Circuits and Signals: Biomedical Applications
Week 7

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Week 7 Agenda:

- Time–Varying Systems
- Capacitors
- Inductors
- Differential Equations
- Steady State and Transient Solutions

Big Picture

Devices

Resistors	Capacitors	Inductors
$v = iR$	$v = \frac{1}{C}\int i\ dt$ $i = C\frac{dv}{dt}$	$i = \frac{1}{L}\int v\ dt$ $v = L\frac{di}{dt}$
R in Ohms	C in Farads	L in Henries
	Voltage Continuous	Current Continuous
	Open to DC	Short to DC

Circuits

RC or RL	LC	RLC
First Order DE	Second Order DE	2nd with Loss
Negative Exponentials	Sinusoids	Lossy Sinusoids

We can do interesting things with time–varying sources.

Differential Equations

$i = C\frac{dv}{dt}$	$v = iR$	$v = L\frac{di}{dt}$	KCL, KVL, *etc.*

- Differential Equation

$$a\frac{d^2z}{dt^2} + b\frac{dz}{dt} + cz + d = 0$$

- Steady State

$$cz + d = 0 = \text{constant}$$

- Transient: (Mostly Switches) Solve DE & BC

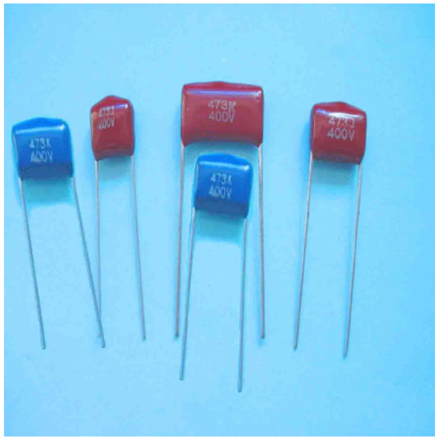
- Steady–State Sinusoids: $\frac{d|z|e^{j\omega t}}{dt} = j\omega|z|e^{j\omega t}$

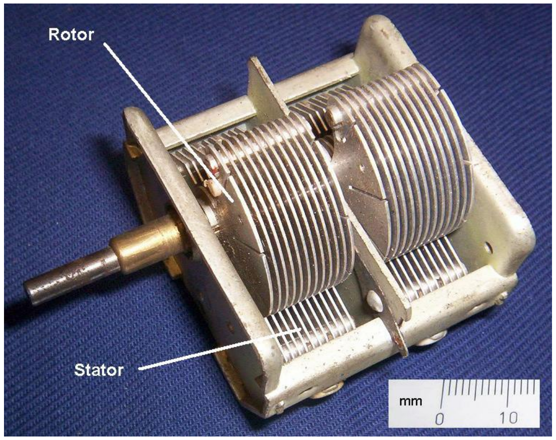
$$-a\omega^2 z + jb\omega z + cz + d = 0$$

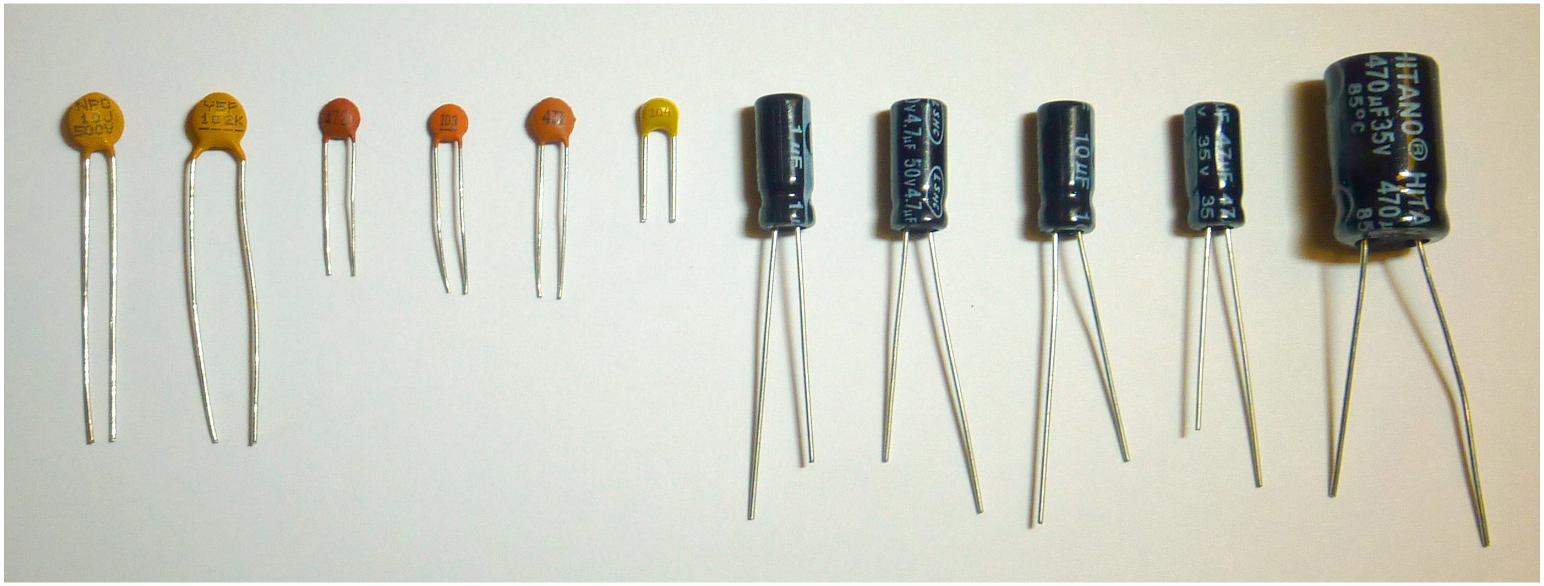
Agenda: Capacitors

- Physical Concepts
- Symbols
- i–v Behavior
- Fabrication
- Power and Energy
- Parallel and Series Combinations
- Steady–State Solutions
- Charge and Discharge

Capacitors (1)

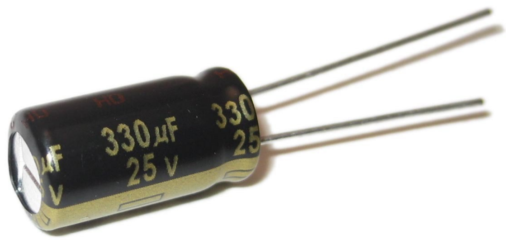






Capacitors (2)

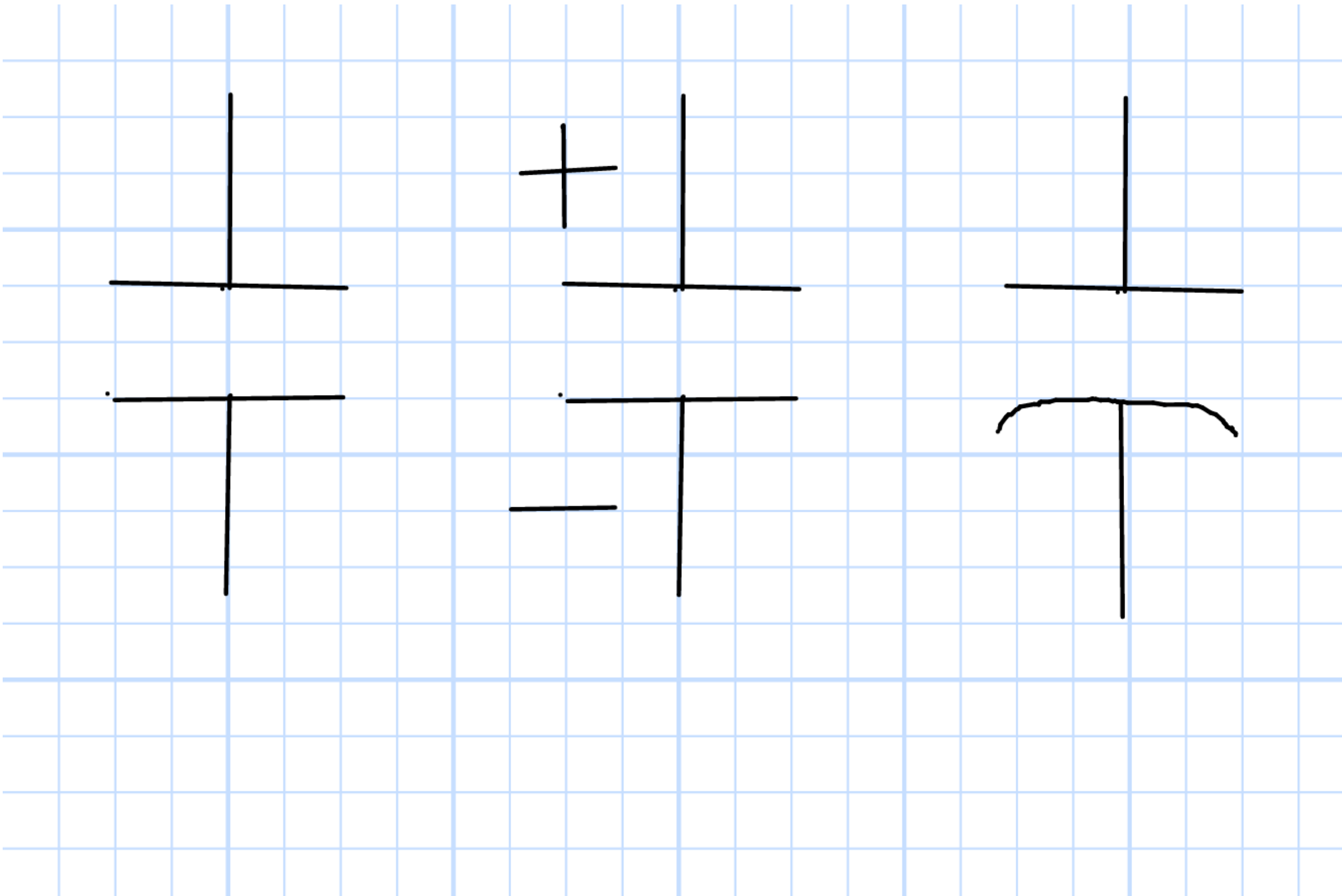
Electrolytics



Big Capacitors

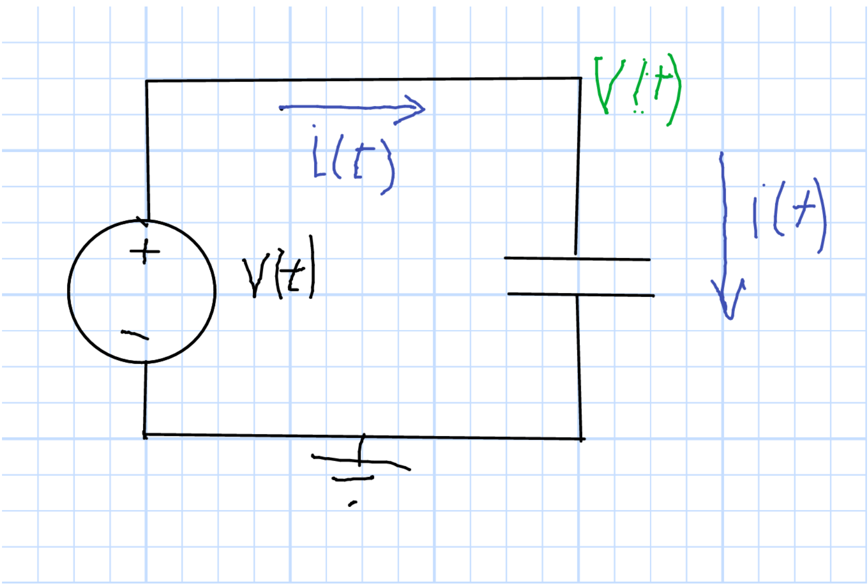
Principal Specifications: Capacitance (Farads), Maximum Voltage

Symbols



Equations

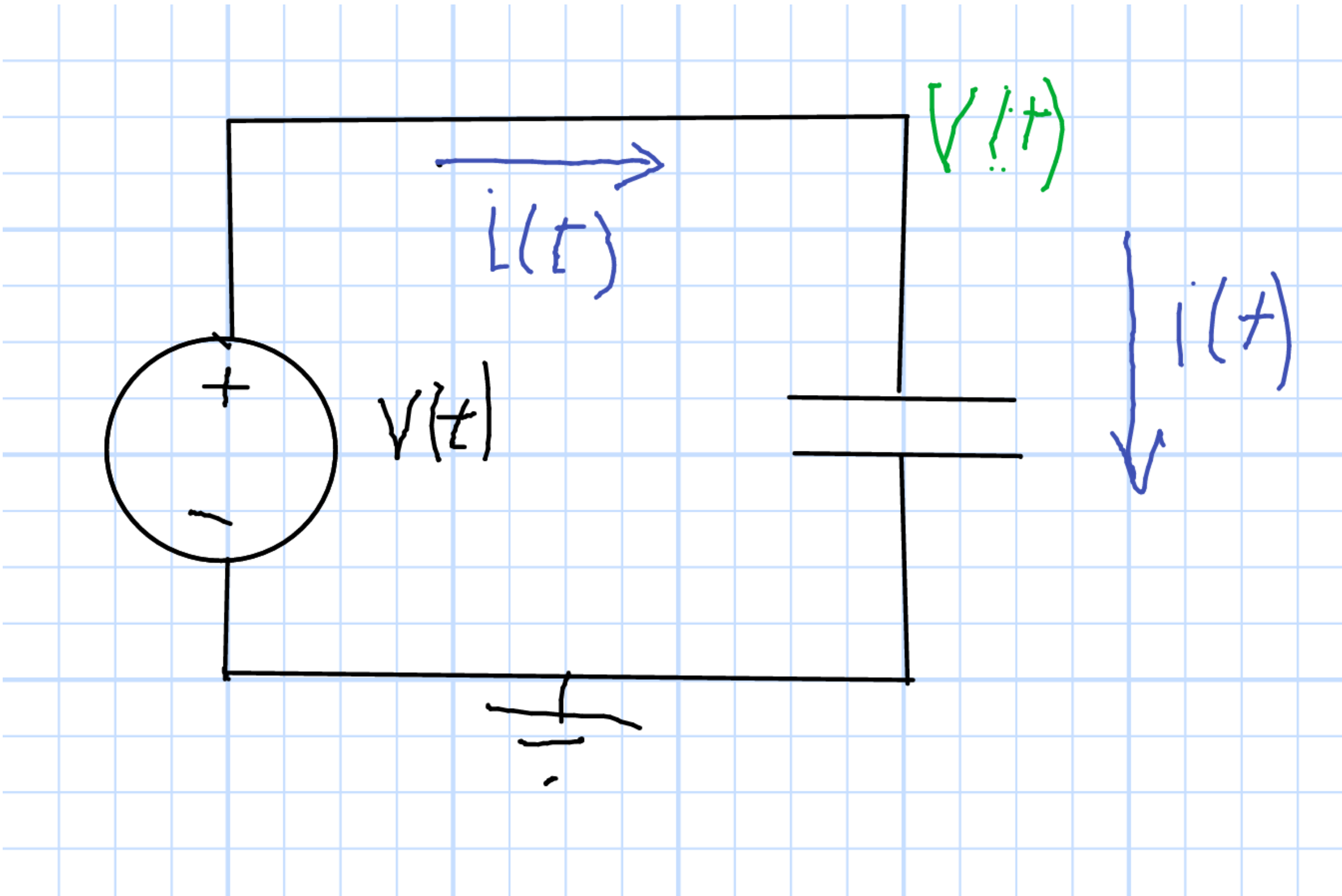
- Charge and Voltage:
  $q = Cv$
- Charge and Current:
  $i = \frac{dq}{dt}$
- Current and Voltage:
  $i = C\frac{dv}{dt}$
- Voltage and Charge:
  $v = \frac{q}{C}$
- Current and Charge:
  $q(t) = \int i(t)dt$
- Voltage and Current:
  $v(t) = \int \frac{i}{C}dt$
- Electrons:
  $n = \frac{Cv}{e}$



$$\frac{dv}{dt} \to \infty : \qquad i \to \infty$$

$$\frac{dv}{dt} \to 0 : \qquad i \to 0$$

Voltage Source

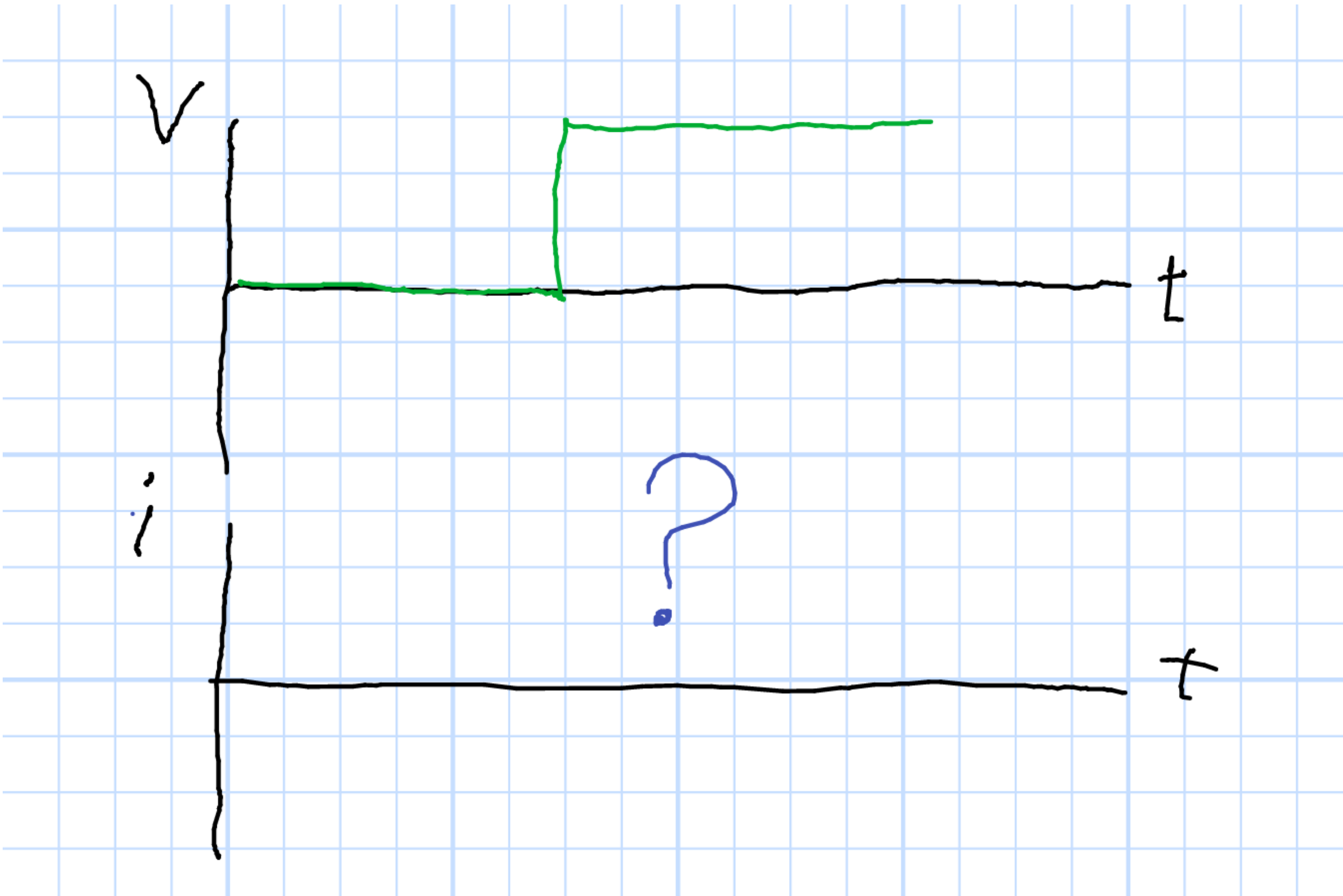


$$i(t) = C\frac{dv(t)}{dt}$$

Example

$$i(t) = C\frac{dv(t)}{dt}$$

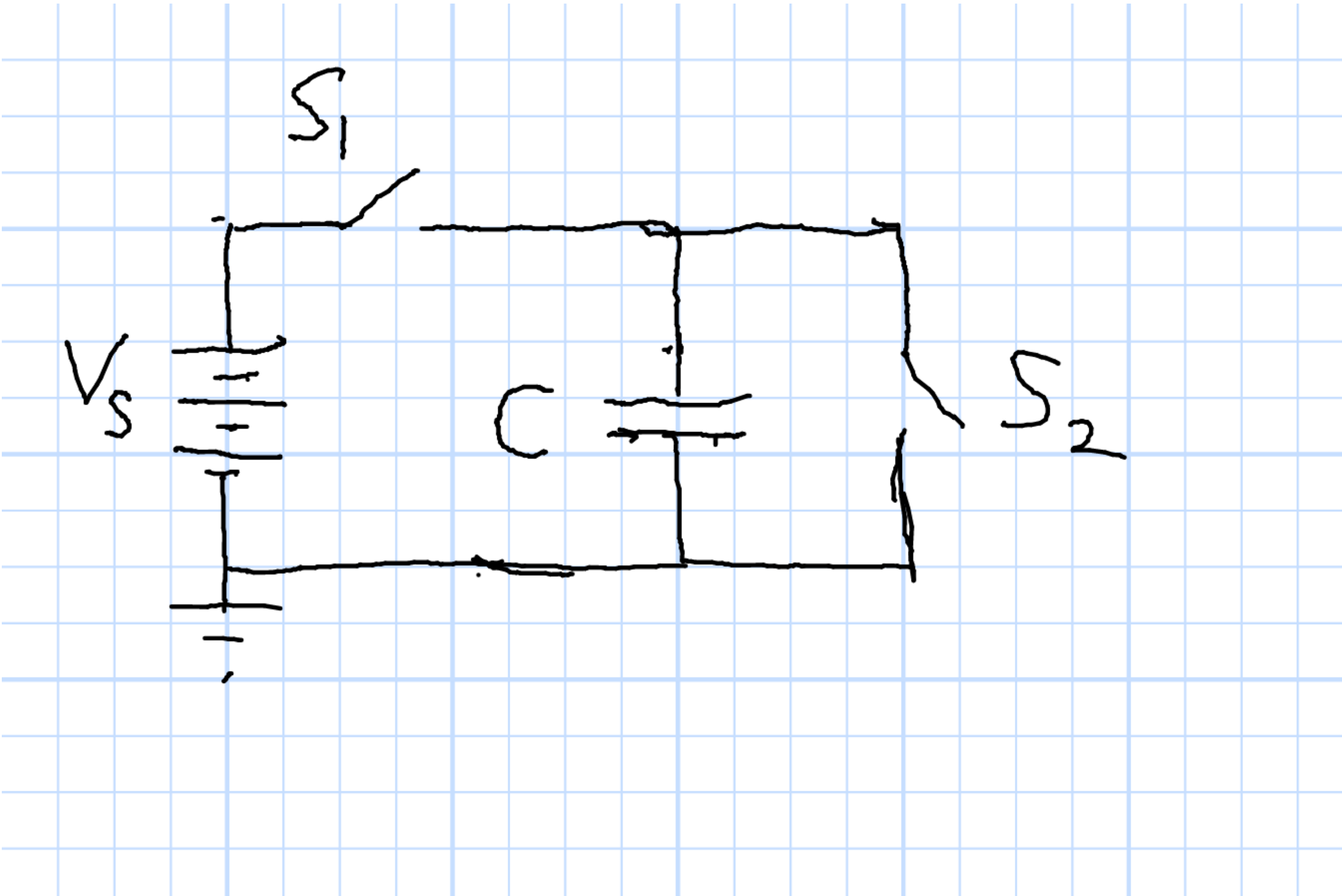
What Will Happen?



