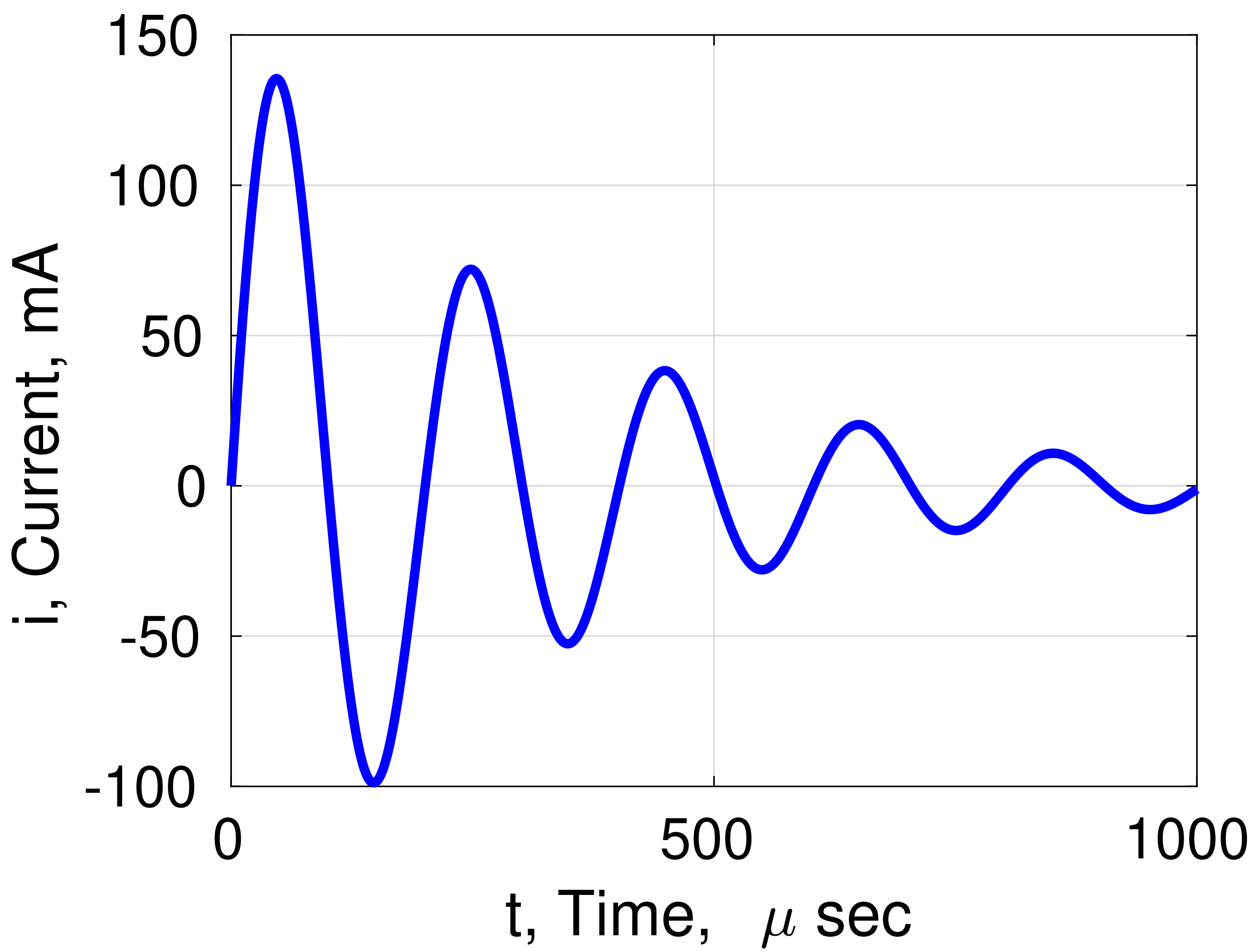
$$\alpha = \omega\zeta \qquad \alpha = \frac{R}{2L}$$

