Circuits and Signals: Biomedical Applications
Week 6

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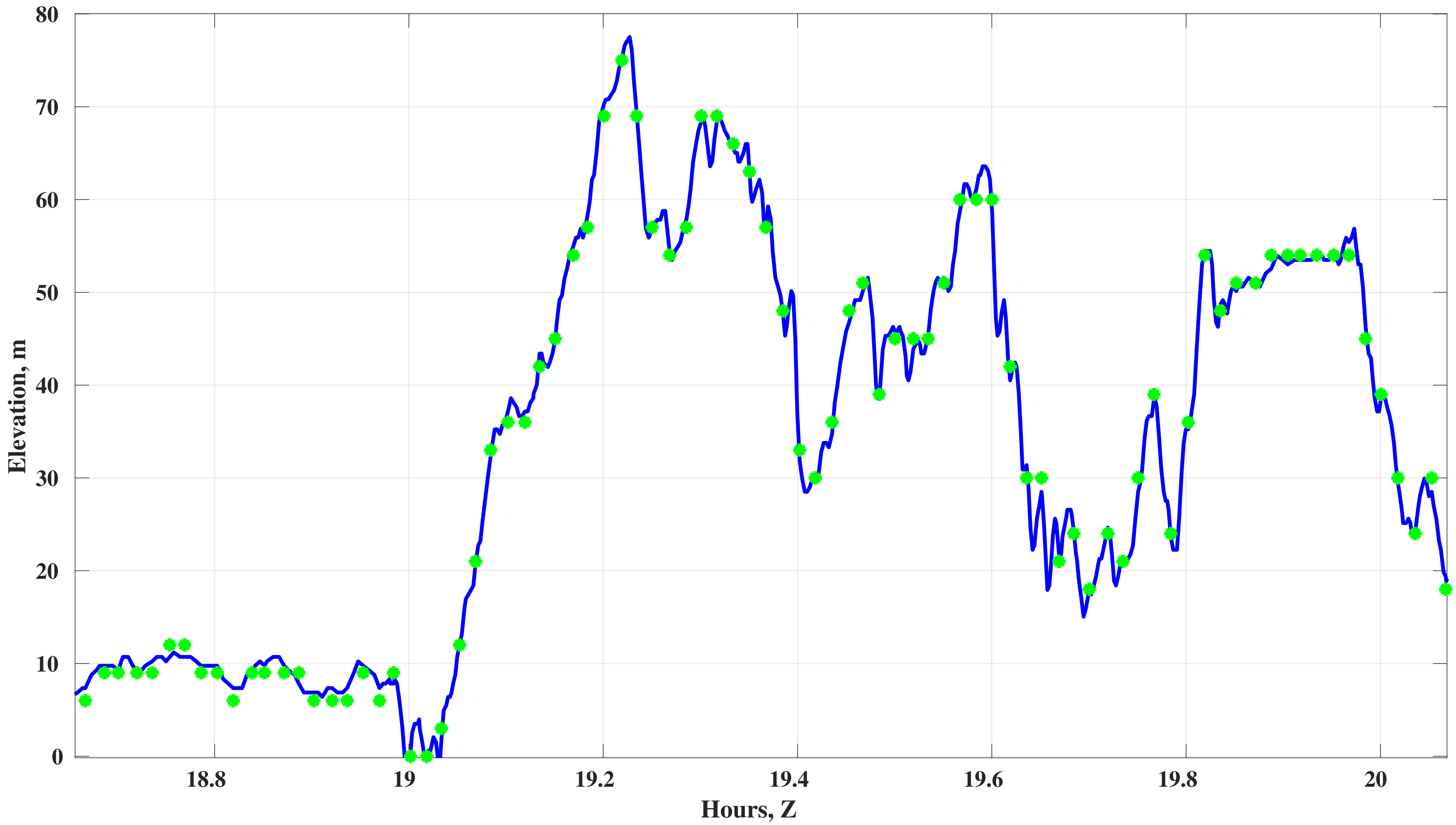
Week 6 Agenda:

- Digital and Analog Data
- Sampling
- Complex Numbers
  - Basics
  - Mathematical Operations
  - Sinusoids