$$i(t) = C\frac{dv(t)}{dt}$$

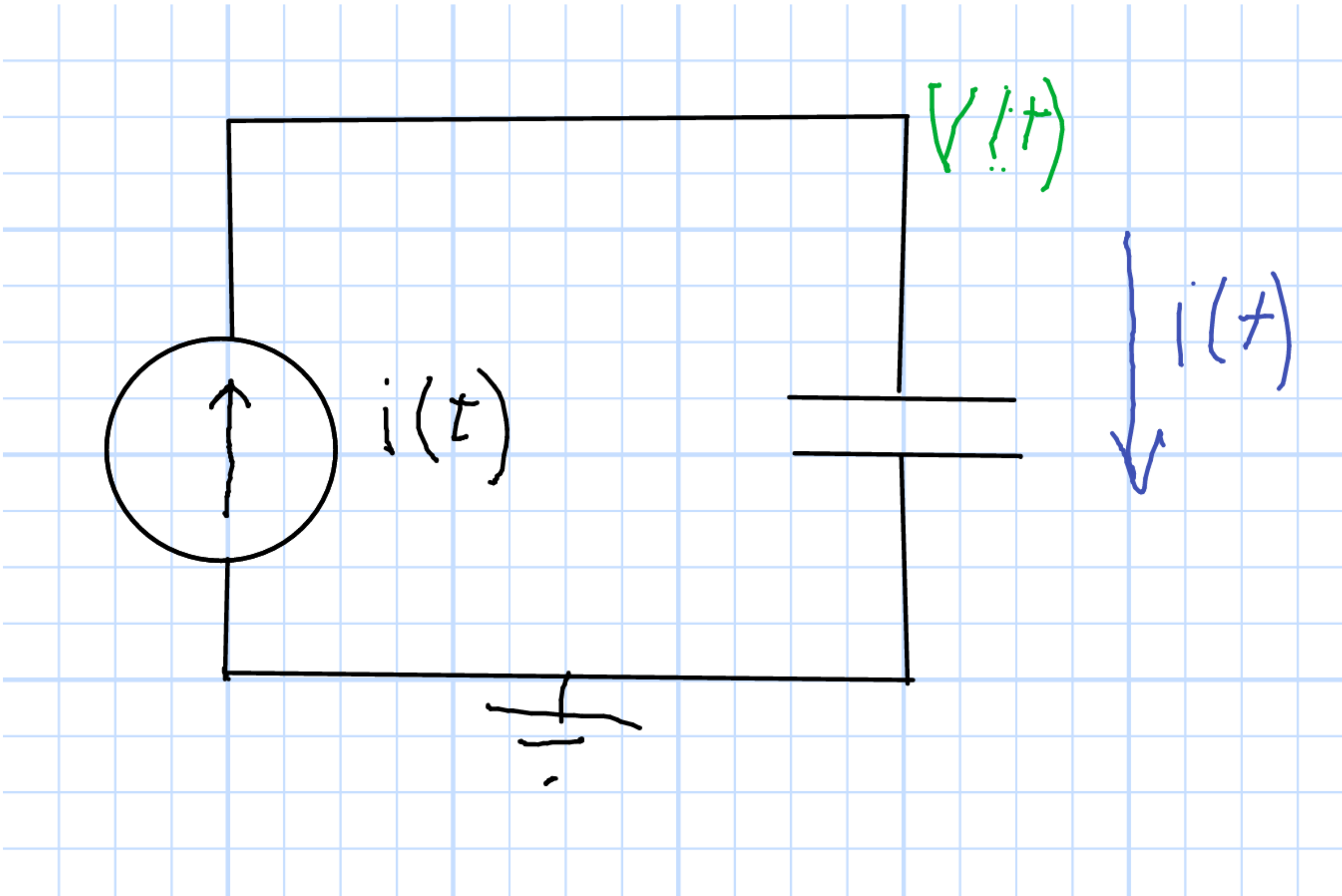
finite i $\rightarrow$ Voltage Continuous in Time

What Will Happen?



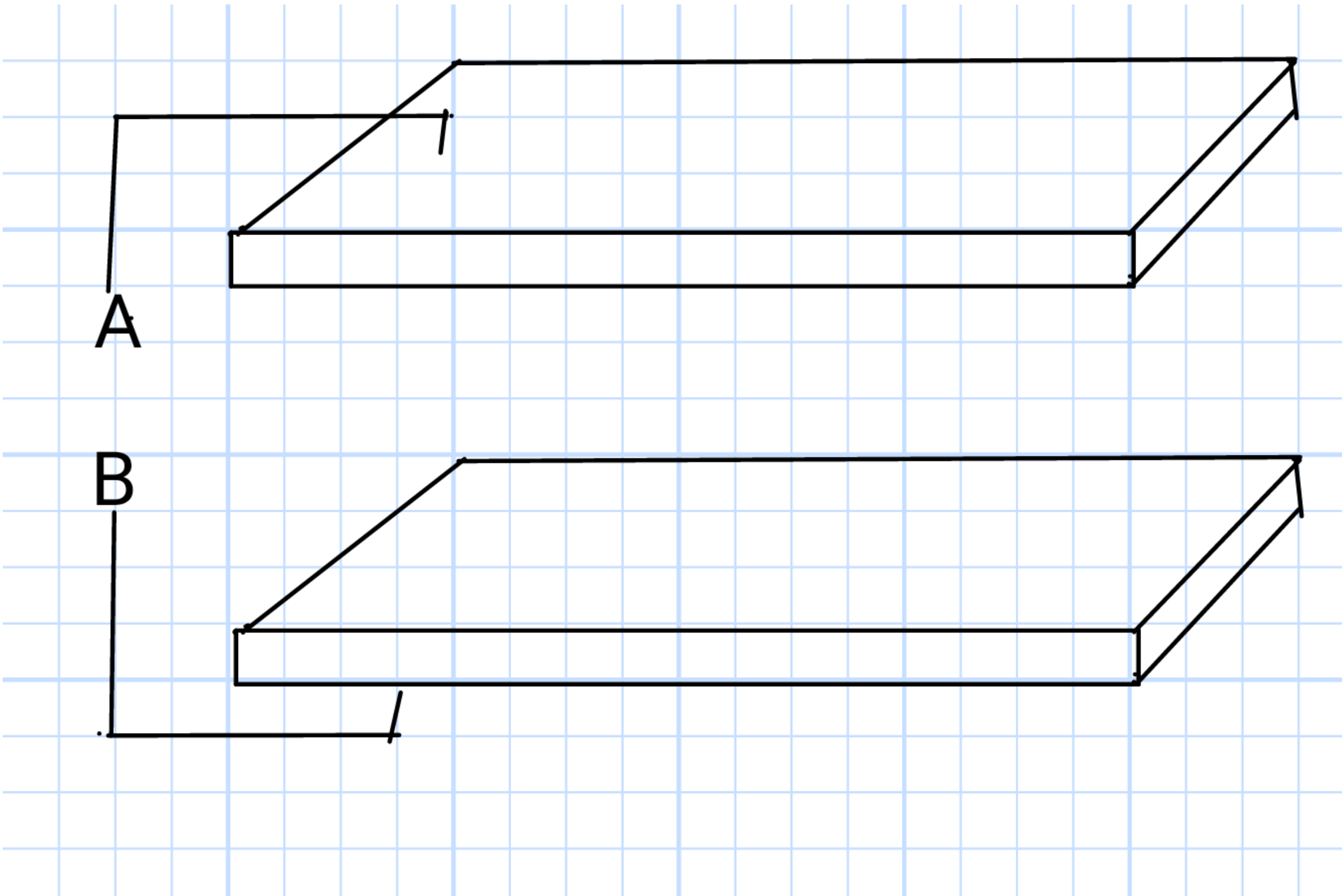
(1) Close S1 (2) Open S1 (3) Close S2

Current Source



$$v(t) = \frac{1}{C}\int_{t_0}^{t} i(t)\,dt + v(t_0)$$

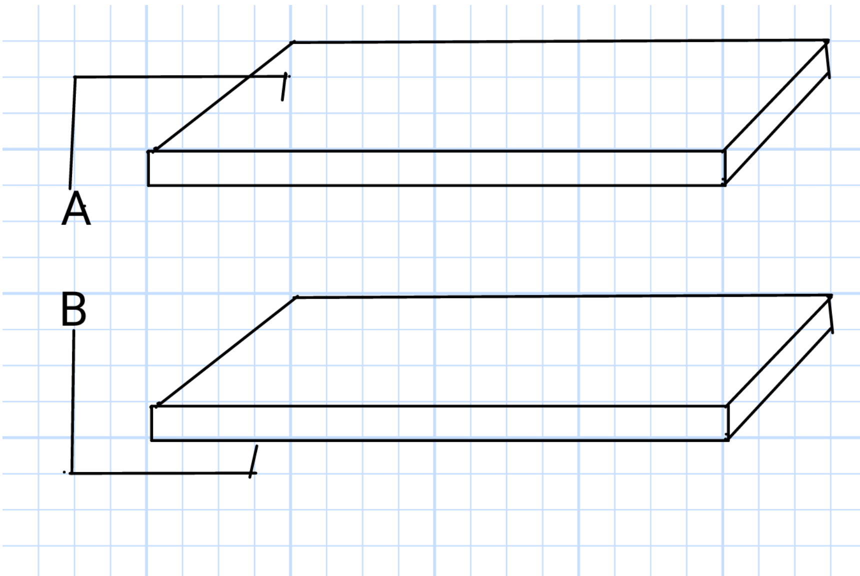
Fabrication



$$C = \frac{\epsilon A}{d}$$

ϵ is the Dielectric Constant

Equations



$$C = \frac{\epsilon A}{d}$$

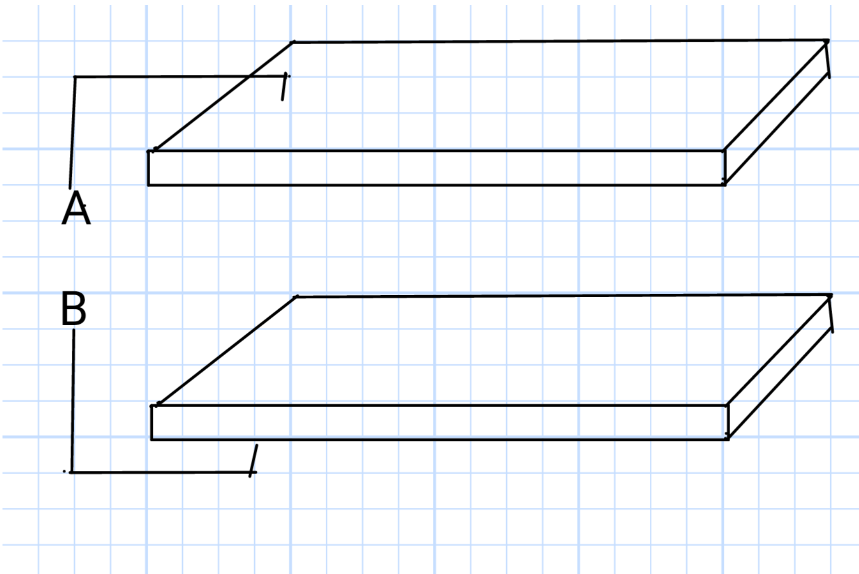
$$\epsilon = \epsilon_r \epsilon_0$$

$$\epsilon_0 = 8.85 \times 10^{-12}\mathrm{F/m}$$

Useful Term: Relative dielectric constant, ϵ_r

$\epsilon_r = 1$ for vacuum. Pretty close for air.

Example



$$\frac{\text{High Voltage}}{\text{Small } d} \rightarrow \text{Breakdown}$$

Dry Air at Sea Level:
$\approx$ 30kV/cm*

Glass: $\approx$ 100kV/cm

$$C = \frac{\epsilon A}{d} = 10\mu\text{f}$$

$\epsilon = 3.9\epsilon_0$ $\quad$ SiO_2

$$\frac{A}{d} = 3 \times 10^5\text{m}$$

$d = 10\mu\text{m}$ $\quad$ $A = 3\text{m}$

Interleave, Roll, or otherwise work to get more A
Use high ϵ_r
Use high Breakdown Voltage

Power and Energy

Power

$$p(t) = v(t)\, i(t) \qquad\qquad p(t) = v(t)\, C \frac{dv(t)}{dt}$$

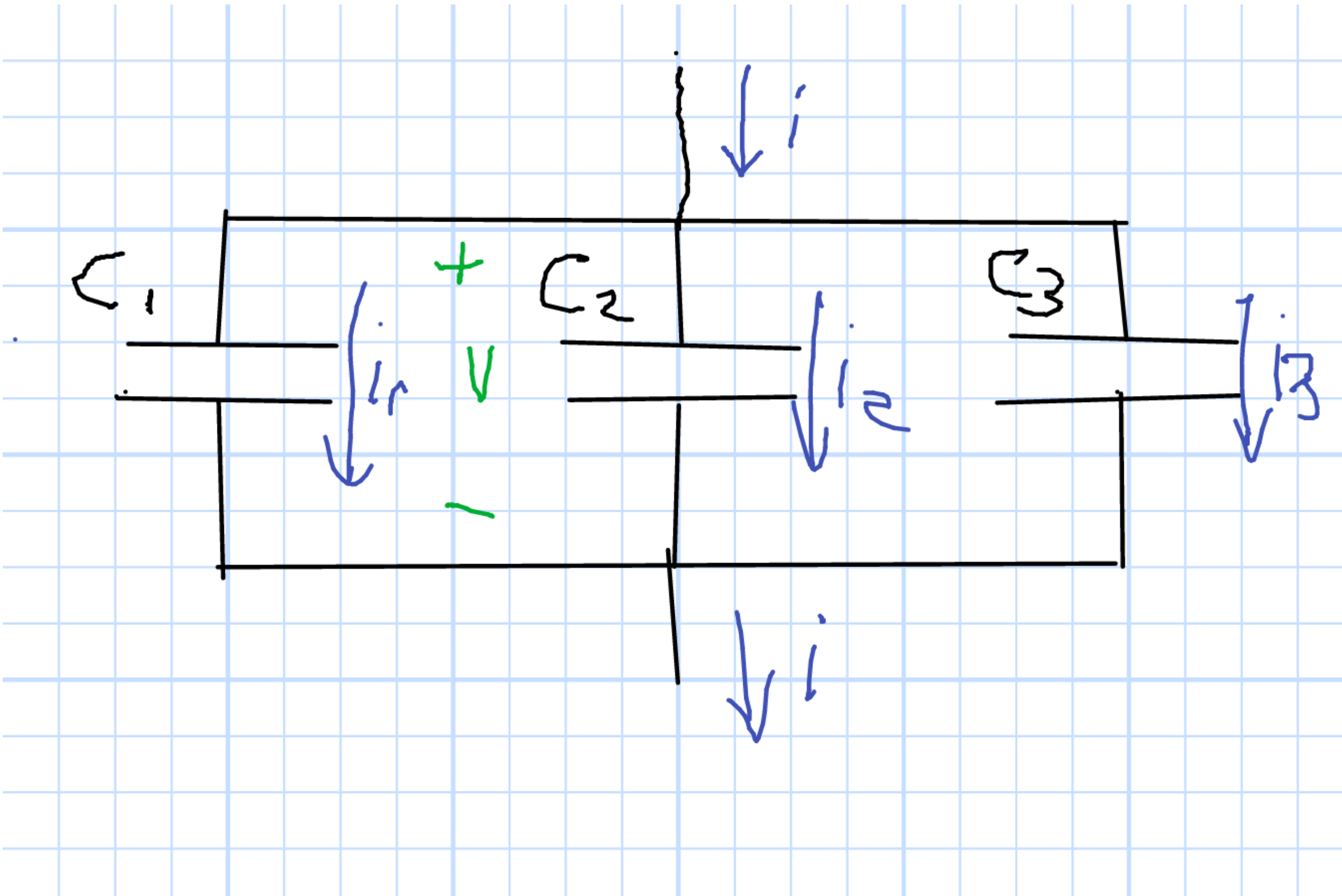
Energy

$$w = \int p(t)\, dt = \frac{v^2 C}{2}$$

Example

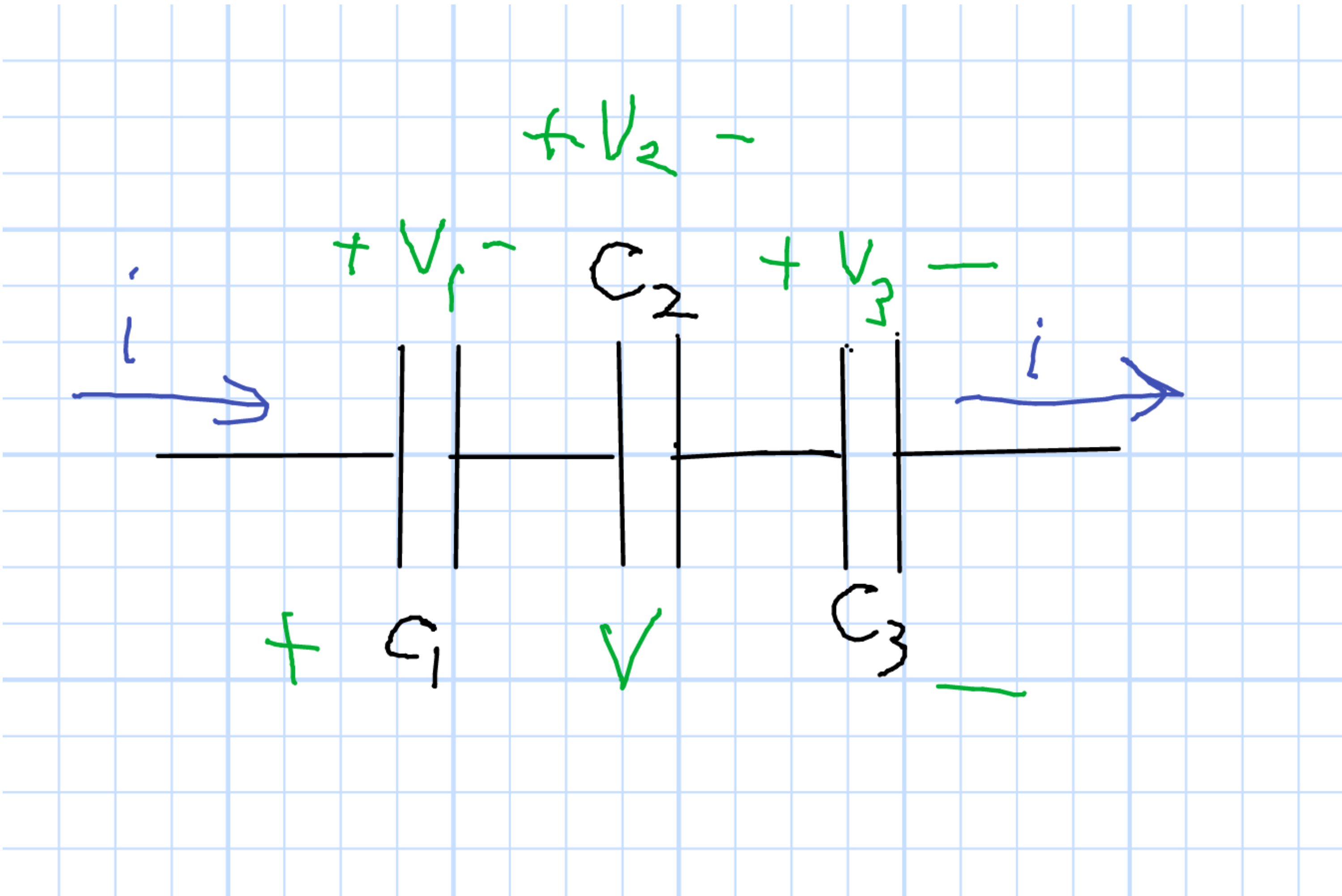
$$w = \frac{(100\text{V})^2\, 100\mu\text{F}}{2} = 500\text{mJ}$$