$$R = 2\alpha L = 6.4\Omega$$

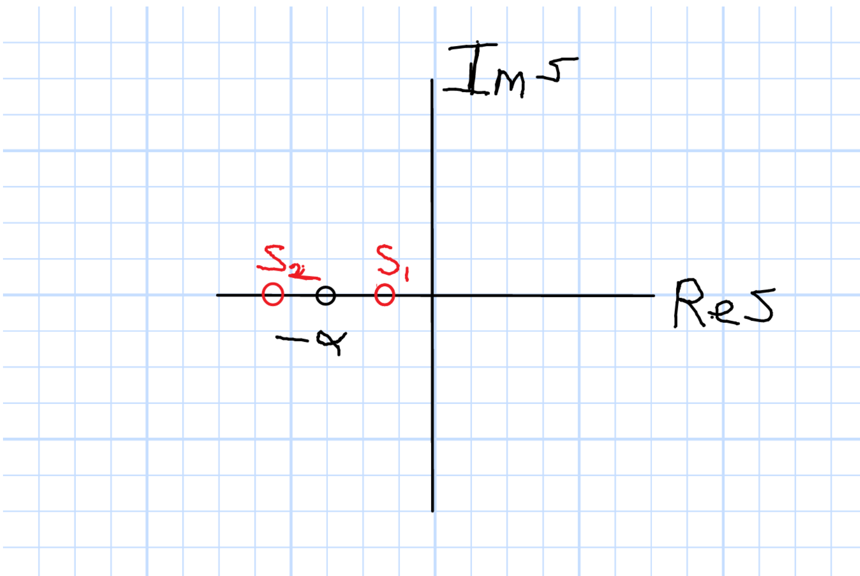
- Parameters

$$\frac{\omega_n}{2\pi} = 4975\text{Hz}$$

Underdamped Current



Overdamped Current



- Solutions for s

$$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2}$$

$$s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2}$$

$$\omega_0 < \alpha$$

- Solution Contains $\sqrt{\text{Positive}}$

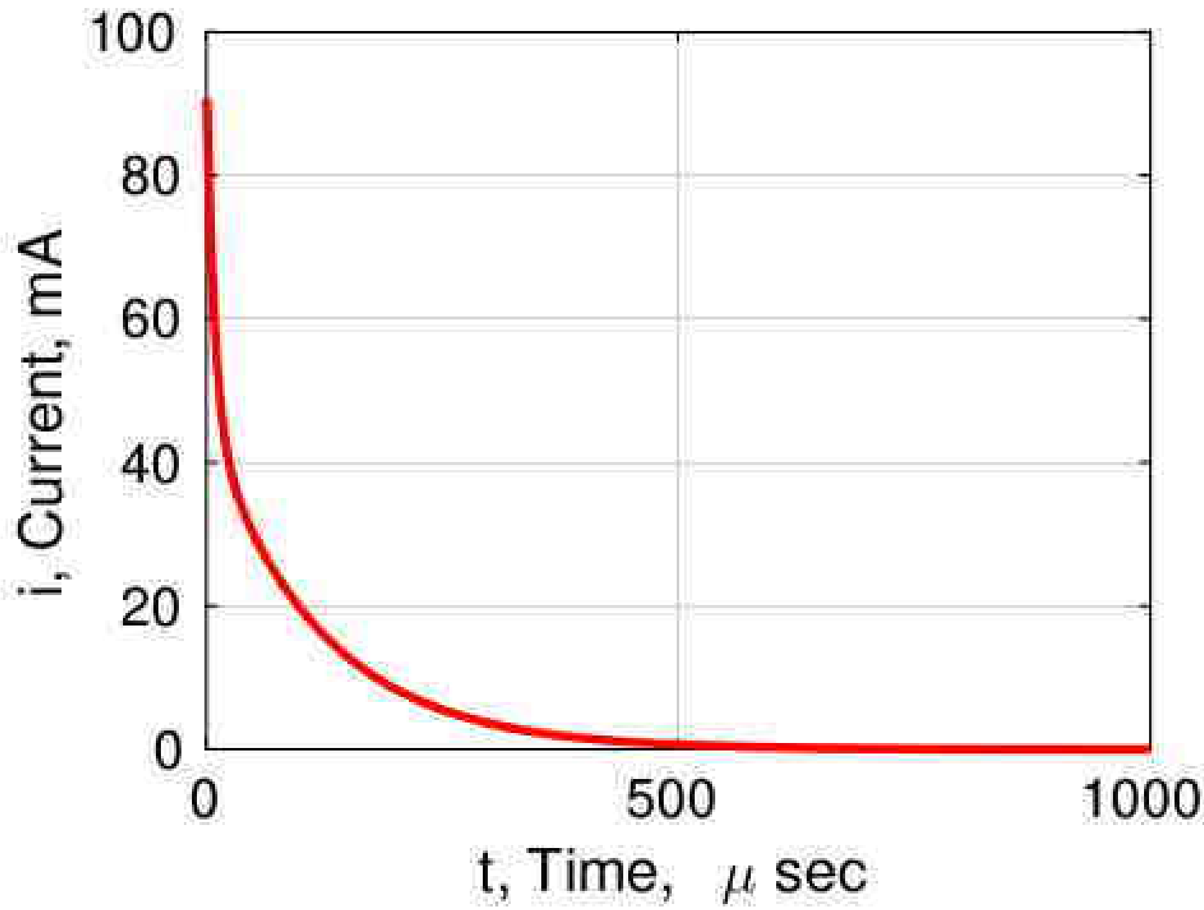
$$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2}$$