Digitization



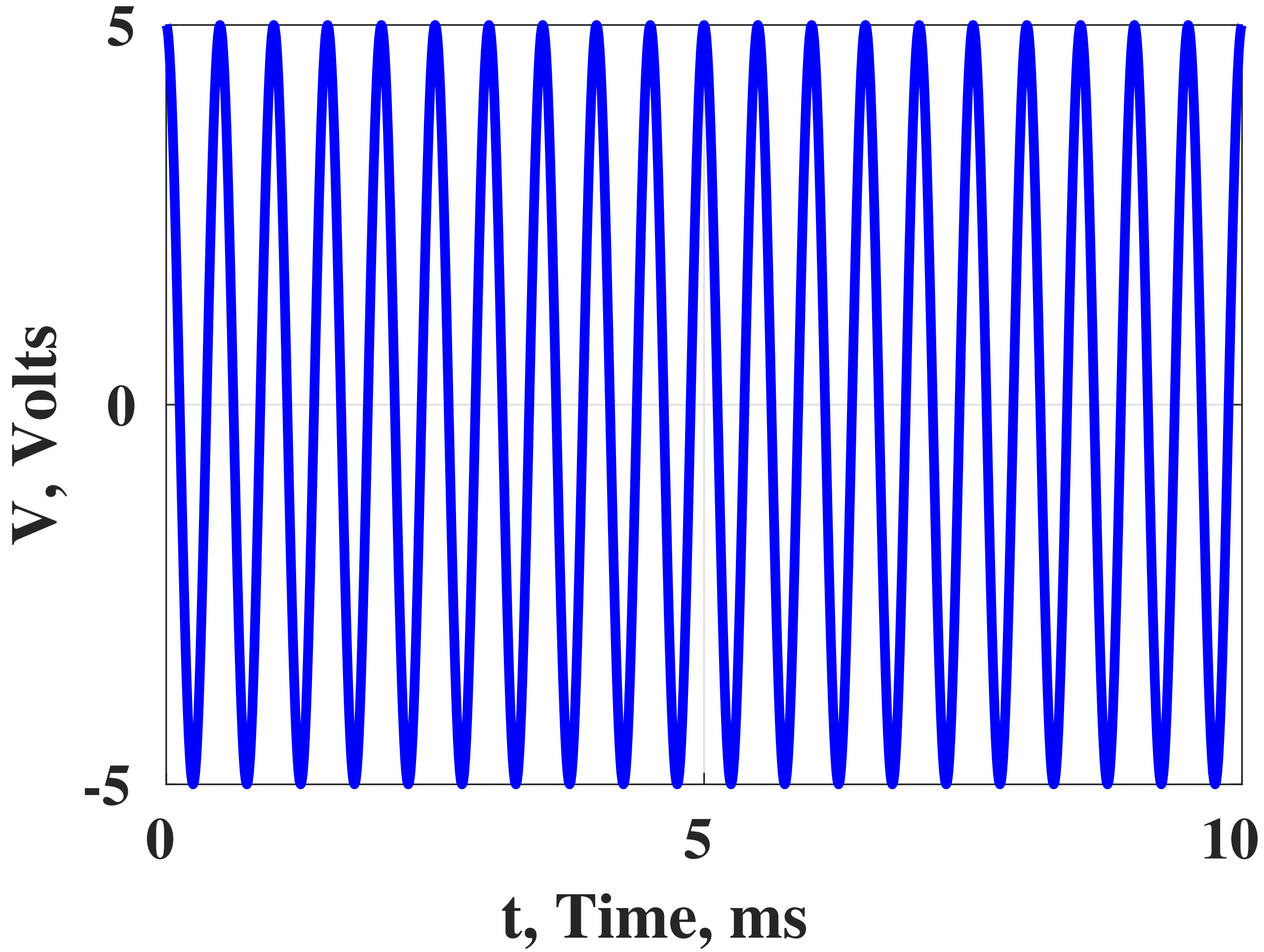
In a computer, all data is digital: Discrete in time, limited resolution.
f_s = 60/hr, Elevation Step Size = 3m: What is dE/dt?
Max(E)? Min(E),

Issues

- f_s Sample Frequency
  - Missing Data
  - Aliasing
  - Sampling Theorem $f_s > 2f_{max}$ (Nyquist)
- Range and Number of Bits
  - Step Size $= \frac{max-min}{2^n}$: *e.g.* 1 count = 3 meters
  - Dynamic Range $\frac{max-min}{step} = 2^n$:
    e.g. $max - min = 3$ meters $\times 2^{14}$ for 14 bits.