Parallel Combinations



$$i_n = C_n \frac{dv_n}{dt} \qquad i = \sum i_n \qquad C = \sum C_n$$

Series Combinations

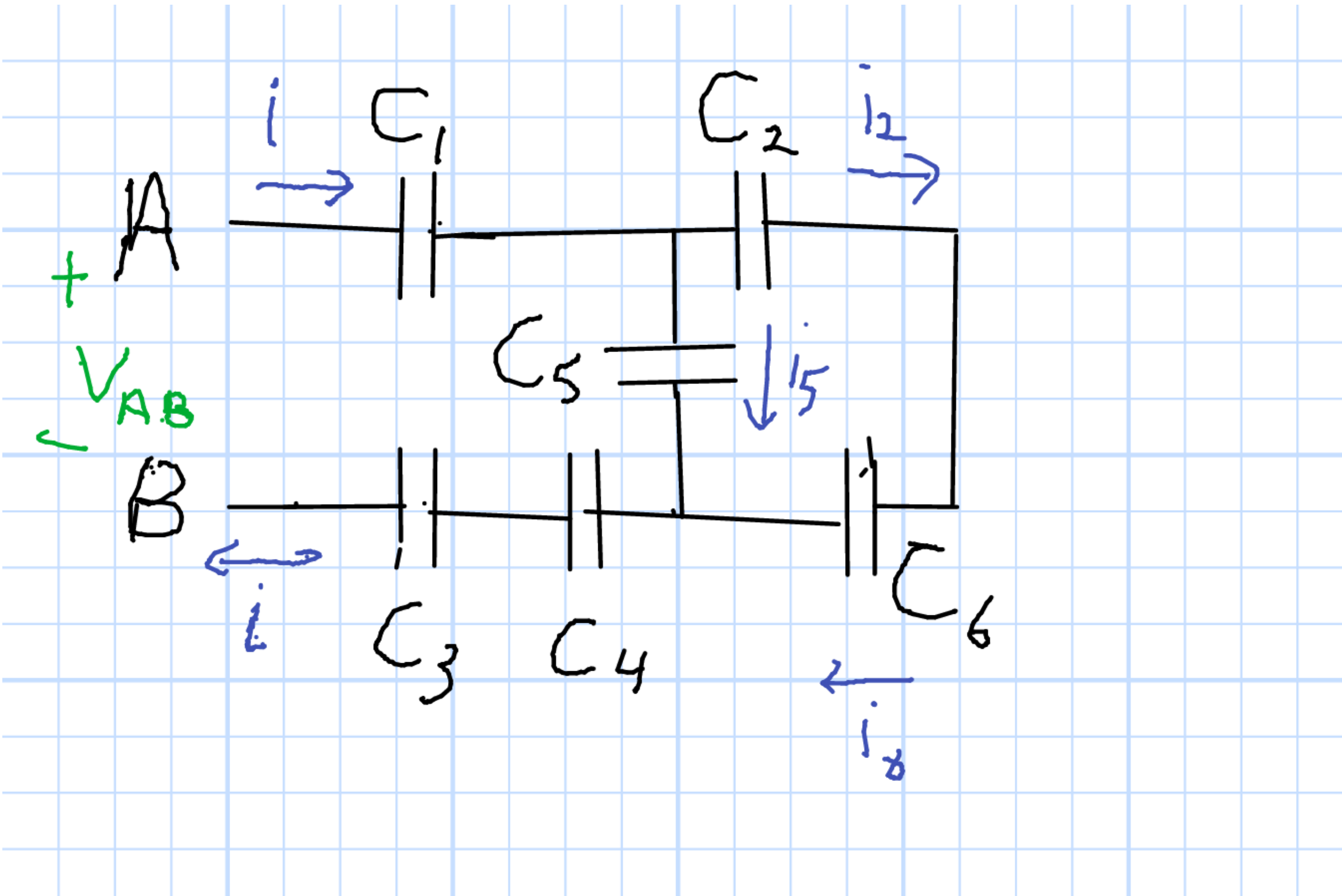


$$\frac{i}{C_n}=\frac{dv_n}{dt} \qquad \frac{dv}{dt}=\sum\frac{dv_n}{dt} \qquad \frac{1}{C}=\sum\frac{1}{C_n}$$

Parallel/Series Summary

	Series	Parallel
Voltage Sources	$v = \sum v_n$	Contradictory
Current Sources	Contradictory	$i = \sum i_n$
Resistors	$R = \sum R_n$	$\frac{1}{R} = \sum \frac{1}{R_n}$
Capacitors	$\frac{1}{C} = \sum \frac{1}{C_n}$	$C = \sum C_n$

Example Problem

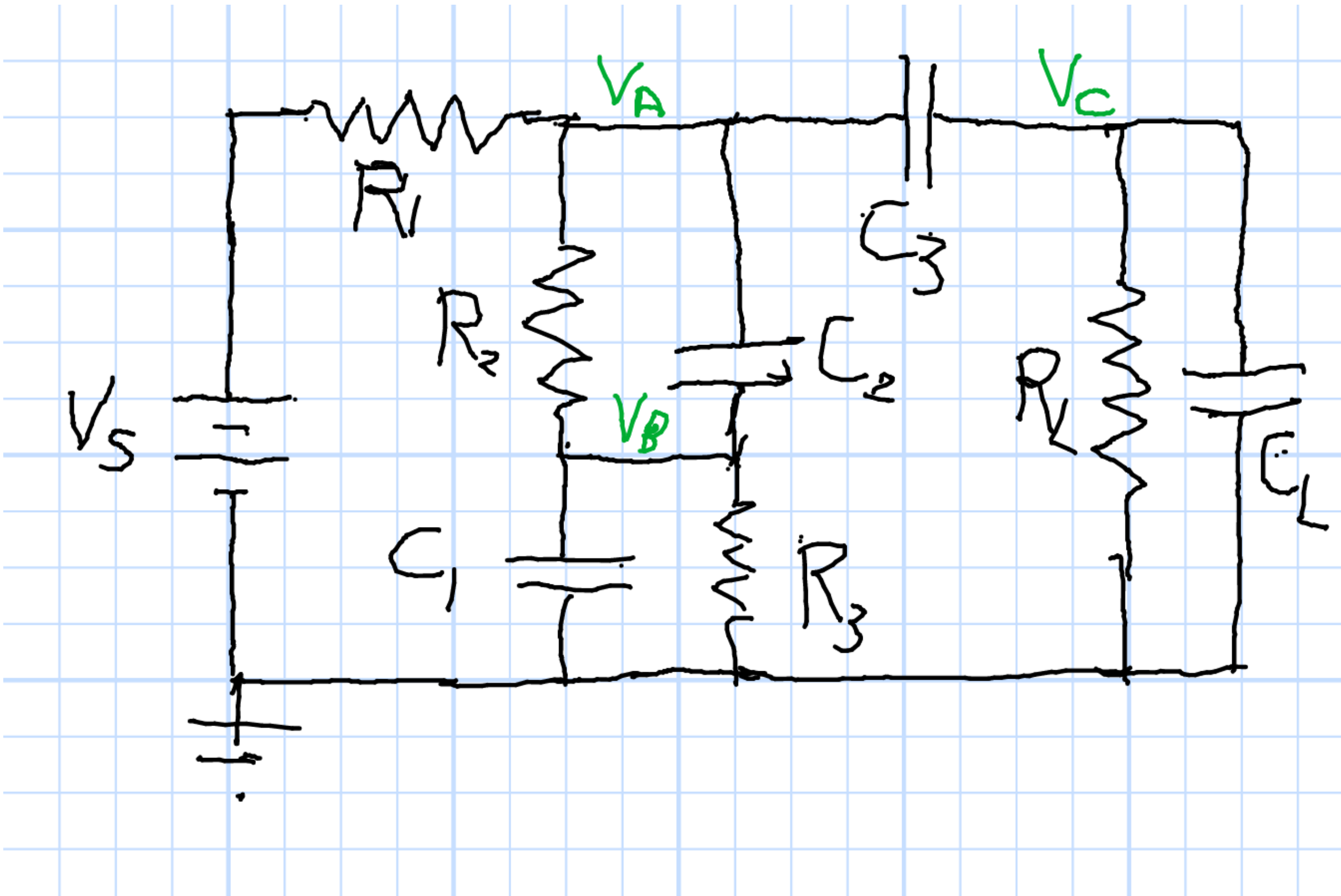


$C_{1:6} = 1\mu\text{F}$ $\qquad$ $C_{AB} = ?$

Steady State

- DC Sources, Resistors, Capacitors (and later Inductors)
- $t \to \infty$ with No Non–DC Sources
- $\frac{d(\text{anything})}{dt} \to 0$
- Specifically $\frac{dv_{capacitor}}{dt} \to 0$
- Therefore $i_{capacitor} = 0$
- Treat Capacitors as Open and Solve

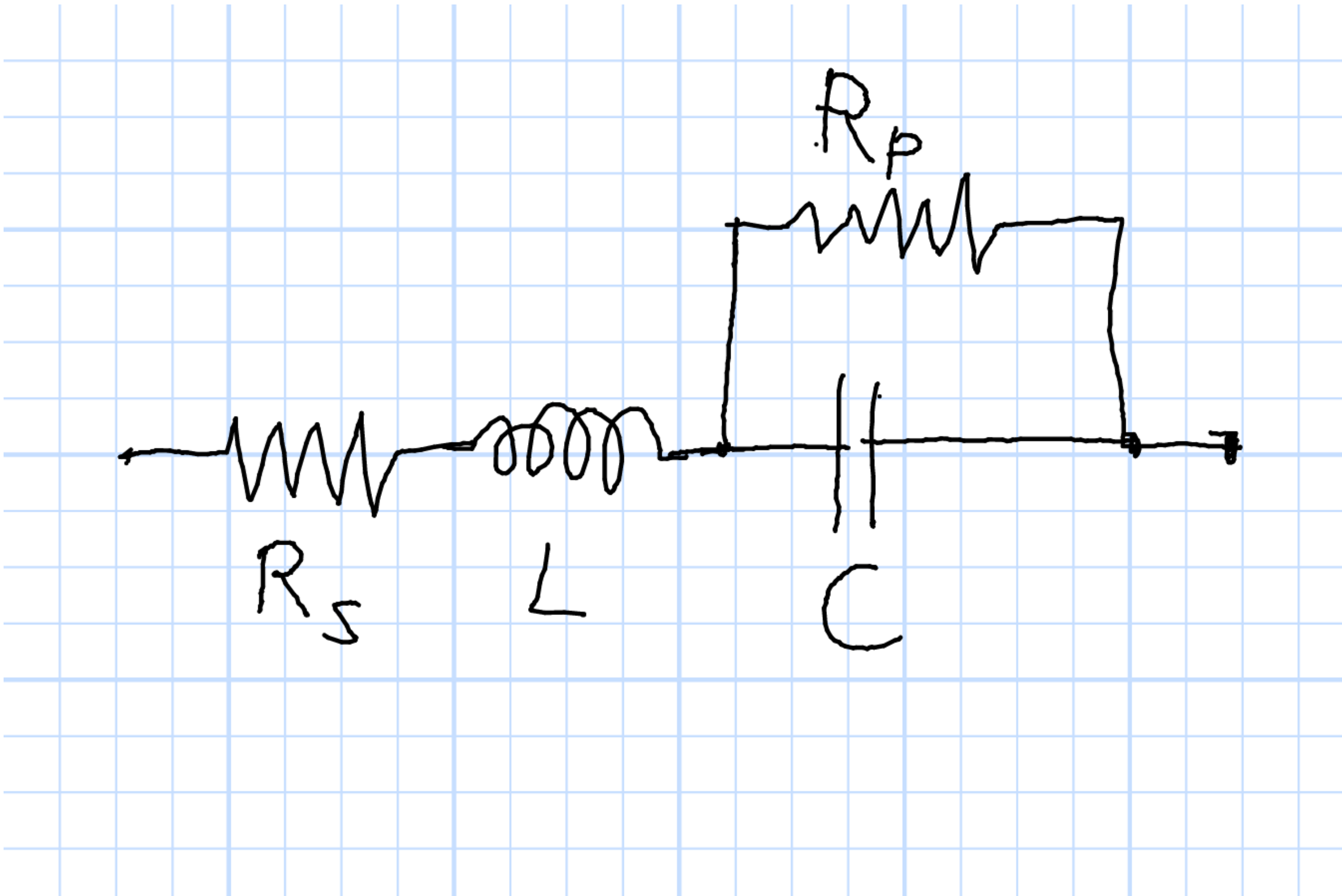
Steady–State Example



Turn on V_s at $t = 0$, Wait a Long Time,

$v_A =?$ $v_B =?$ $v_C =?$ $i_L =?$

Real Capacitors



R_s, L Low. R_p High.

Don't Try this at Home, Kids!

High–Voltage Capacitor:

$v = 10\text{kV} \qquad C = 10,000\mu\text{F} \qquad R_p = 10\text{G}\Omega$

Discharge Time at Constant Current:

$$q = Cv = 10 \text{ Coulombs}$$

$$i\,(0) = \frac{v}{R_P} = 1\mu\text{A}$$

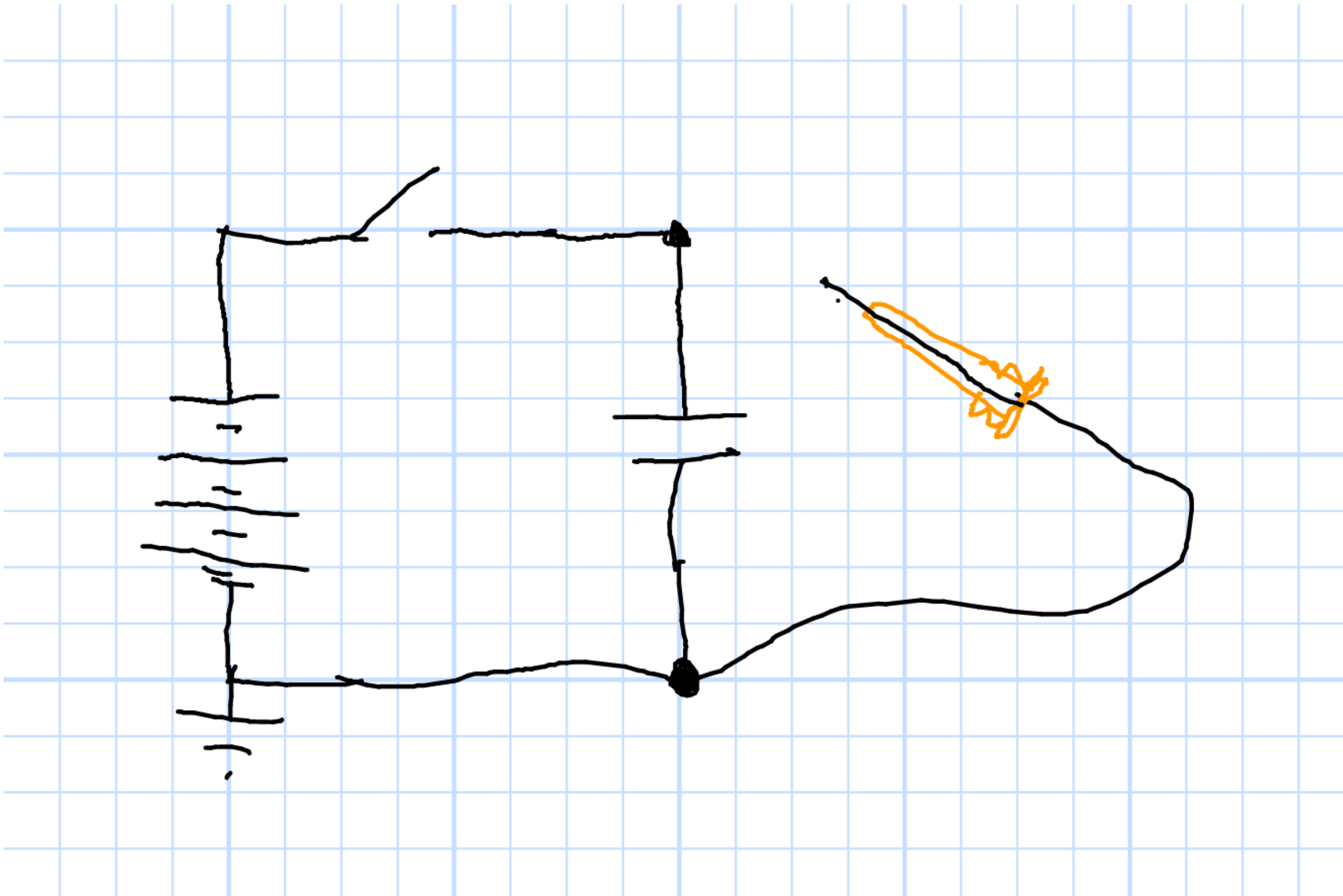
$$t = \frac{q}{i} = 10^7\text{sec} = 116 \text{ Days}$$

Exponential decay will be slower. . .

Unless you put a low resistor across it.

Then very bad things can happen.

"Shorting Bar"



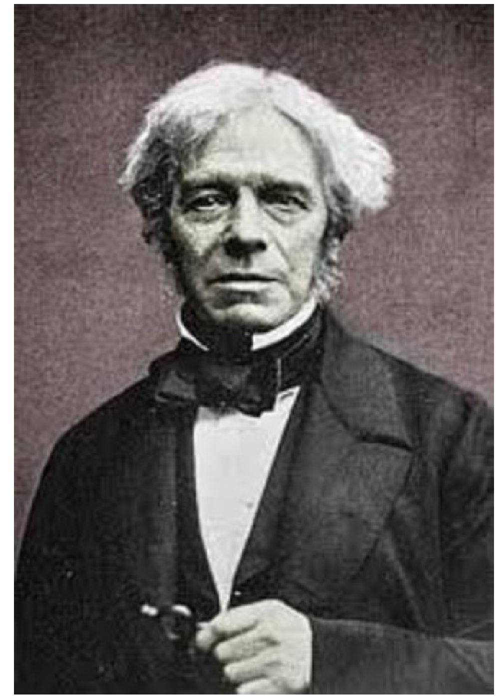
Agenda: Inductors

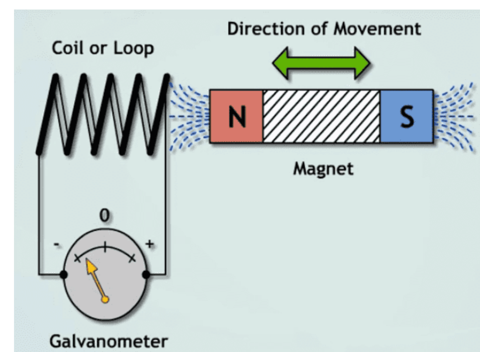
- Physical Concepts
- Symbols
- i–v Behavior
- Fabrication
- Power and Energy
- Parallel and Series Combinations
- Steady–State Solutions
- "Instantaneous" Current Change

The Inductor

- Coil of Wire
- Air or Ferromagnetic Core
- Current $\rightarrow$ Magnetic Field (Electromagnet)
- Changing Field $\rightarrow$ Voltage (Faraday's Law)
- Voltage Opposes Change in Current