$$s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2}$$

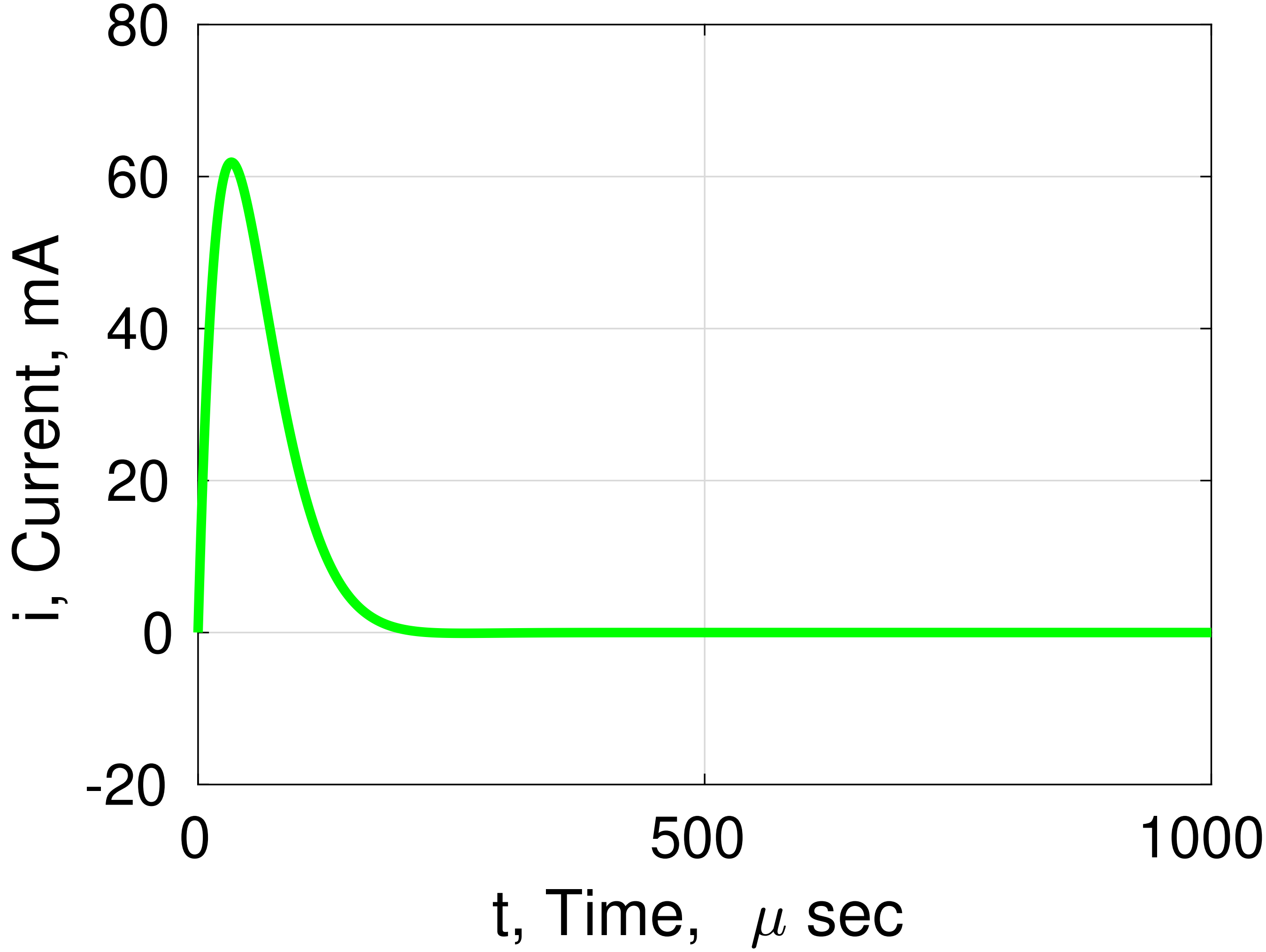
$$i(t) = K_1 e^{-\alpha + \sqrt{\alpha^2 - \omega_0^2}} + K_2 e^{-\alpha - \sqrt{\alpha^2 - \omega_0^2}}$$