$$v(t) = L\frac{di(t)}{dt}$$

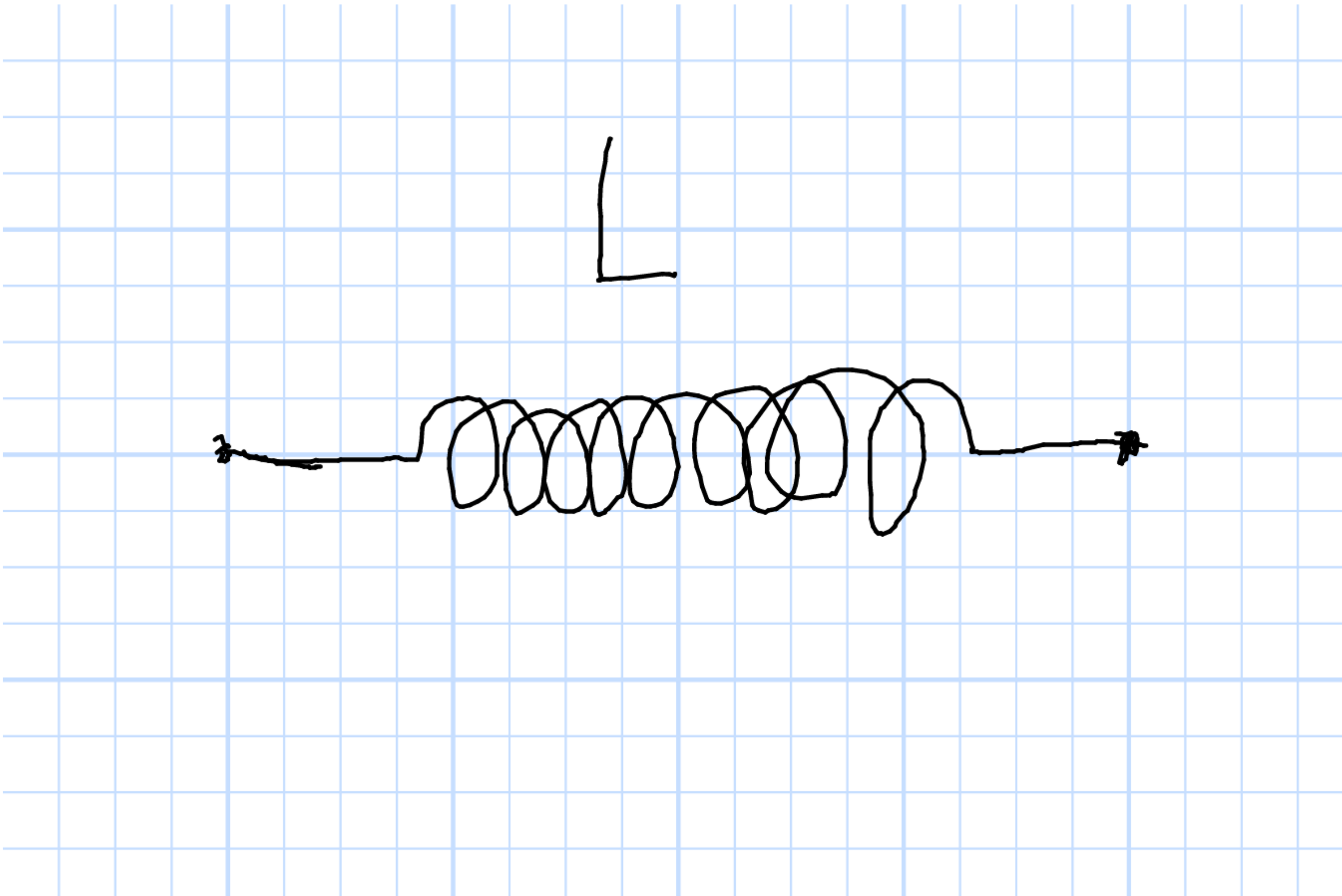




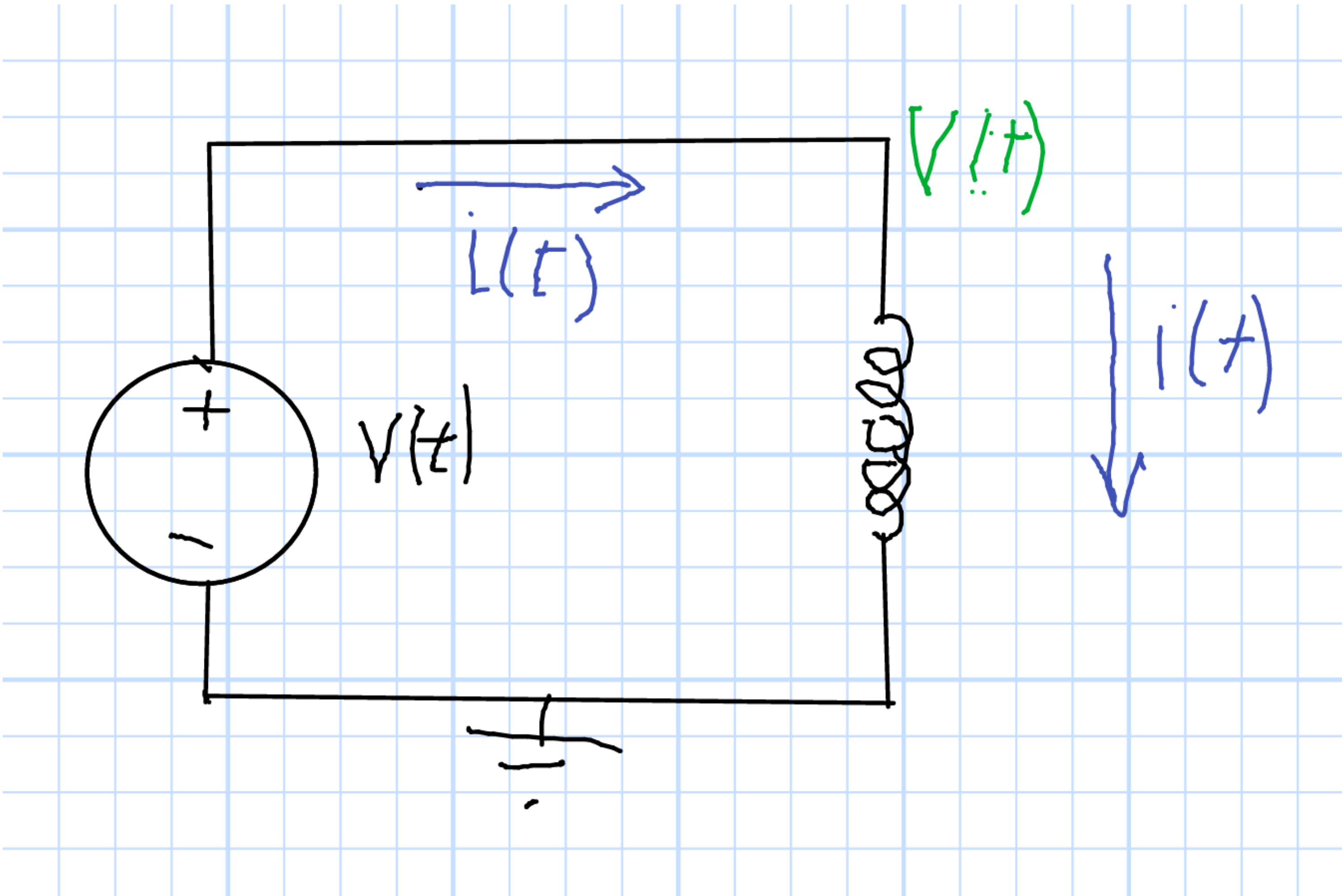
timetoast.com

www.electrical4u.net

Symbol

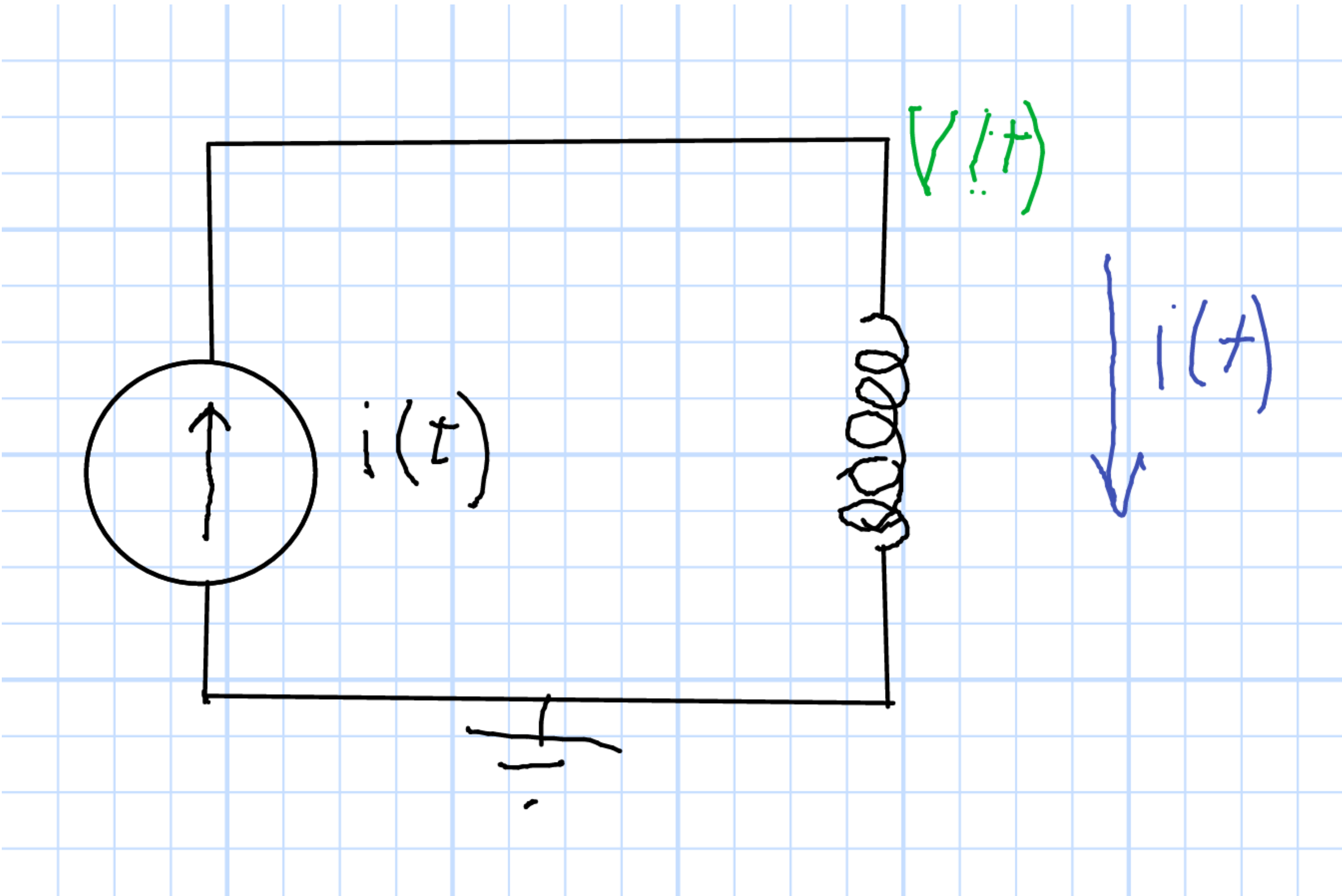


Voltage Source



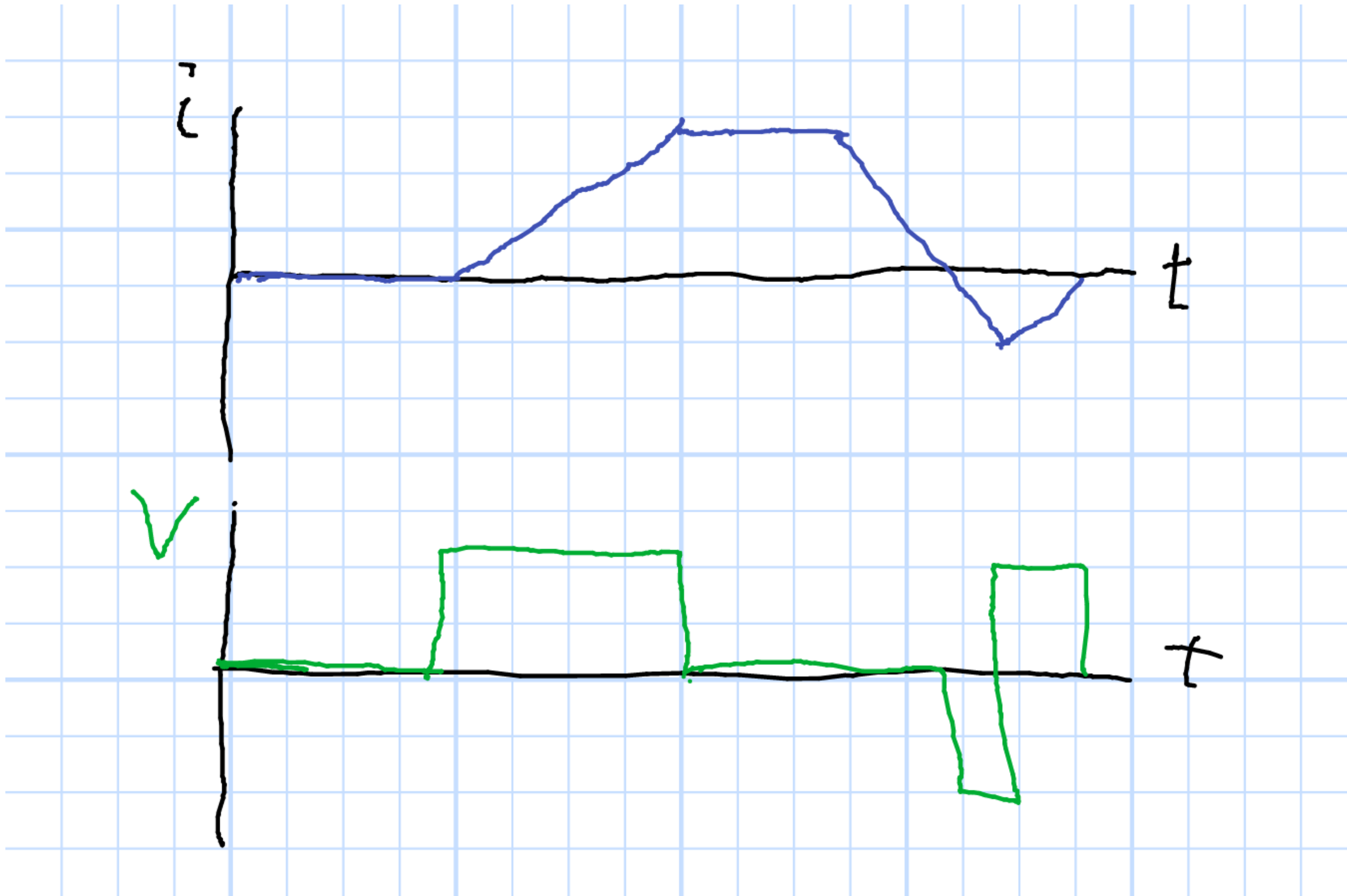
$$i(t) = \int v(t)\ dt$$

Current Source



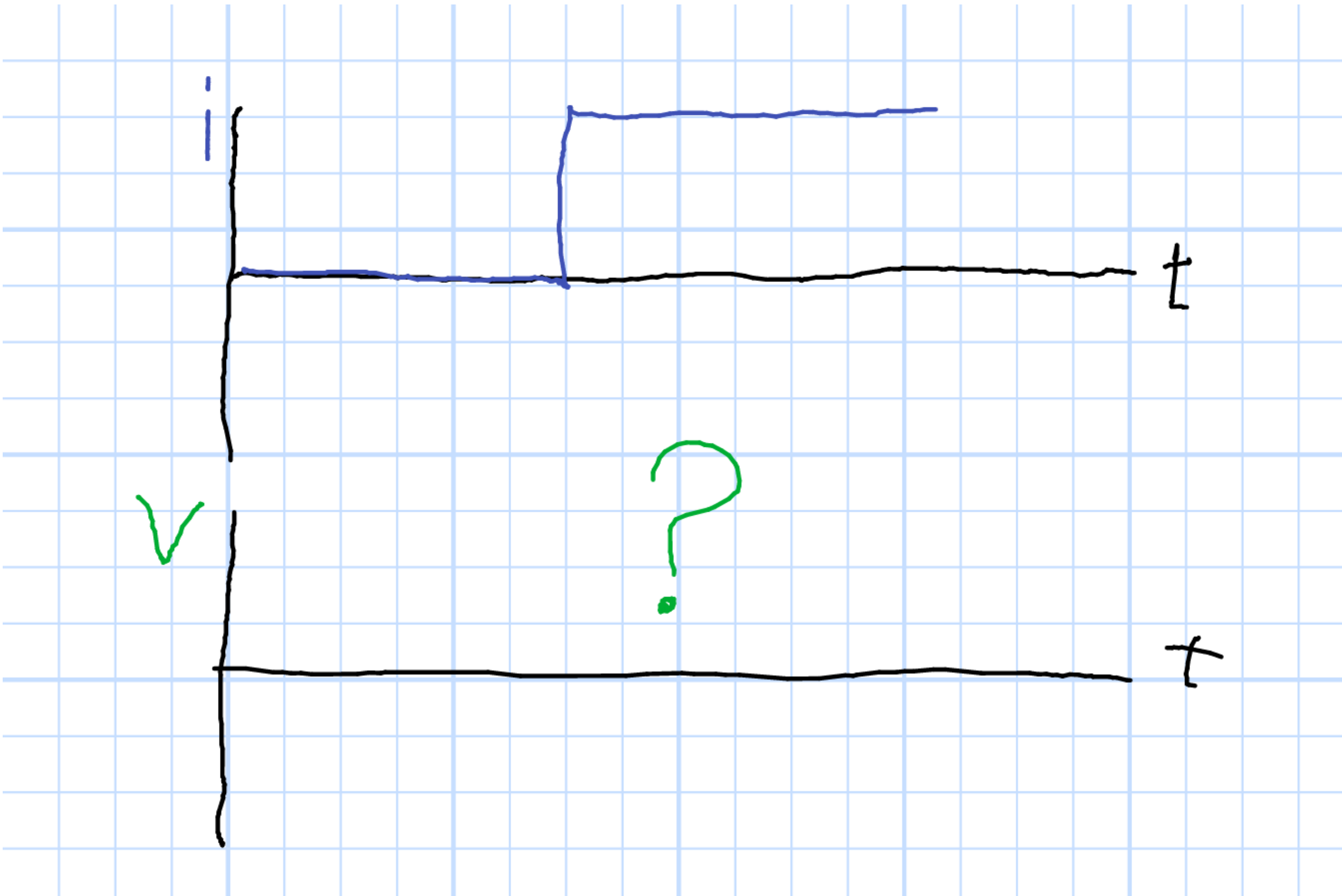
$$v(t) = L\frac{di(t)}{dt}$$

i–v Behavior



$$v(t) = L\frac{di(t)}{dt}$$

Step Current



Voltage is Finite so Current is Continuous

Values

$$v = L\frac{di}{dt}$$

Typical Values:

$$\text{Volts} = L\frac{\text{mA}}{\text{ms}}$$

$$L \text{ in } \frac{\text{Vs}}{\text{A}} = \text{Henries} = \text{H}$$

mH, μH Common in RF.
kH Do Exist.

Fabrication

- Coil of Wire (Many Turns)
- Field of a solenoid

$$B(t) = \frac{\mu N}{\ell} i(t)$$

- Inductance of a solenoid

$$v(t) = \frac{\mu A N^2}{\ell} \frac{di(t)}{dt} \qquad L = \frac{\mu A N^2}{\ell}$$

- Air, Iron, Ferrite Core (Increased Field)
- Solenoid, Toroid, Helmholz Coils *etc.*
- Many Options

$\mu = \mu_r \times 1.26 \times 10^{-6}$H/m

Inductors

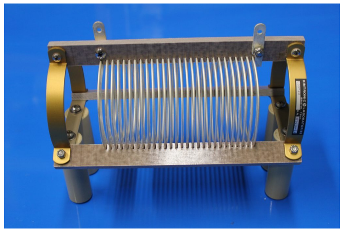
indiamart.com

components101.com

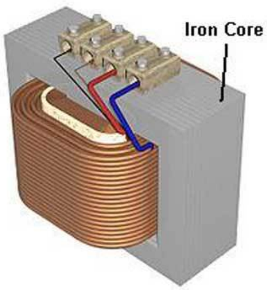
toroids.com

falconacoustics.co.uk

More Inductors & Transformers

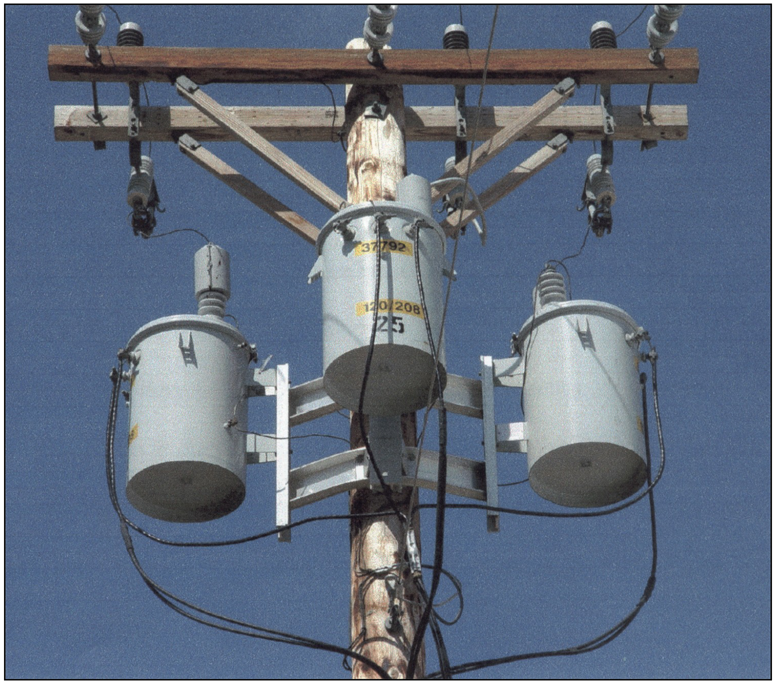


kintronic.com



polytechnichub.com

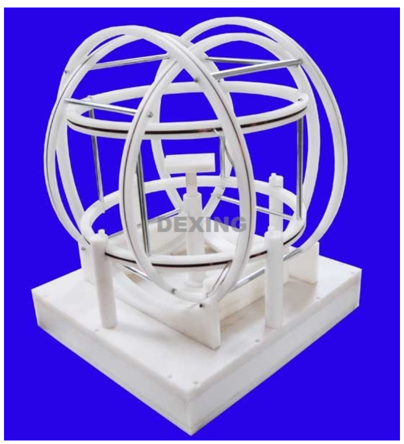
globalspec.com



coloradocountrylife.coop

electricianinperth.com.au

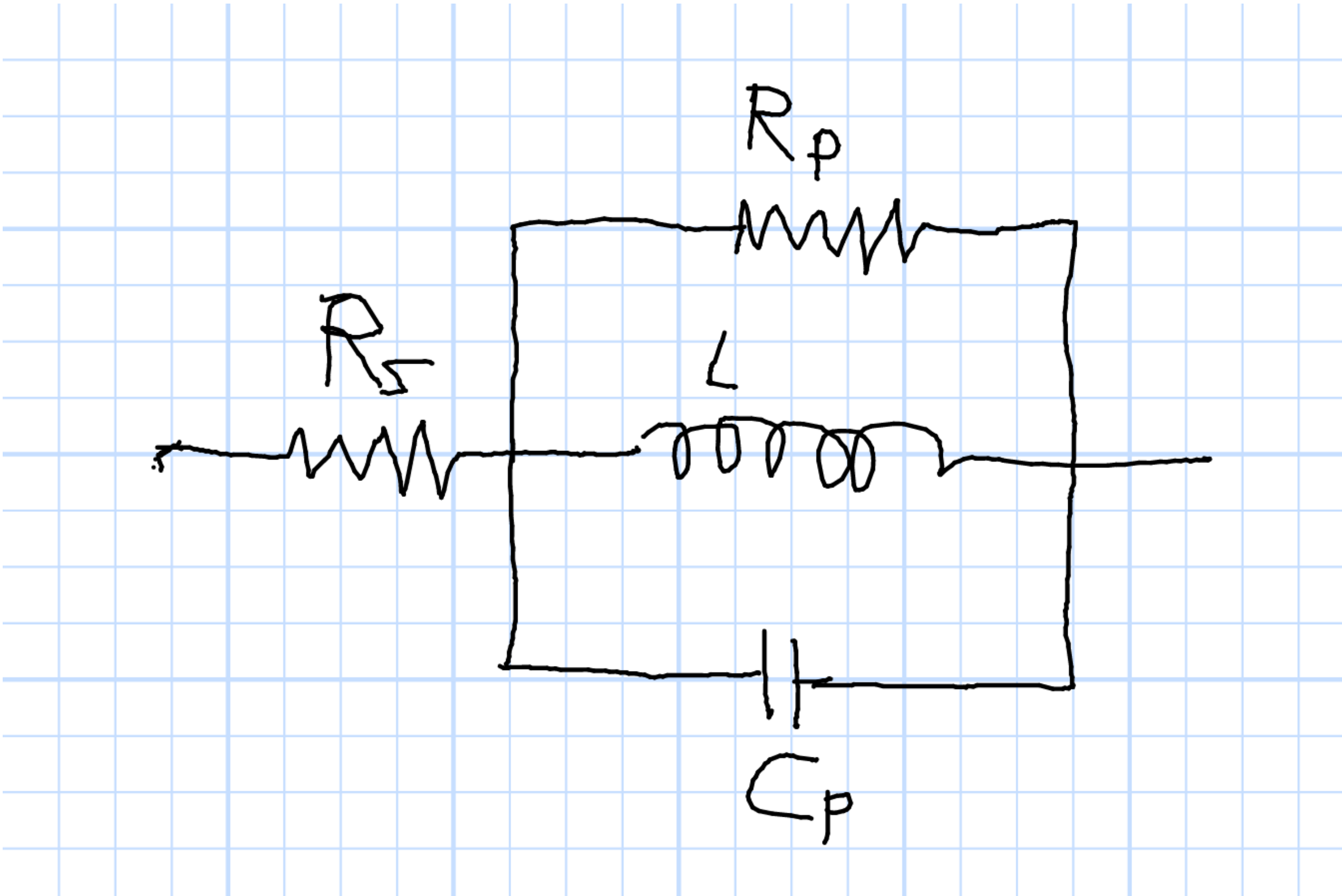
Helmholz Coils



3bscientific.com

magnetic-instrument.com

Real Inductors



R_s, C_p Small. R_p Large

Power and Energy

$$v(t) = L\frac{di(t)}{dt}$$

$$p(t) = i(t)\,v(t) = i(t)\,L\frac{di(t)}{dt}$$

$$w = \int p(t)\,dt = \frac{i^2 L}{2}$$

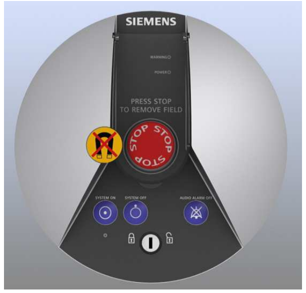
Example: Still Another Cup of Coffee

$$w = 42\text{kJ} \qquad i^2 L = 84\text{kJ}$$

$$L = 6\text{H} \qquad i = 118\text{A}$$

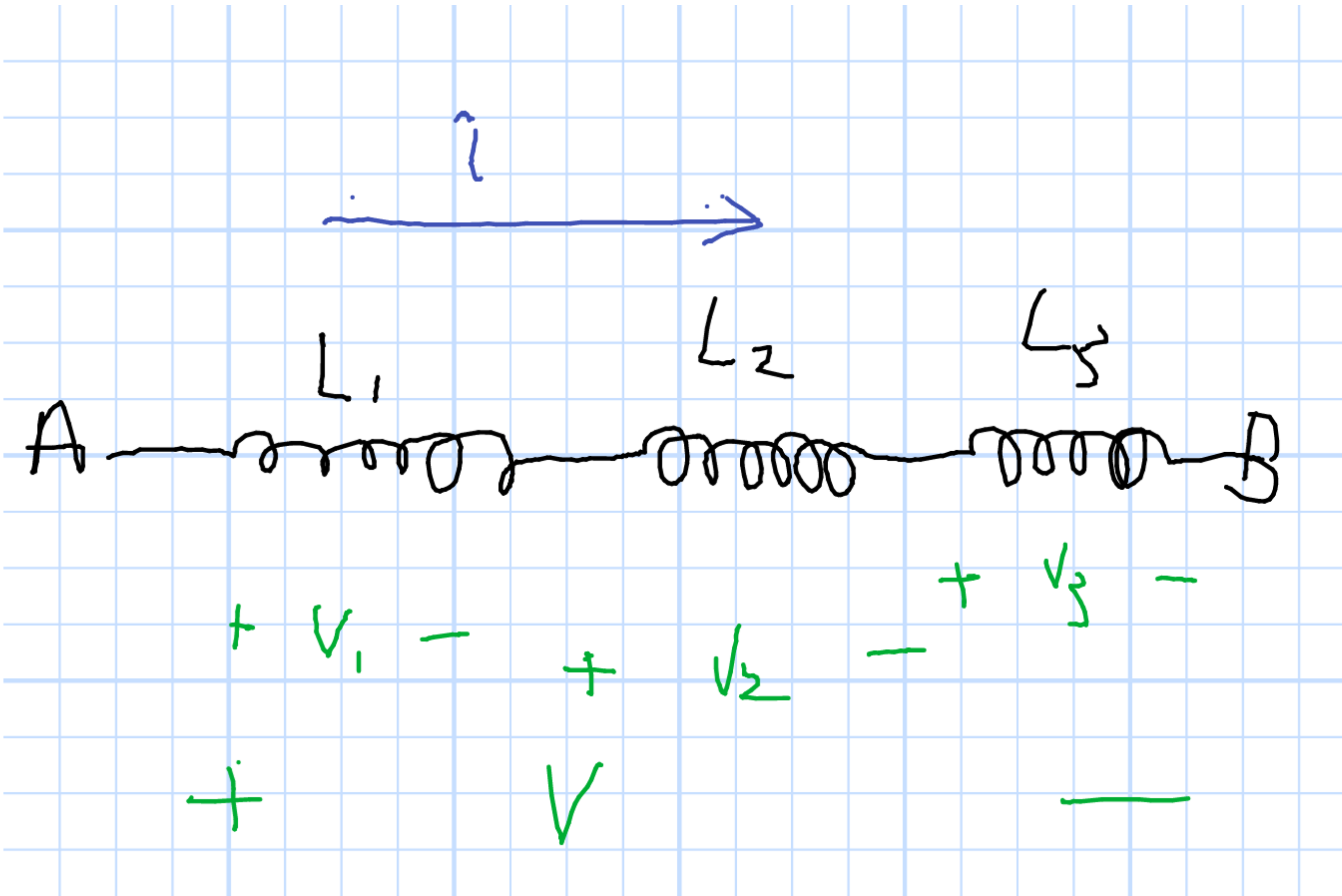
Note: At DC $v \approx 0$, so $p \approx 0$, except during turn–on and turn–off. These times can be exciting!

MRI Magnet Quench



fickr.com Superconducting magnet in use, Low T, R_s, v, High i.
In quench, $-di/dt \uparrow$, $T \uparrow$, $R_s \uparrow$, High v, i, p.

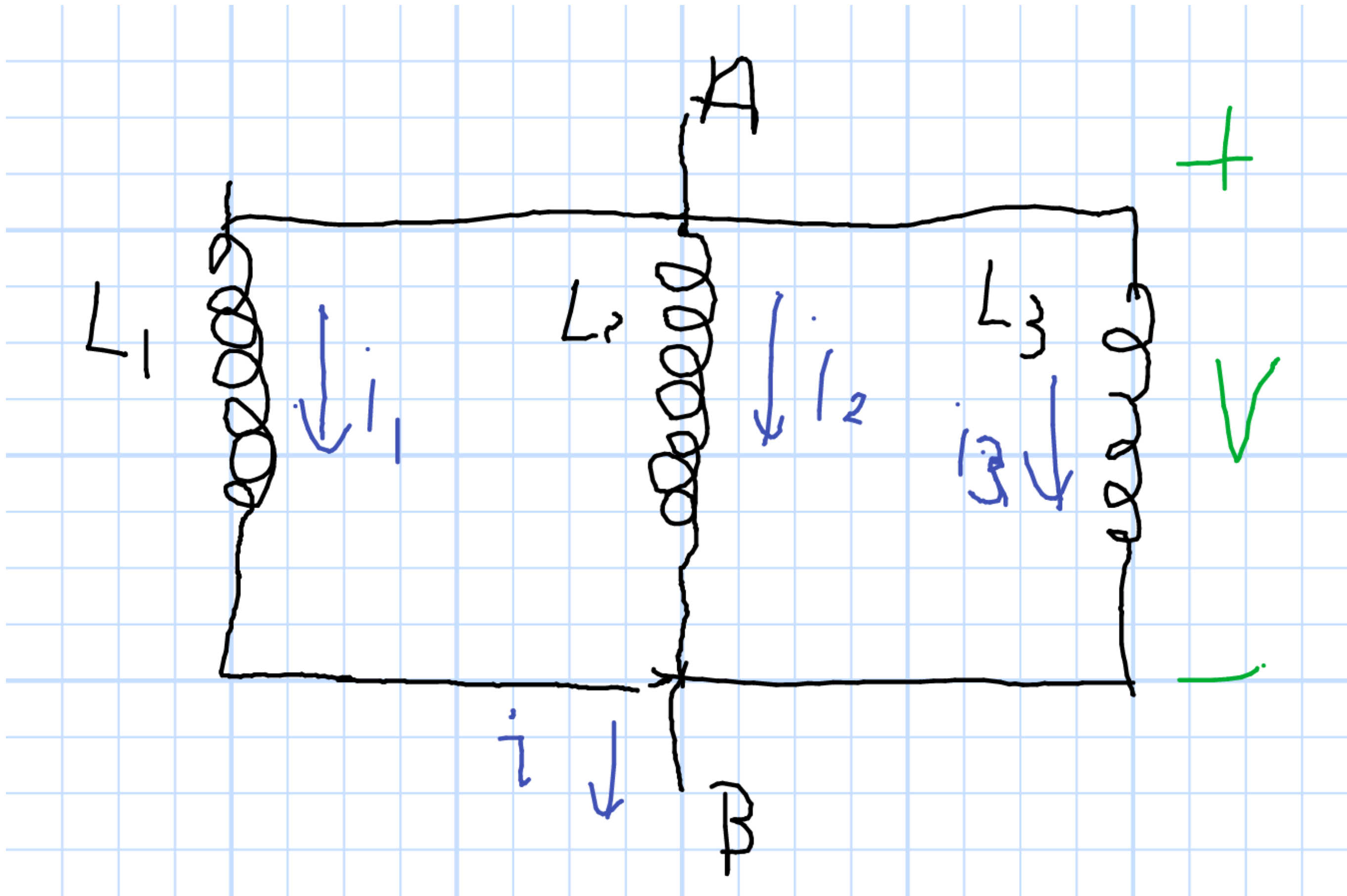
Inductors in Series



$$v_n = L_n \frac{di}{dt} \qquad v = \sum v_n \qquad L = \sum L_n$$

Just Like Resistors

Parallel Inductors



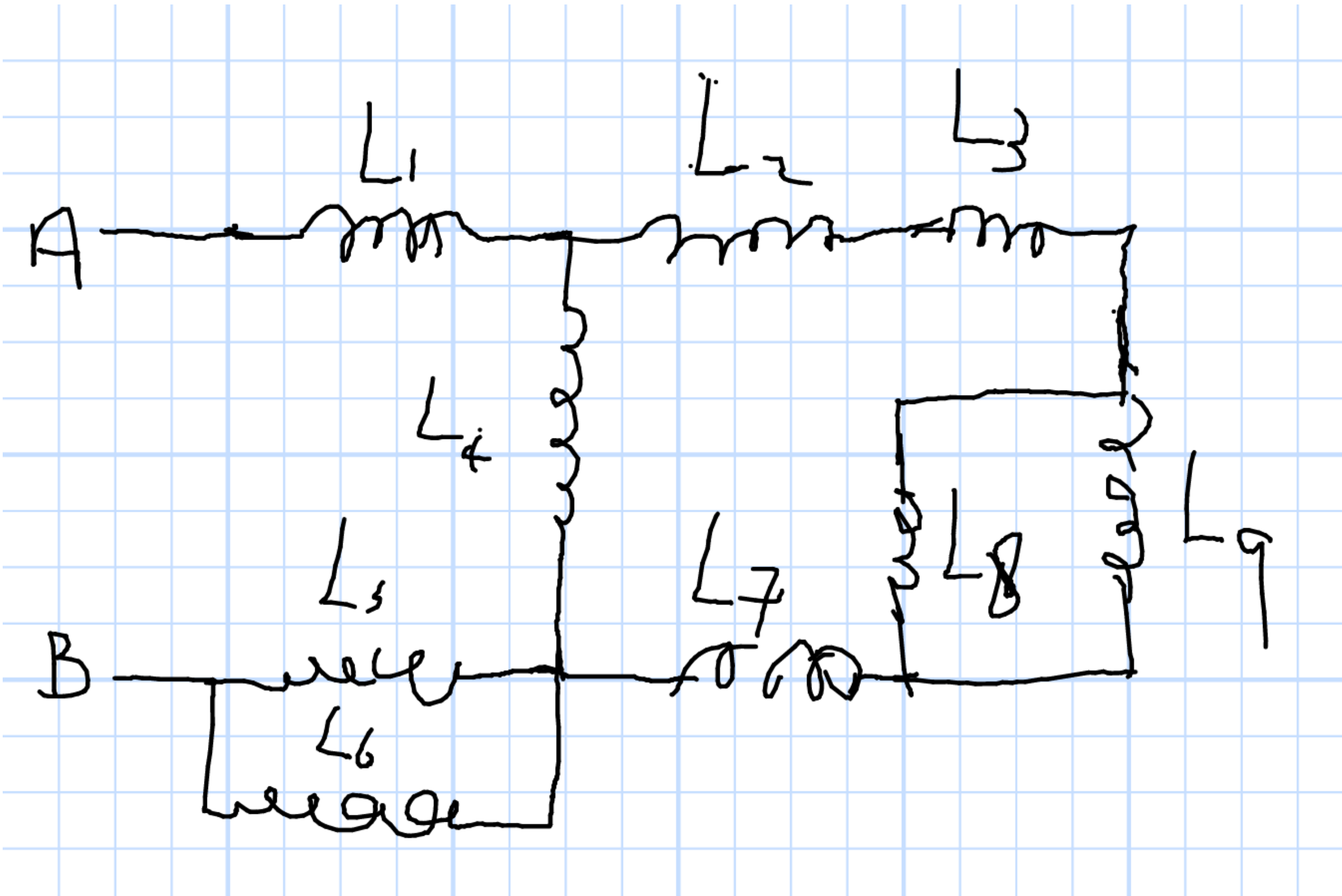
$$\frac{1}{L_n}\frac{di_n}{dt} = v \qquad \frac{di}{dt} = \sum \frac{di_n}{dt} \qquad \frac{L}{=}\sum \frac{1}{L_n}$$

Just Like Resistors Again

Parallel/Series Summary

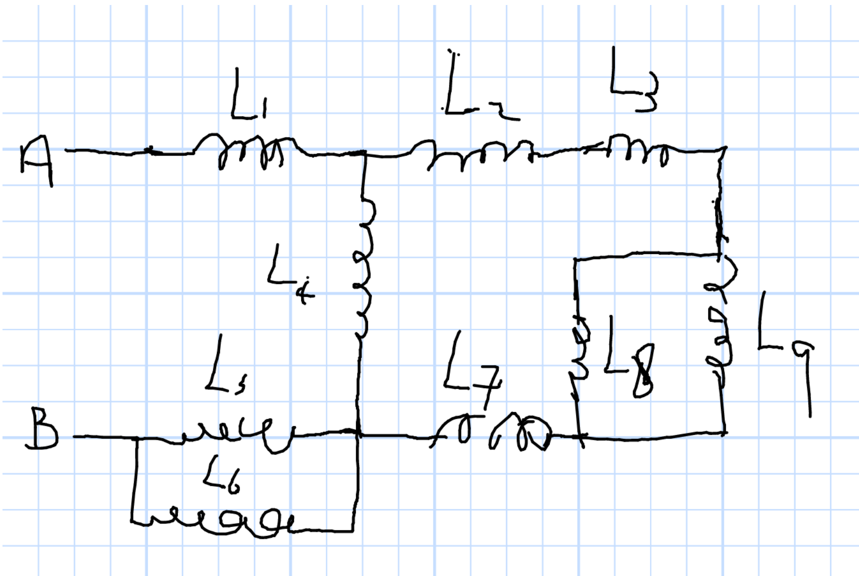
	Series	Parallel
Voltage Sources	$v = \sum v_n$	Contradictory
Current Sources	Contradictory	$i = \sum i_n$
Resistors	$R = \sum R_n$	$\frac{1}{R} = \sum \frac{1}{R_n}$
Inductors	$L = \sum L_n$	$\frac{1}{L} = \sum \frac{1}{L_n}$
Capacitors	$\frac{1}{C} = \sum \frac{1}{C_n}$	$C = \sum C_n$

Parallel/Series Example (1)



$$L_{AB} = L_1 + \{L_4 \parallel [L_2 + L_3 + (L_8 \parallel L_9) + L_7] + [L_5 \parallel L_6]\}$$

Parallel/Series Example (2)



$$L_{1:9} = 1\text{mH}$$

$$L_{23897} = 1 + 1 + \frac{1}{2} + 1 = 3.5\text{mH}$$

$$L_{423897} = 1 \parallel 3.5 = 778\mu\text{H}$$

$$L_{AB} = 1 + 0.778 + \frac{1}{2} = 2.28\text{mH}$$

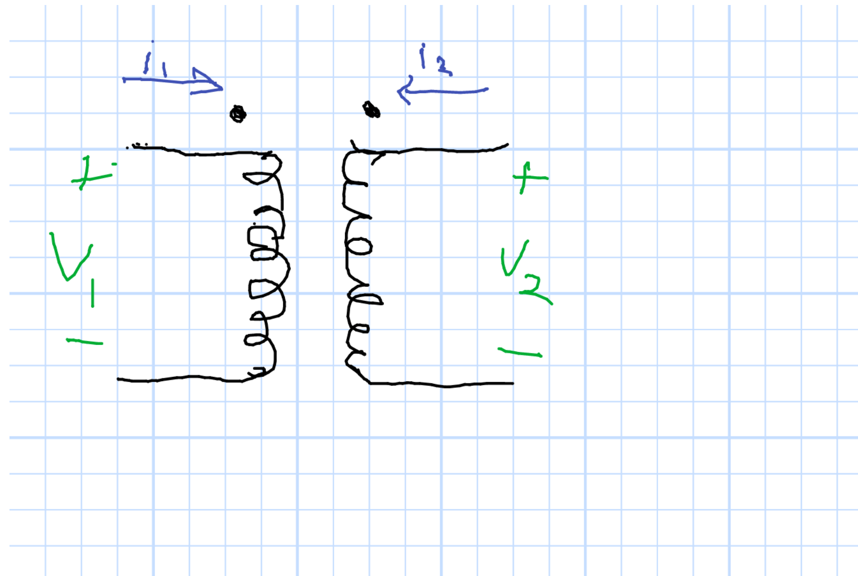
Mutual Inductance

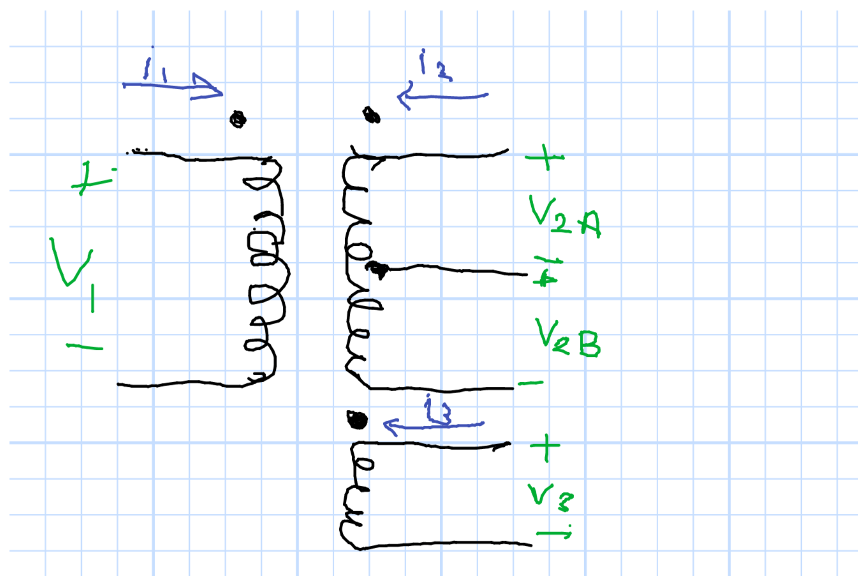
- Two or More Coils
- Same Core

$$v_1(t) = L_1 \frac{di_1(t)}{dt} + M \frac{di_2(t)}{dt}$$

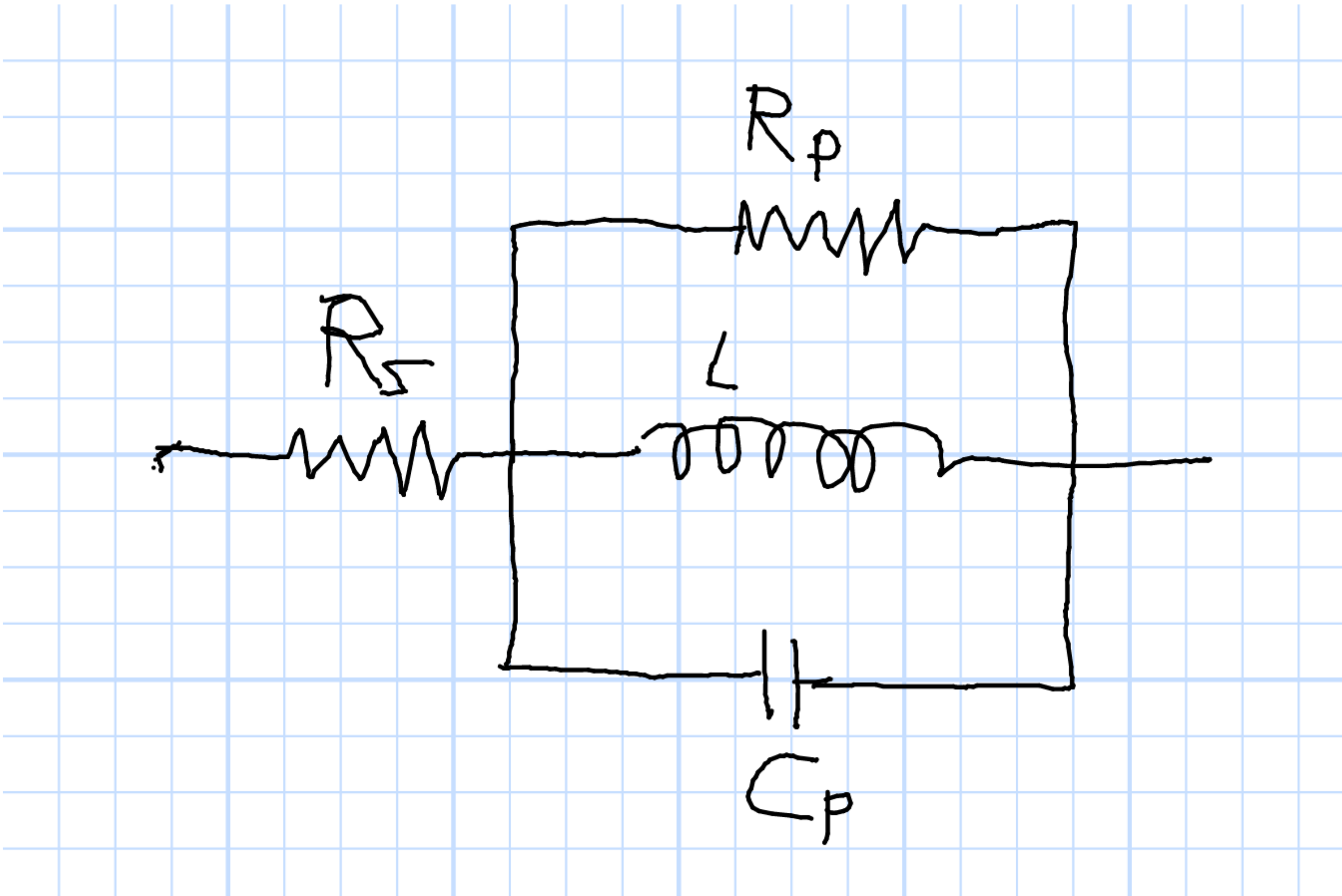
$$v_2(t) = L_2 \frac{di_2(t)}{dt} + M \frac{di_1(t)}{dt}$$

- M Same Units as L
- Transformers
  - AC Only
  - Higher Frequency
    $\rightarrow$ Smaller