$$i(t) = K_1 e^{(-1+0.87)\times 6.2\times 10^4} + K_2 e^{(-1-0.87)\times 6.2\times 10^4}$$

Overdamped Example



Critically Damped Example



Application

- Scanning Lidar
- Rapid Scan from Left to Right
  - Move the Laser Beam 10 Meters
  - Sample 1000 Times for 1 centimeter steps
- Quick Return From Right to Left
  - Time Constant τ

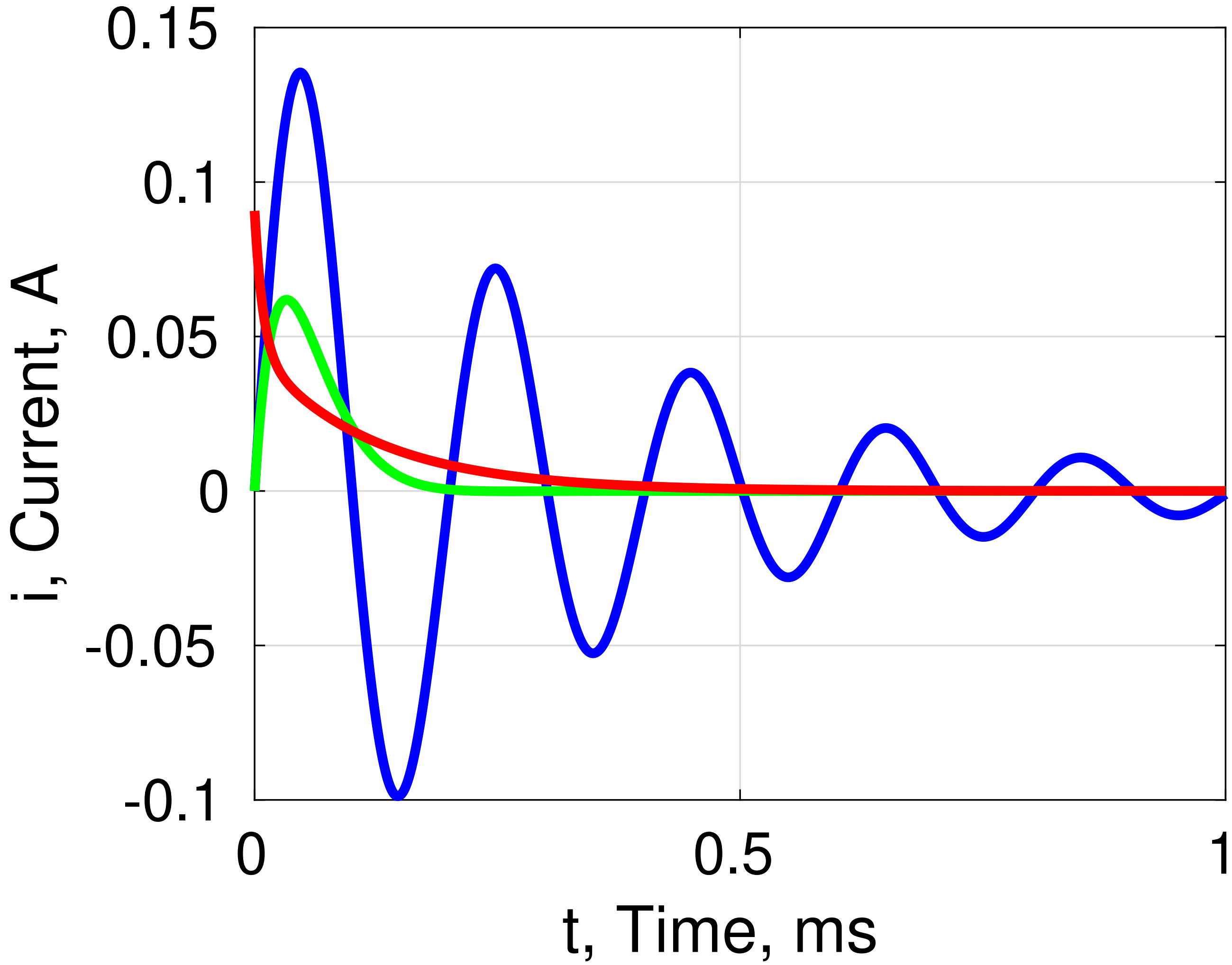
$$10\text{m} \times e^{-t/\tau} < 10^{-2}\text{m}$$