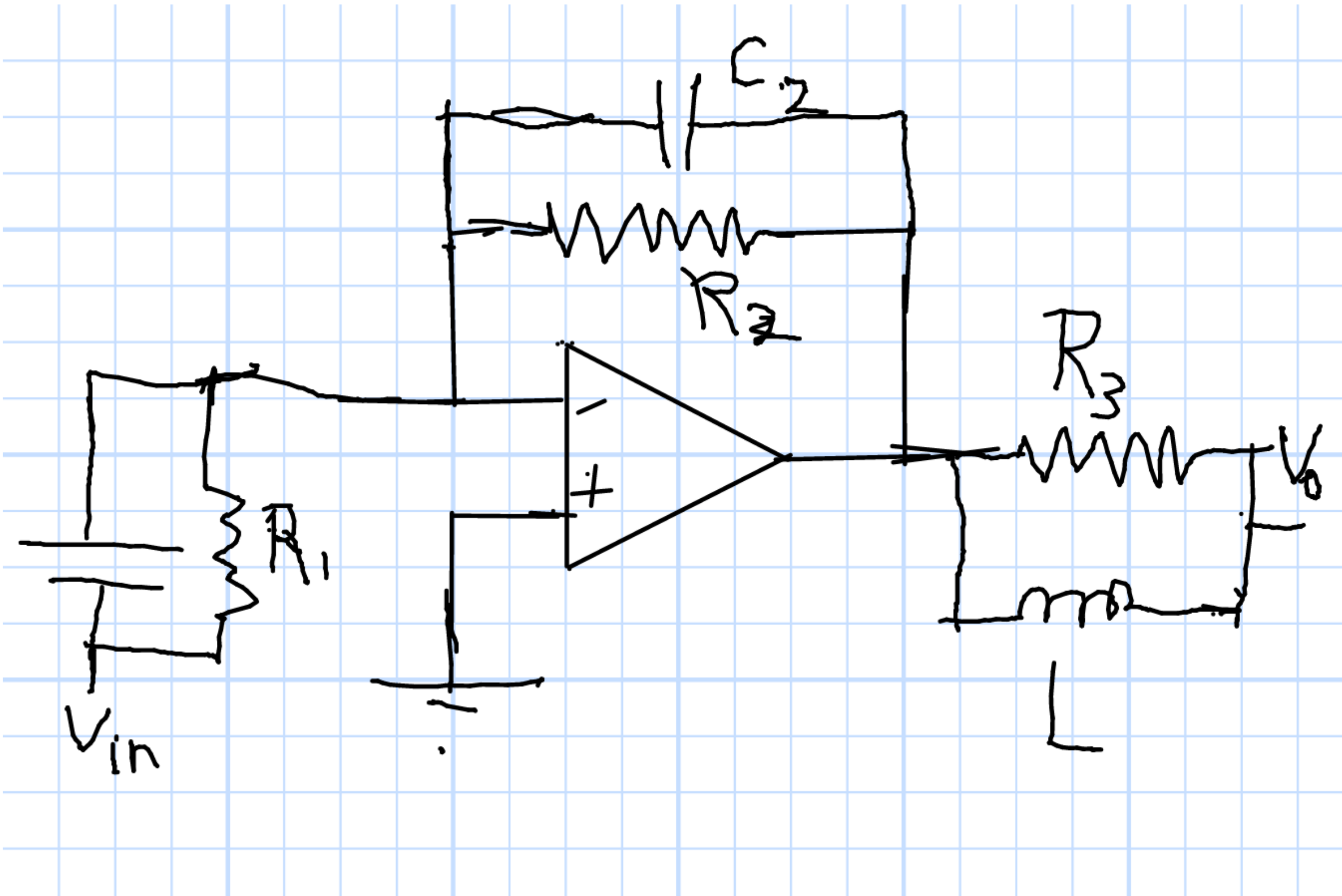


Inductors at DC (Steady State)



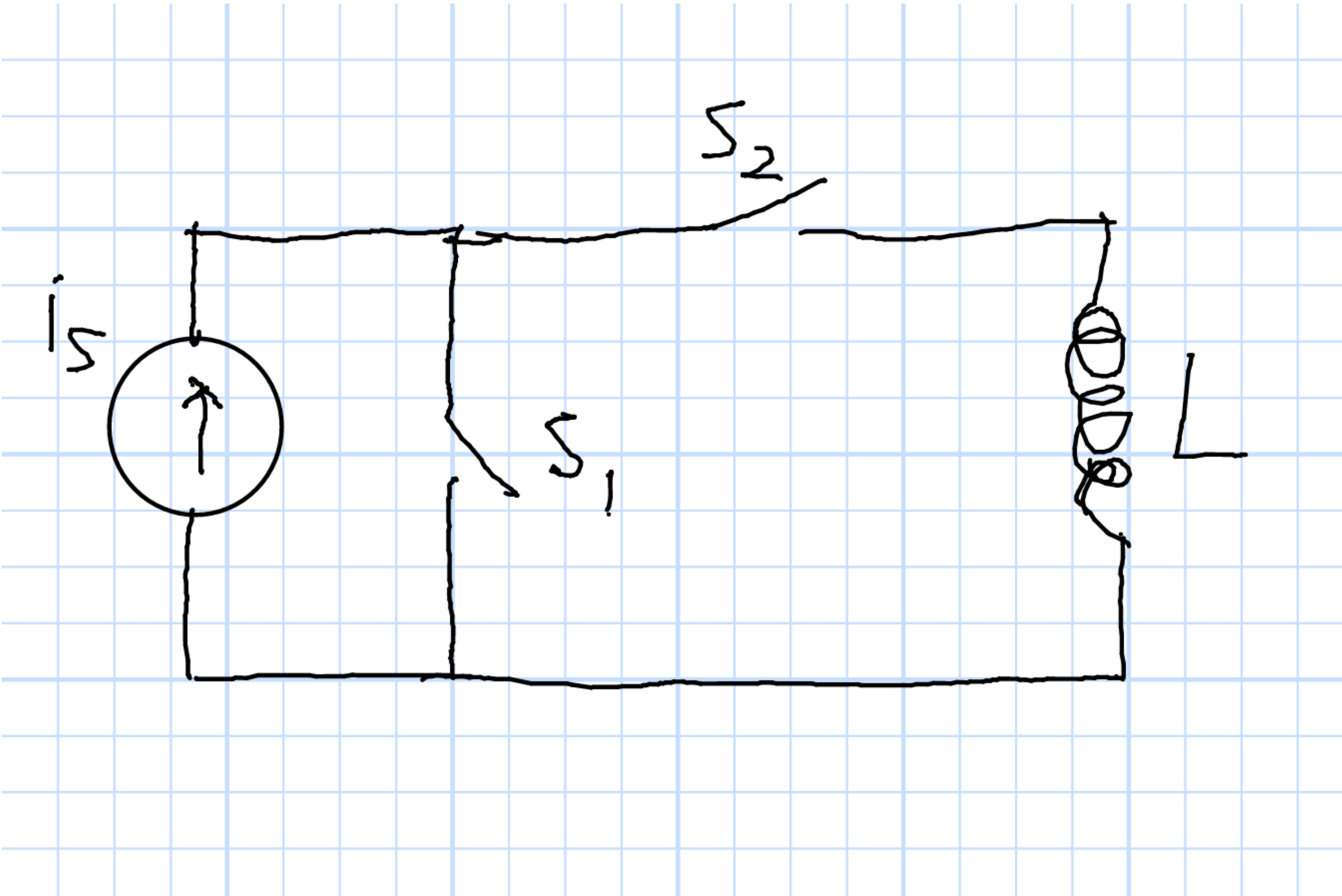
C_p Open, L Short, R_p Large (ignore). All that's left is R_s (just the resistance of the wire). $v = iR_s \to 0$

Steady State



Steady State (Short L, Open C): $v_o = -v_{in}R_2/R_1$ and $R_{out} = 0$

What Happens?



S_1, S_2 Closed. Open S_1, Wait, Open S_2

Jacob's Ladder

https://www.youtube.com/watch?v=PXiOQCRiSp0

Agenda:
First–Order Circuits

- RC Circuits
- Boundary Conditions
- Steady State Solutions
- Charge and Discharge a Capacitor
- RL Circuits
- Some Examples

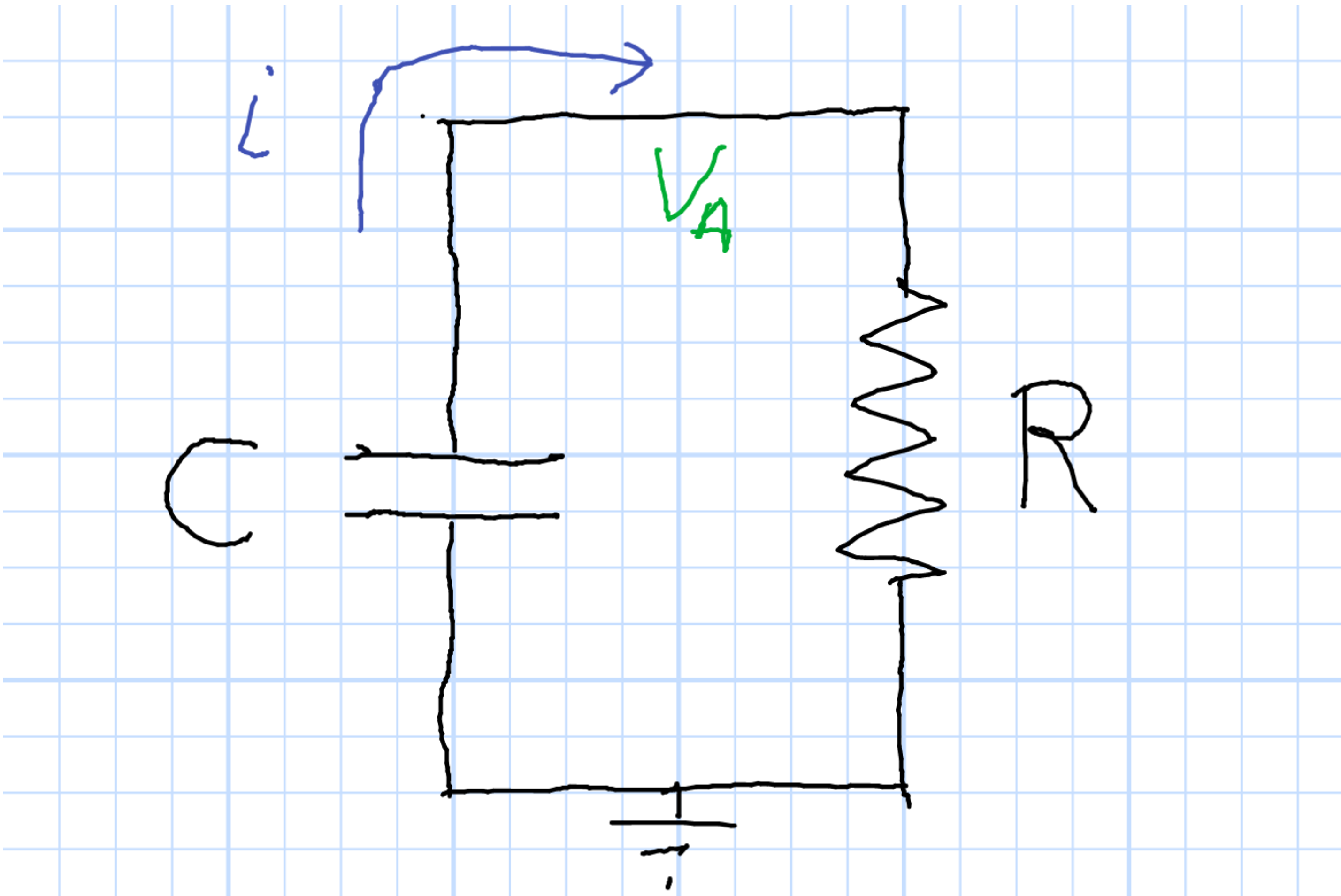
Time–Varying Sources

- Transient Analysis (Now)
  - Differential Equations
  - First Order for RL, RC
  - Second Order for RLC
  - Circuits Usually Involve Switches
  - Transient and Steady–State Solutions
- Sinusoidal Solution (Later)
  - Phasor Analysis ($\frac{d}{dt} = j2\pi f$)
  - Complex Impedance
  - "Easy" Solutions
  - Fourier Series and Transforms

Transient Solution Approach

- Write the Differential Equation (KCL, KVL, Component Eqns.)
- Postulate a Solution: Exponential, Sinusoid, Constant
- Solve for Some Unknowns
- Solve Steady–State Problem for Final Condition
- Use Continuity for Initial Condition

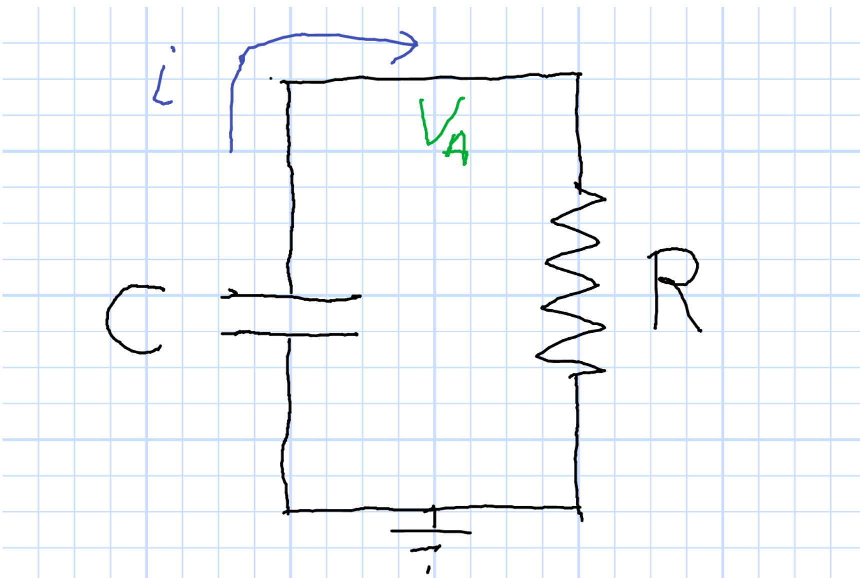
RC Circuit



Start with $v_A \neq 0$ (eg. use a switch)

$$i = C\frac{dv}{dt} \qquad i = -C\frac{dv_A}{dt} \qquad v_A = iR$$

RC Equations



$$i = -C\frac{dv_A}{dt}$$

$$i = \frac{v}{R}$$

- Differential Equation

$$v_A = -RC\frac{dv_A}{dt}$$

- Test Solution

$$v_A = k_1 e^{st} + k_2$$

- Substitute

$$k_1 e^{st} + k_2 = -RC\frac{d}{dt}\left(k_1 e^{st} + k_2\right)$$

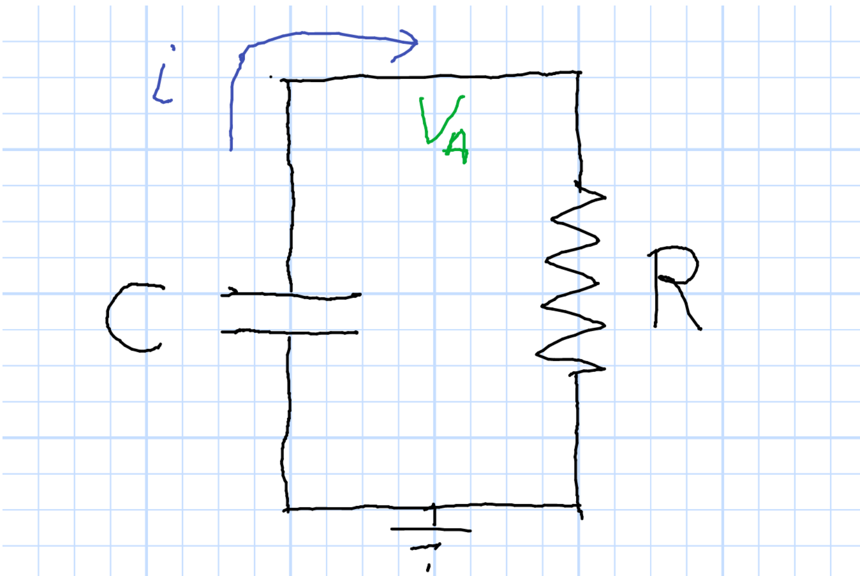
- Take the Derivative

$$k_1 e^{st} + k_2 = -RCsk_1 e^{st}$$

- Group

$$k_1\left(1 + RCs\right)e^{st} - k_2 = 0$$

RC Solution



$$i = -C\frac{dv_A}{dt}$$

$$i = \frac{v}{R}$$

$$v_A = k_1 e^{st} + k_2$$

- From Previous Page

$$k_1\left(1 + RCs\right)e^{st} - k_2 = 0$$

- True for All Time (Above is zero term–by–term

$$k_2 = 0 \qquad s = -\frac{1}{RC}$$

- Solution

$$v_A = k_1 e^{-t/(RC)} + 0$$

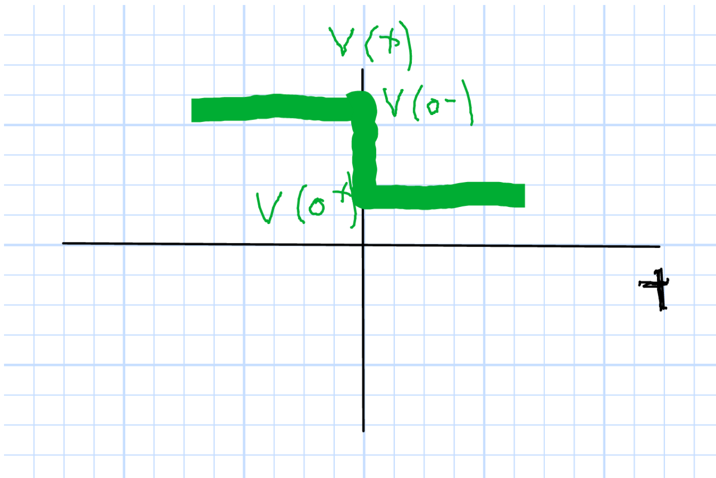
- Time Constant

$$v_A = k_1 e^{-t/\tau}$$

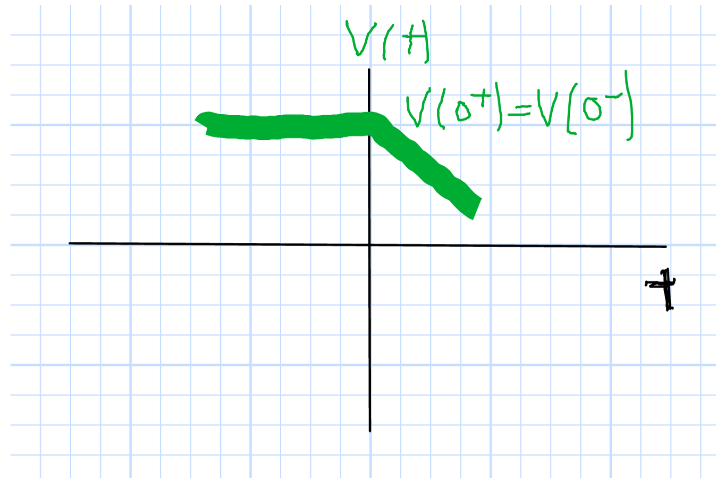
$$\tau = RC$$

- Still One Unknown (k_1)