$$t > \tau \log_e \frac{10\text{m}}{10^{-2}\text{m}} \approx 7\tau = \frac{7}{\alpha}$$

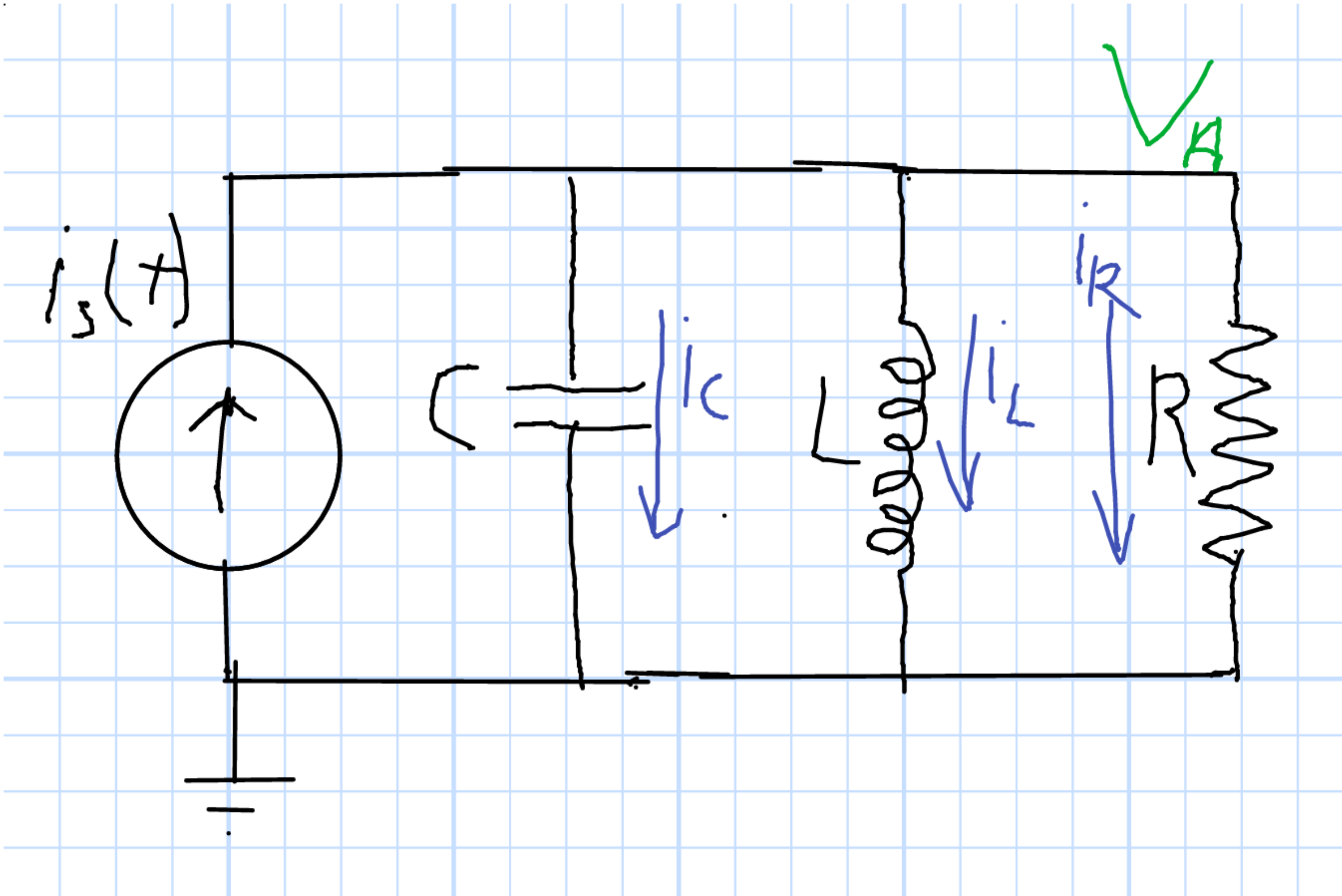
  - Critical Damping

$$\zeta = 1 \quad \rightarrow \quad \alpha = \omega_0 \qquad t > \frac{7}{\omega_0}$$

Three Damping Examples



RLC Parallel Circuit



$$C\frac{dv\left(t\right)}{dt}+\frac{1}{L}\int v\left(t\right)dt+\frac{1}{R}v\left(t\right)=i_s\left(t\right)$$

Equation for i_L

- From Previous Page

$$C\frac{dv(t)}{dt}+\frac{1}{L}\int v(t)\,dt+\frac{1}{R}v(t)=i_s(t)$$

- Inductor Equation

$$v(t)=L\frac{di_L(t)}{dt}$$

- Substitute

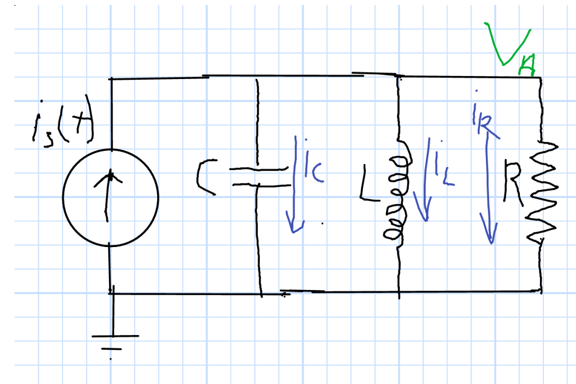
$$LC\frac{d^2i_L(t)}{dt^2}+i_L(t)+\frac{L}{R}\frac{di_L(t)}{dt}=i_s(t)$$

- Divide by LC, Reorder

$$\frac{d^2i_L(t)}{dt^2}+\frac{1}{RC}\frac{di_L(t)}{dt}+\frac{1}{LC}i_L(t)=i_s(t)$$

- Define ω_0, α

$$\frac{d^2i_L(t)}{dt^2}+2\alpha\frac{di_L(t)}{dt}+\omega_0^2i_L(t)=i_s(t)$$



$$\alpha=\frac{1}{2RC}$$

$$\omega_0=\frac{1}{\sqrt{LC}}$$

$$\zeta=\frac{\alpha}{\omega_0}$$

Solutions (1)

- From Previous Page

$$\frac{d^2 i_L(t)}{dt^2} + 2\alpha \frac{d i_L(t)}{dt} + \omega_0^2 i_L(t) = i_s(t)$$