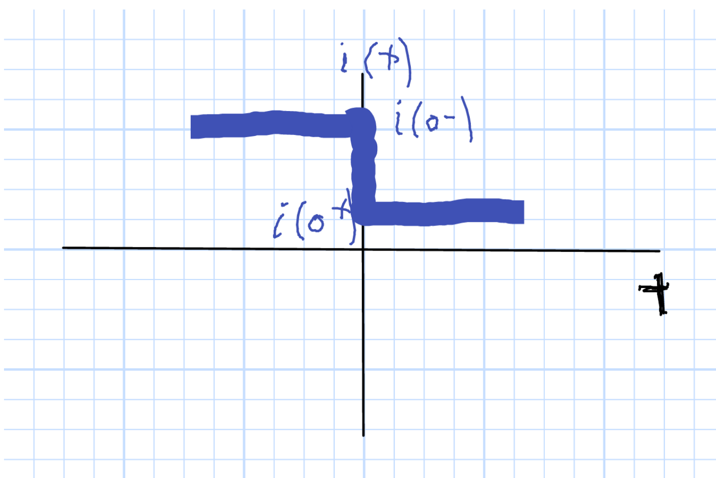
General Boundary Conditions



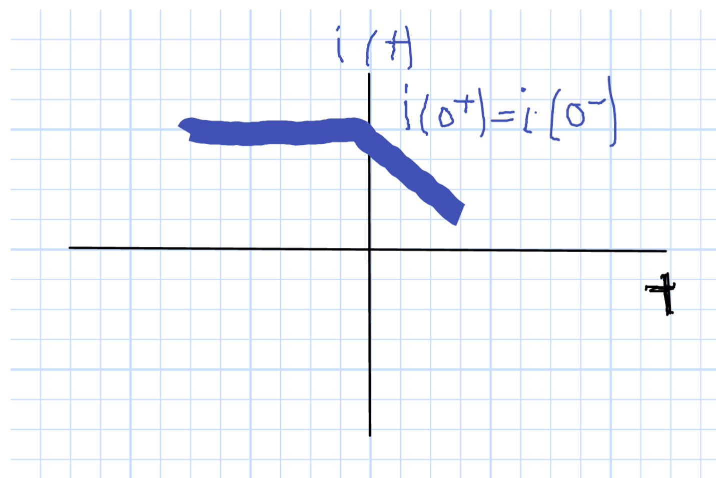
Not Valid for Capacitors



OK

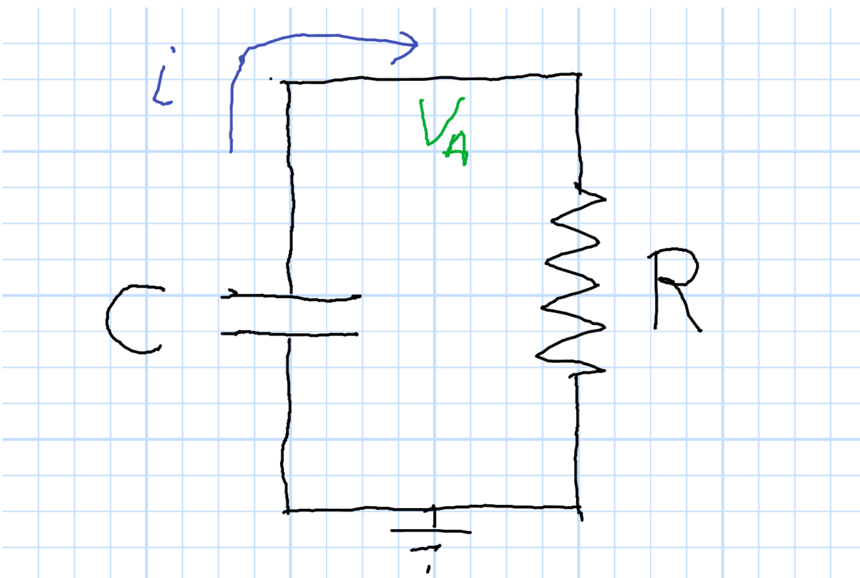


Not Valid for Inductors



OK

Initial Conditions



$$i = -C\frac{dv_A}{dt}$$

$$i = \frac{v}{R}$$

$$v_A = k_1 e^{st} + k_2$$

- From Earlier Page

$$v_A = k_1 e^{-t/\tau}$$

$$\tau = RC$$

- Original Voltage $V\left(0^-\right)$
- Boundary Condition

$$V\left(0^+\right) = V\left(0^-\right)$$

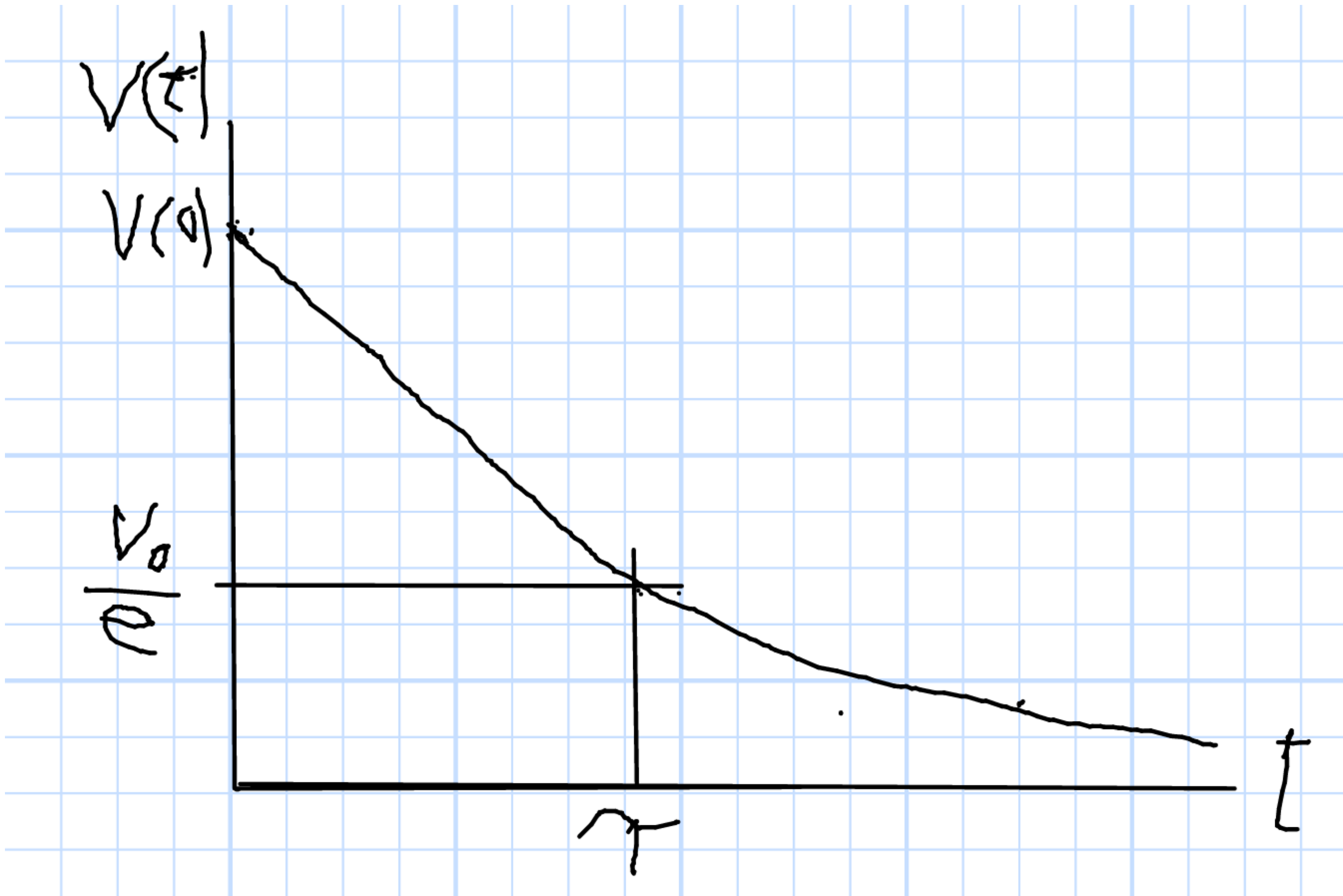
- At $t = 0$

$$k_1 e^{-0/\tau} = k_1 = V\left(0^+\right)$$

- Solution

$$v_A = V\left(0^-\right) e^{-t/\tau}$$

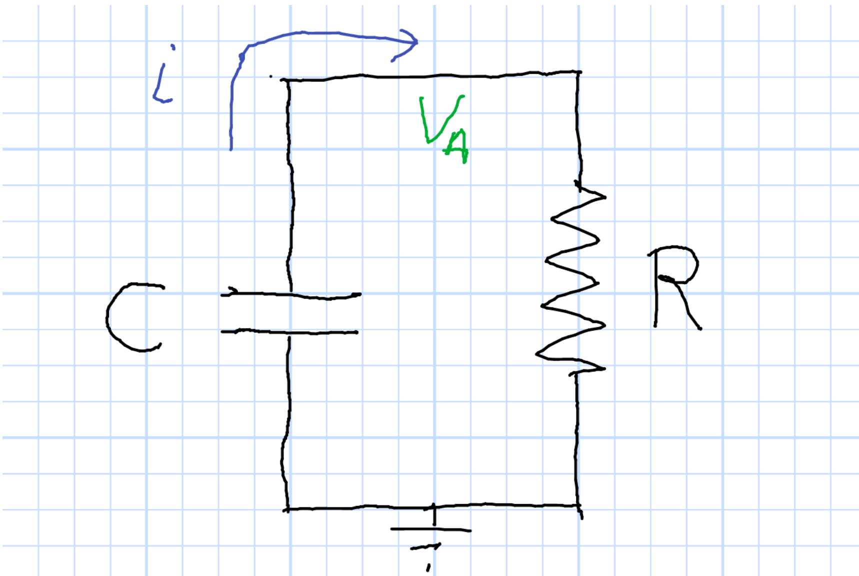
Exponential Solutions



$$v_a = v_a(0)\, e^{-t/\tau} \qquad v_a(\tau) = v_a(0) \times \frac{1}{e} \approx v_a(0) \times 0.3679$$

$$v_a(2\tau) \approx v_a(0) \times 0.1353 \qquad v_a(10\tau) \approx v_a(0) \times 4.540 \times 10^{-5}$$

Steady–State Solution



$$i = -C\frac{dv_A}{dt}$$

$$i = \frac{v}{R}$$

$$v_A = k_1 e^{st} + k_2$$

- Steady State

$$t \to \infty$$

- Anything that is going to happen has happened

$$\frac{d\text{Anything}}{dt} = 0$$

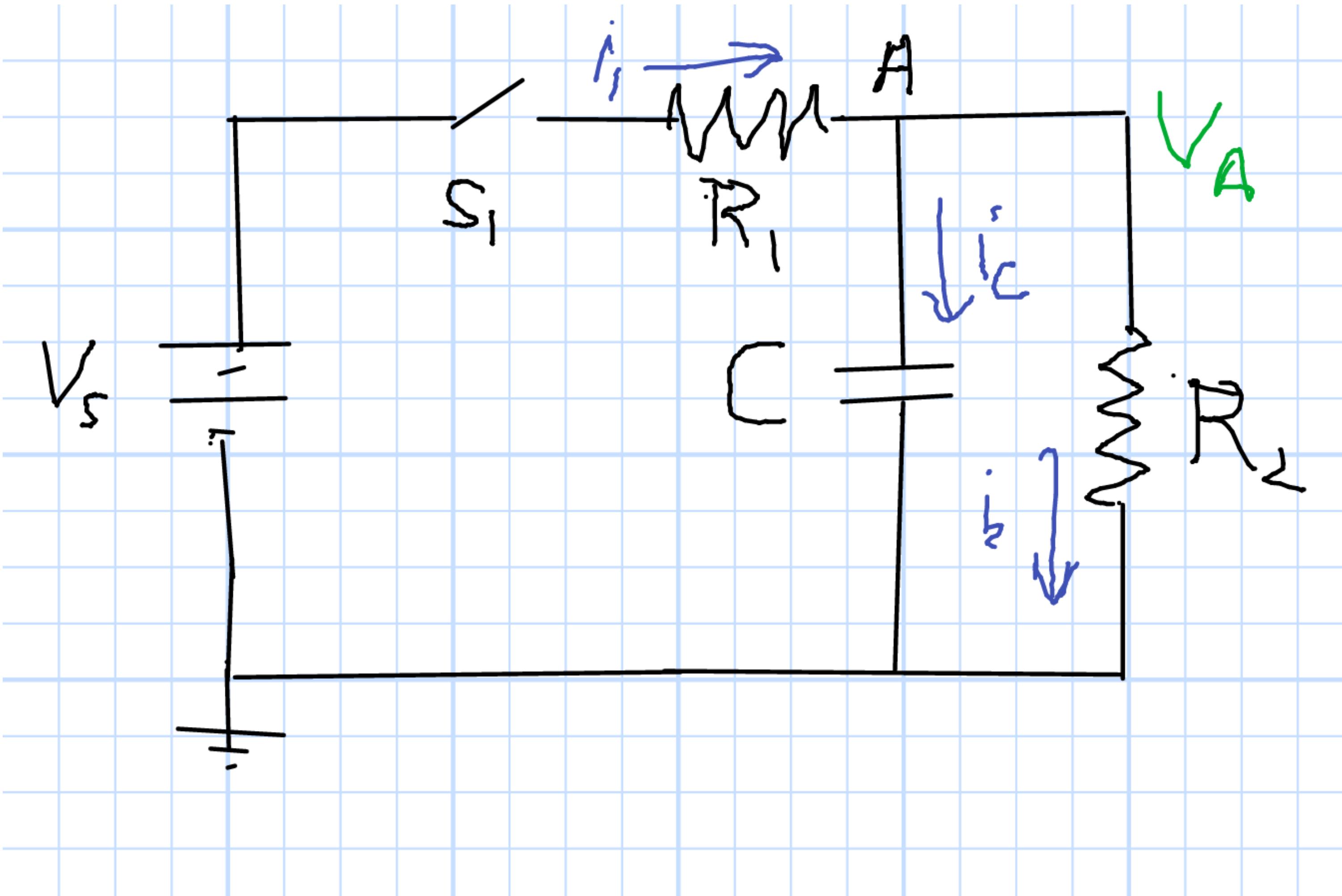
- Transient Solution is Zero

$$\frac{dv_A}{dt} = 0 \qquad i = 0$$

- Solution

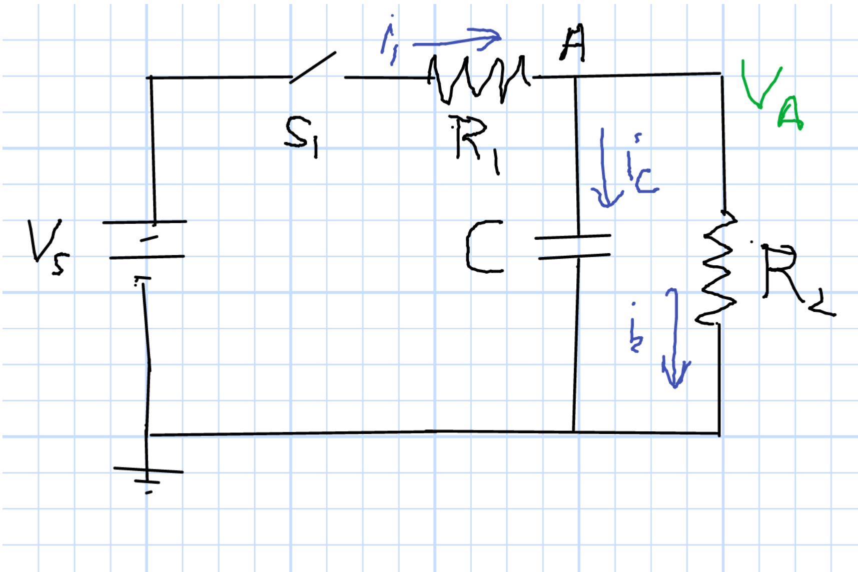
$$k_2 = 0 = v_{A\infty}$$

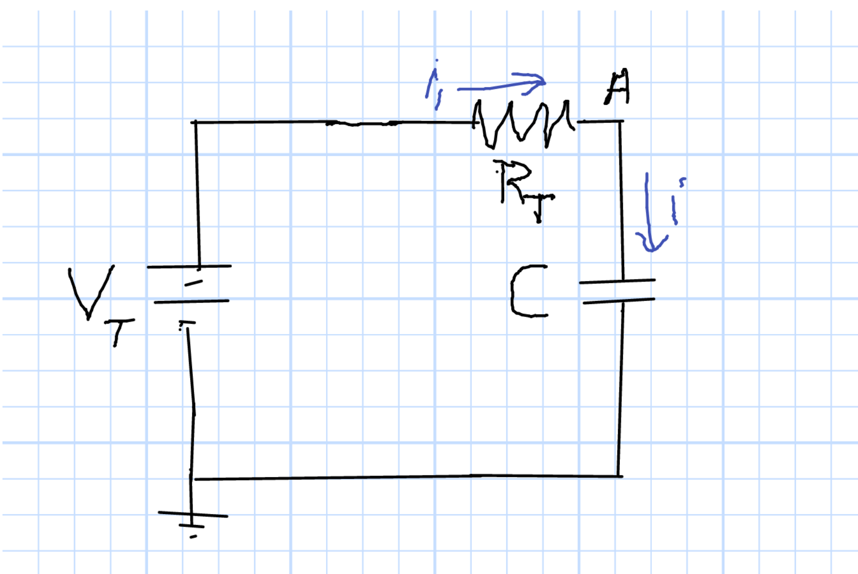
Charge and Discharge



Close S_1 at $t = 0$. Open S_1 at $t = t_1$. What will happen?

Charge!





Thévenin Equivalent Charging

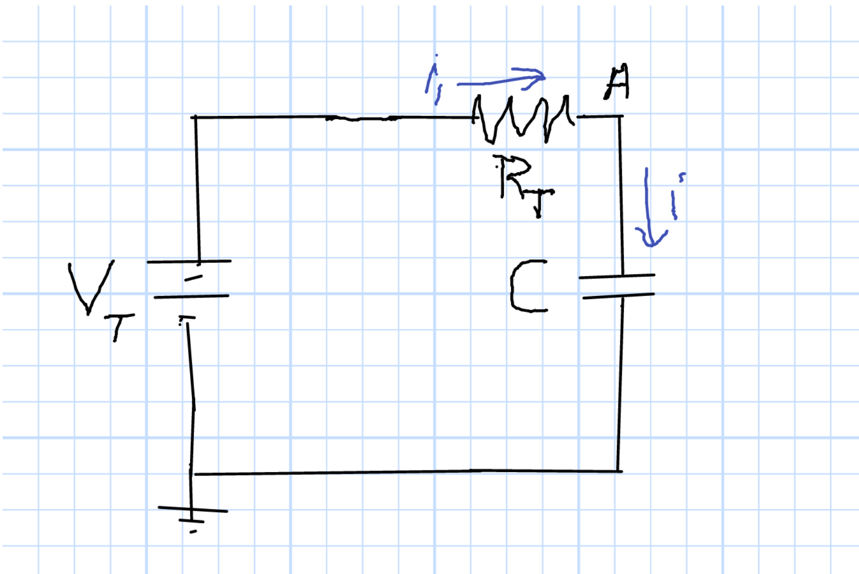
$$v_T = v_S \frac{R_2}{R_1 + R_2}$$

$$R_T = R_1 \parallel R_2$$

Assume

$$v_A(0) = 0$$

Charging Equations



$$i = \frac{v_T - v_A}{R_T} = C\frac{dv_A}{dt}$$

$$v_T - v_A = R_T C\frac{dv_A}{dt}$$

$$v_A + R_T C\frac{dv_A}{dt} = v_T$$

Proposed Solution

$$v_A = k_1 e^{st} + k_2$$

$$k_1 e^{st} + k_2 + R_T C\frac{d}{dt}\left(k_1 e^{st} + k_2\right) = v_T$$

$$k_1 e^{st}\left(1 + R_T C s\right) + k_2 = v_T$$

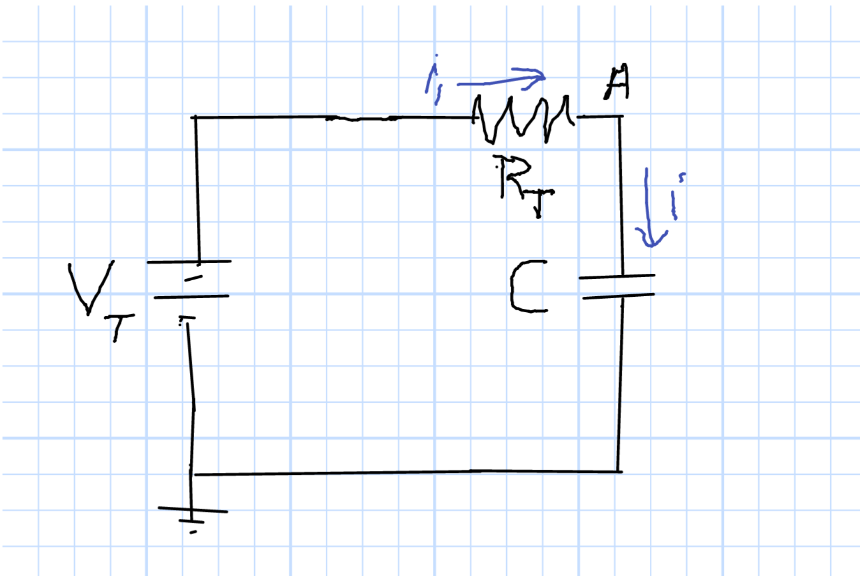
$$s = \frac{-1}{R_T C} \qquad k_2 = v_T$$

$$v_T = v_S \frac{R_2}{R_1 + R_2}$$

$$R_T = R_1 \parallel R_2$$

$$v_A(0) = 0$$

Charging Solution



$$v_T = v_S \frac{R_2}{R_1 + R_2}$$

$$R_T = R_1 \parallel R_2$$

$$v_A(0) = 0$$

- From Previous Page

$$v_A = k_1 e^{st} + k_2$$

$$s = \frac{1}{R_T C} \qquad k_2 = v_T$$

$$v_A = k_1 e^{-t/(R_T C)} + v_T$$

- Initial Condition

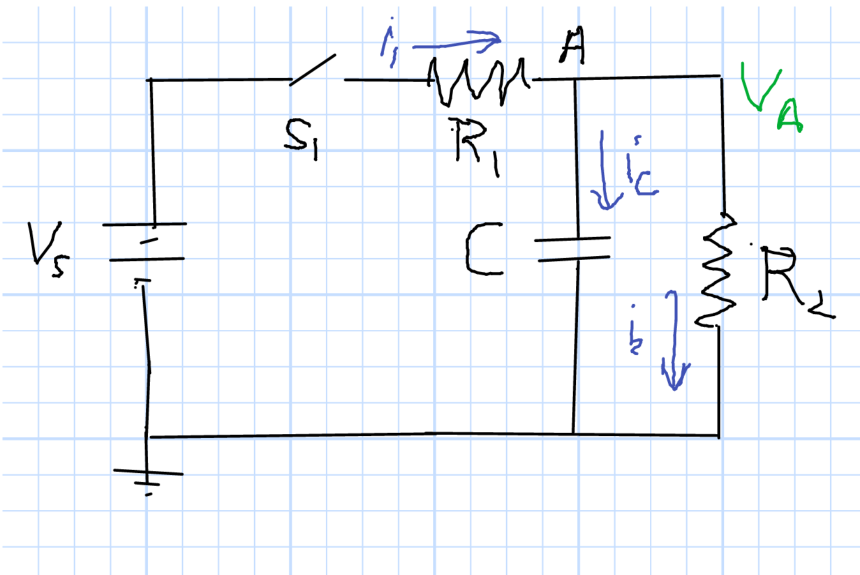
$$v_A(0) = k_1 + v_T$$

$$v_A(0) = 0 \qquad k_1 = -v_T$$

- Solution

$$v_A = v_T \left(1 - e^{-t/(R_T C)}\right)$$

Charging Result



$$v_T = v_S \frac{R_2}{R_1 + R_2}$$

$$R_T = R_1 \parallel R_2$$

$$v_A(0) = 0$$

- From Previous Page

$$v_A = v_T \left(1 - e^{-t/(R_T C)}\right)$$

- Use v_T and R_T

$$v_A = v_s \frac{R_2}{R_1 + R_2} \times$$

$$\left\{1 - e^{-t/((R_1 \| R_2) C]}\right\}$$

- Assume $R_1 \ll R_2$

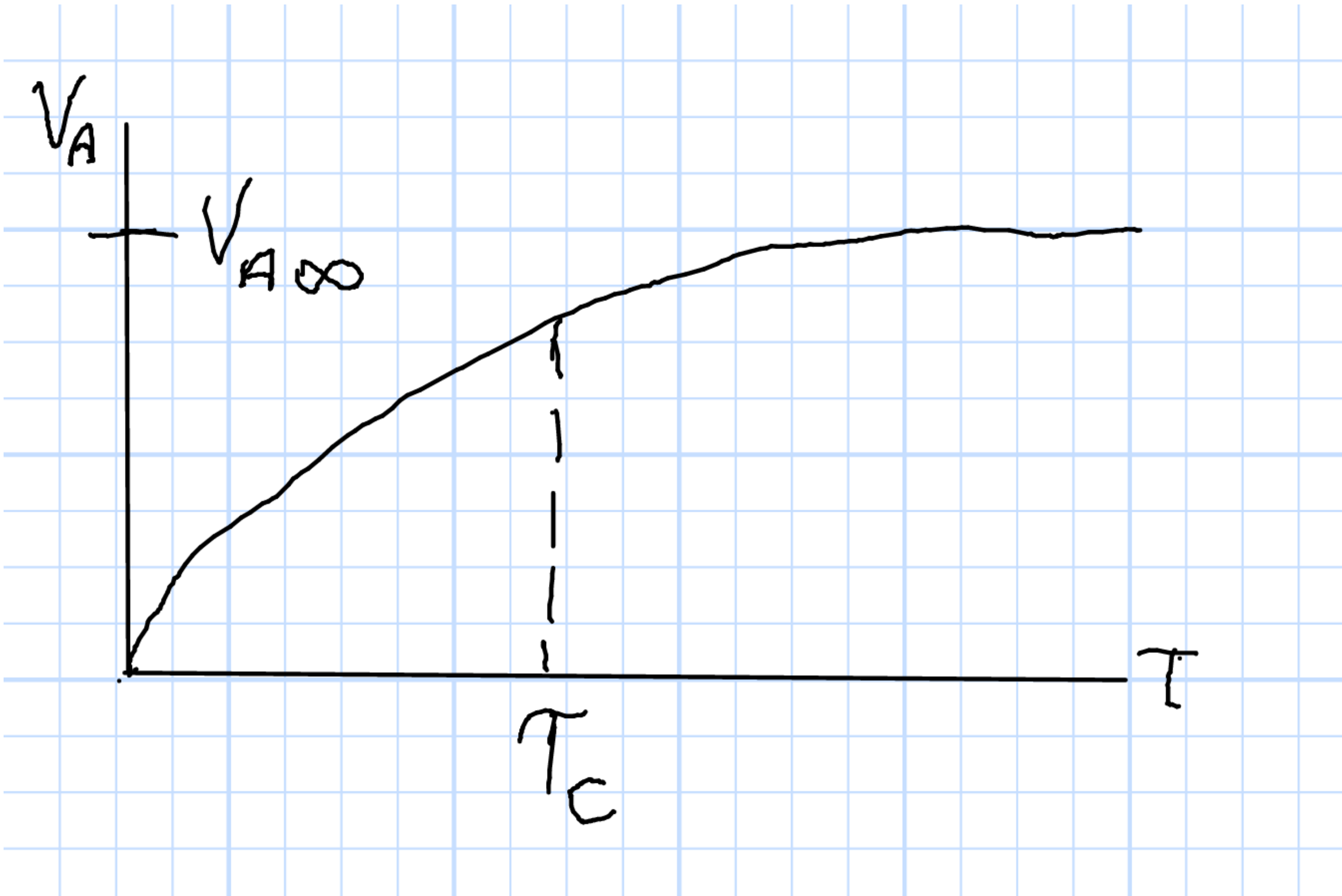
$$v_A \approx v_s \times \left\{1 - e^{-t/(R_1 C]}\right\}$$

- Example

$$R_1 = 100\Omega \qquad C = 100\mu\text{F}$$

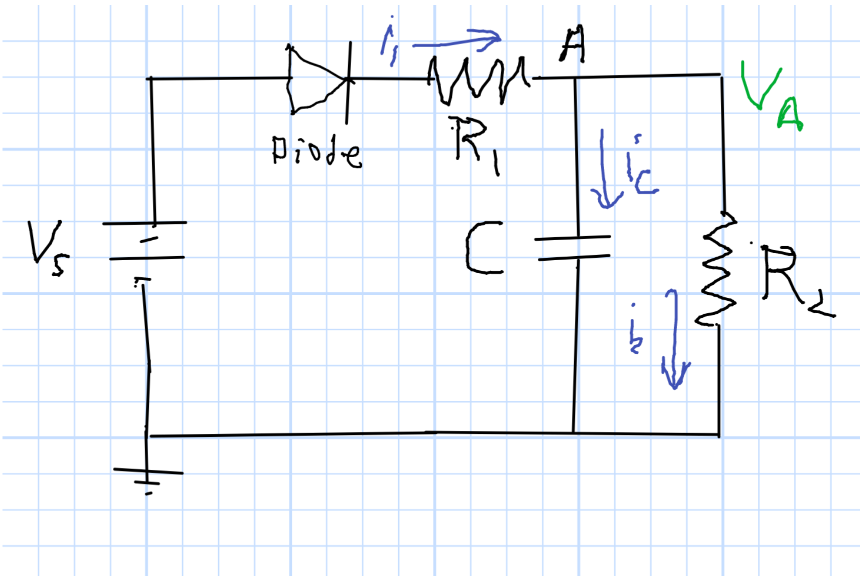
$$\tau \approx R_1 C = 10\text{ms}$$

Charging Voltage



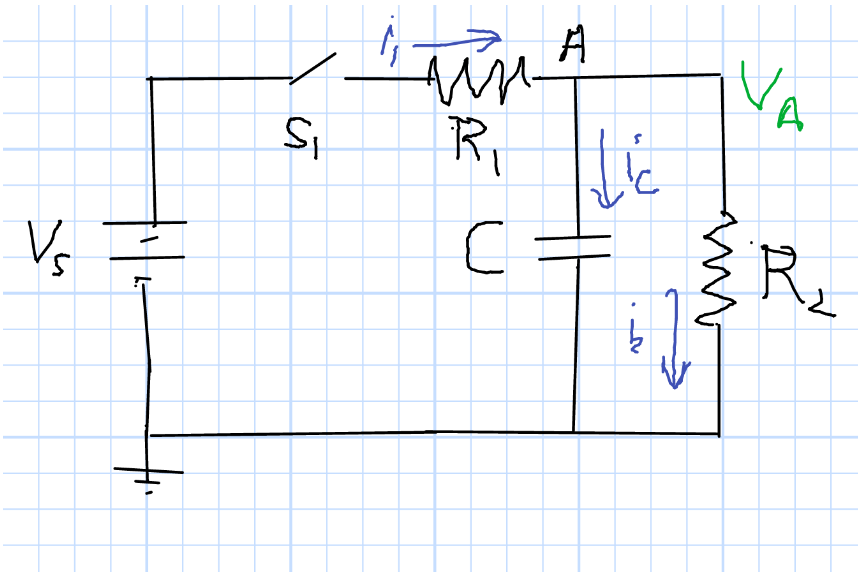
$$v_{A\infty} = v_T \approx v_s \qquad \tau_C \approx R_1 C \qquad v_A\left(\tau_C\right) \approx \left(1 - 0.3679\right) v_s = 0.6321 v_s$$

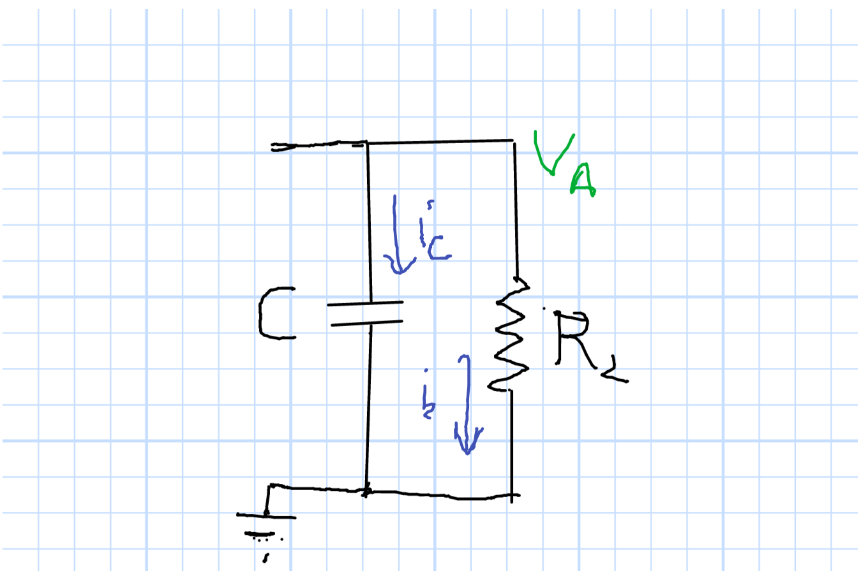
Diode: The Magic Switch



- $i > 0$ and $v \approx 0$ (Directon of the "Arrow")
- $v > 0$ and $i \approx 0$ (Directon of the "Arrow")
- Switches "ON" to Charge when $v_s > v_A$
- Switches "OFF" to Discharge when $v_s < v_A$

Discharge!





- Thévenin Equivalent

$$v_T = 0 \qquad R_T = R_2$$

- Initial Voltage
- Open S_1 at $t = t_1$

$$v_A(t_1) = v_s \frac{R_2}{R_1 + R_2} \times$$

$$\left\{1 - e^{-t_1/((R_1 \| R_2)C]}\right\}$$

- We've seen this before

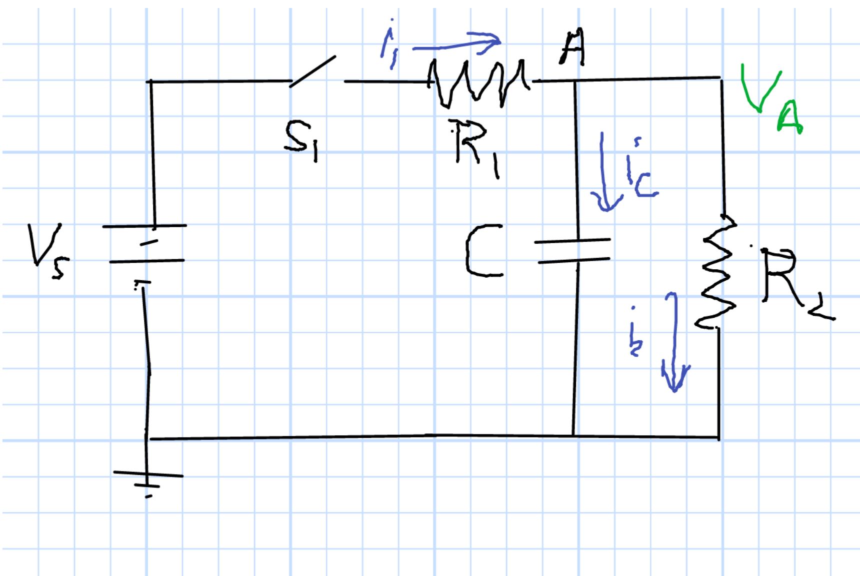
$$v_A = v_A(t_1)\, e^{-(t-t_1)/(R_2 C)}$$

- Example: $R_2 = 10\text{k}\Omega$

$$v_A = v_A(t_1)\, e^{-(t-t_1)/\tau_D}$$

$$\tau_D = 1\text{s}$$

Summary



- Charge

$$v_A = v_s \frac{R_2}{R_1 + R_2} \left\{1 - e^{-t/\tau_C}\right\}$$

- Charging Time Constant

$$\tau_C = (R_1 \parallel R_2)\, C$$

- End of Actual Charge

$$v_C = v_s \frac{R_2}{R_1 + R_2} \left\{1 - e^{-t_1/\tau_C}\right\}$$

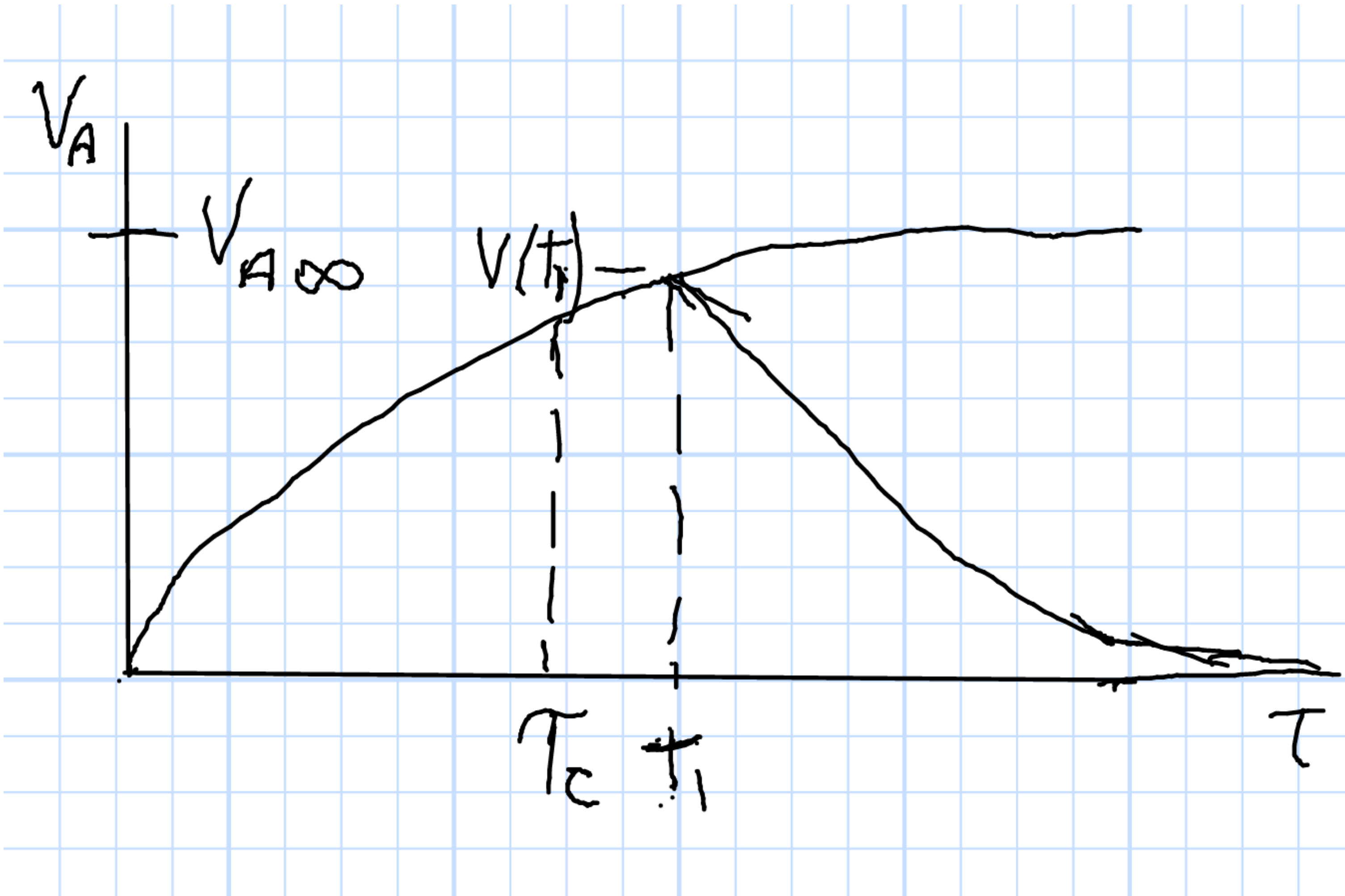
- Discharge

$$v_A = v_C\,(t_1)\, e^{-(t-t_1)/\tau_D}$$

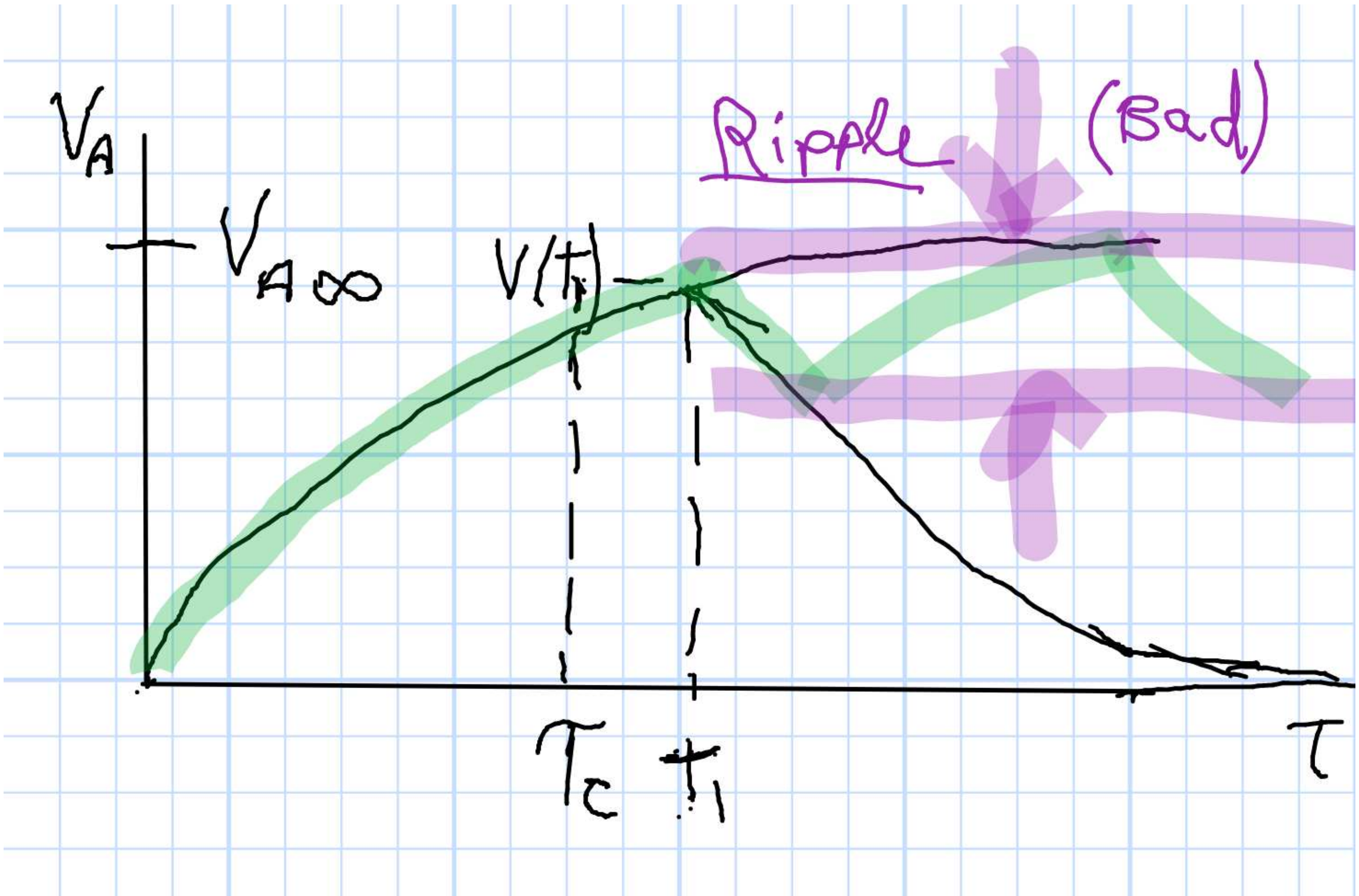
- Discharge Time Constant

$$\tau_D = R_2 C$$

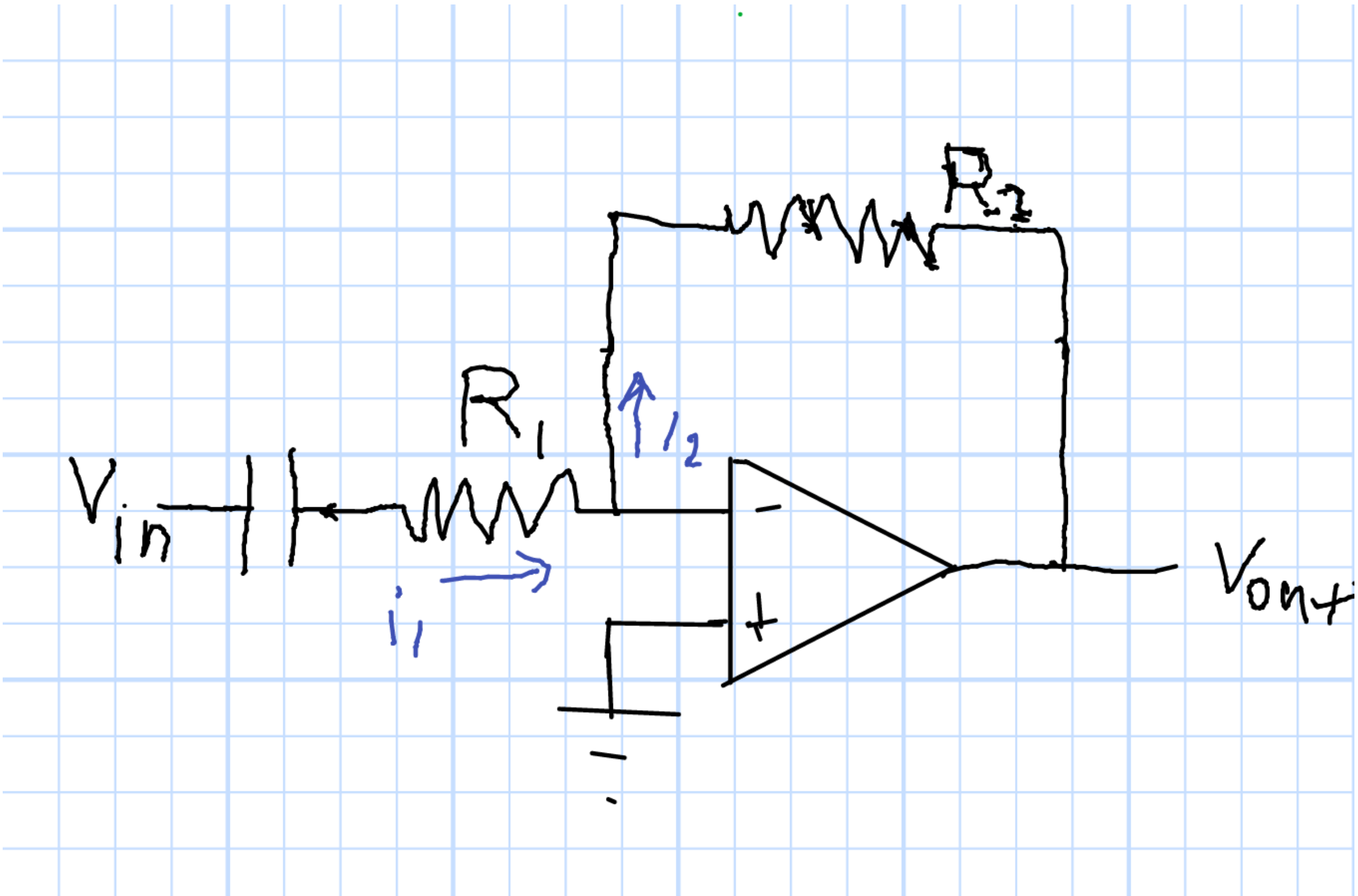
Charging and Discharging Voltage



Repeated Charging and Discharging Voltage



AC Coupled Amplifier



Steady State?, Transient?