- Step Current ($i_s\left(0^-\right) = 0$, $i_s\left(0^+\right) = 100\text{mA}$)

- Particular Solution (Steady State)

$$i_L(t) = i_s\left(0^+\right)$$

- Complementary Solution (Transient) $i_L(t) = Ke^{st}$

$$s^2 i_L(t) + 2\alpha s i_L(t) + \omega_0^2 i_L(t) = 0$$

Solutions (2)

- Complementary Solution From Previous Page

$$s^2 i_L(t) + 2\alpha s i_L(t) + \omega_0^2 i_L(t) + i_s(0) = 0$$

- Overdamped Solutions (Add Forced Solution)

$$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2} \qquad s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2}$$

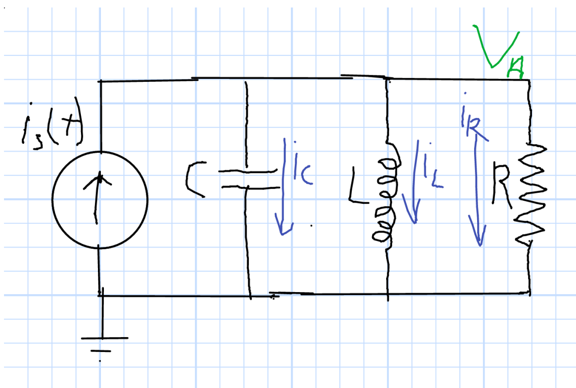
$$i_L(t) = K_1 e^{-s_1 t} + K_2 e^{-s_2 t} + i_s(0+)$$

- Underdamped Solutions $\omega_n = \sqrt{\omega_0^2 - \alpha^2}$

$$s_1 = -\alpha + j\omega_n \qquad s_2 = -\alpha - j\omega_n$$

$$i(t) = K_1 e^{-\alpha t}\cos\omega_n t + K_2 e^{-\alpha t}\sin\omega_n t + i_s(0+)$$

Initial Conditions



- Inductor Requires

$$i_L\left(0^+\right) = i_L\left(0^-\right) = 0$$

- Capacitor Requires v_A Continuous

$$\frac{di_L}{dt} = 0 \text{ at } t = 0^+$$

- Underdamped Equation from Previous Page

$$i\left(t\right) = K_1 e^{-\alpha t} \cos \omega_n t + K_2 e^{-\alpha t} \sin \omega_n t + i_s\left(0+\right)$$

- Initial Current

$$i\left(0\right) = K_1 + i_s\left(0+\right) = 0$$

- $di_L/dt = 0$ at $t = 0$ (d/dt by Product Rule)

$$-\alpha K_1 + \omega_n K_2 = 0$$

Plots

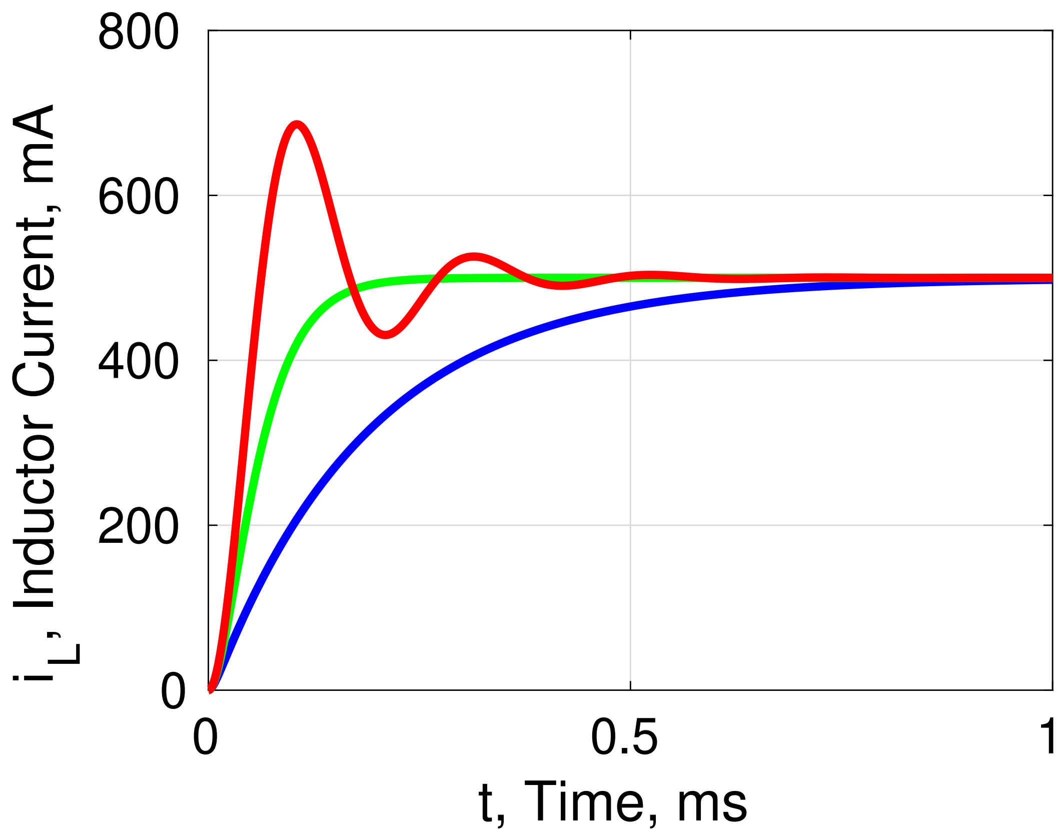
$$\omega_0 = 2\pi \times 5\text{kHz}$$

$$C = 1\mu\text{F}$$

$$L = 1\text{mH}$$

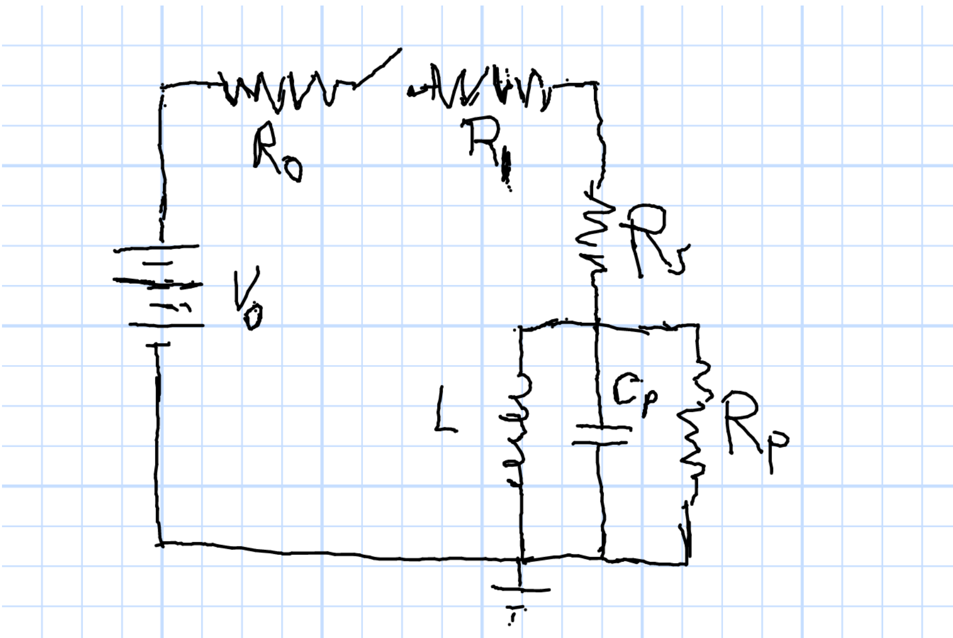
$$\alpha = \omega_0\zeta \qquad \alpha = \frac{1}{2RC}$$

$\zeta = 0$	$R \rightarrow \infty$
$\zeta = 0.3$	$R = 53.0\Omega$
$\zeta = 1$	$R = 15.9\Omega$
$\zeta = 3$	$R = 5.3\Omega$



Application: Magnet

Source and Magnet Coil



Norton Equivalent