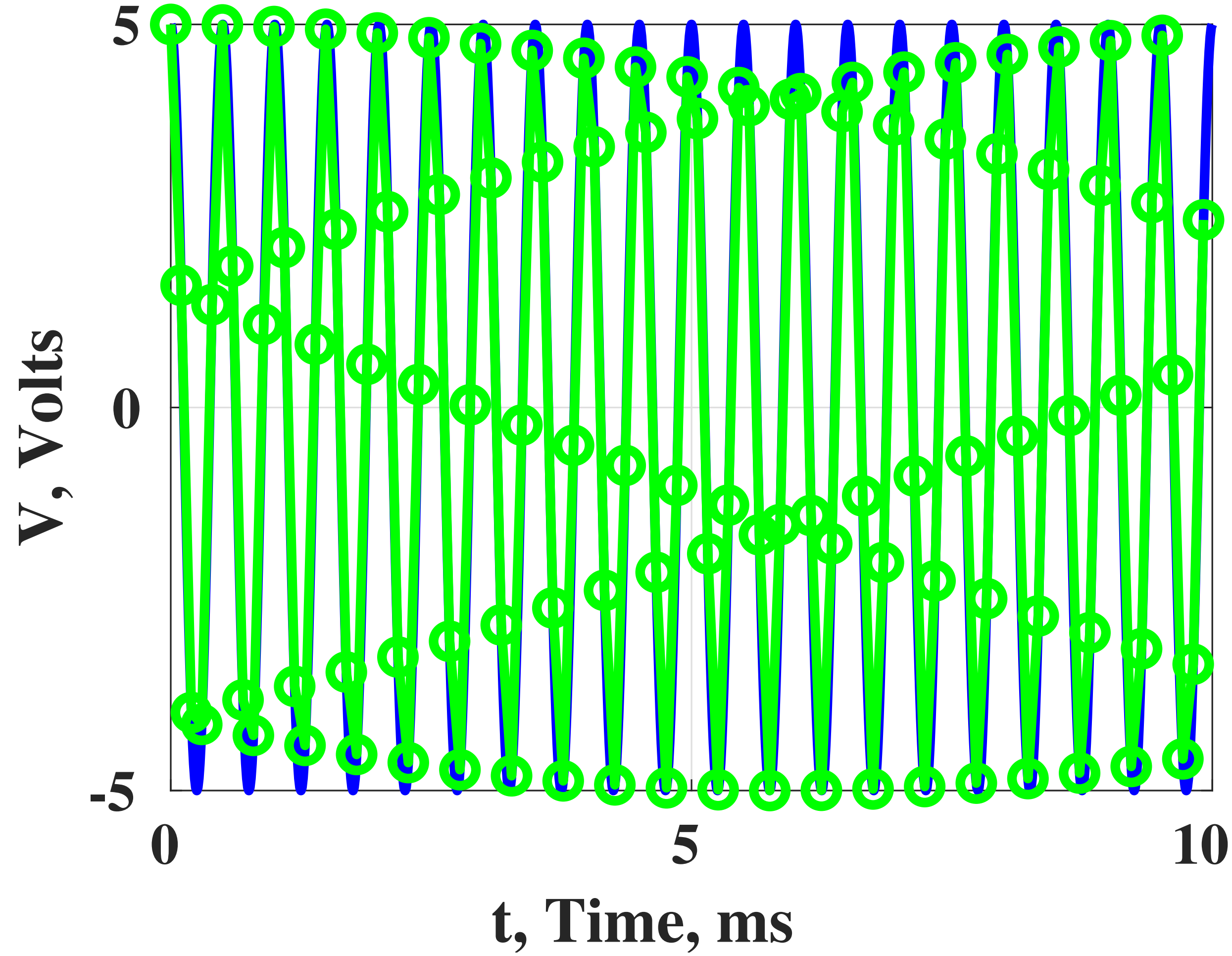
Sine Wave

New Concept: Nyquist Theorem (Sample at twice highest frequency)



$f_0 = 2$kHz (Sampled at 100kHz)

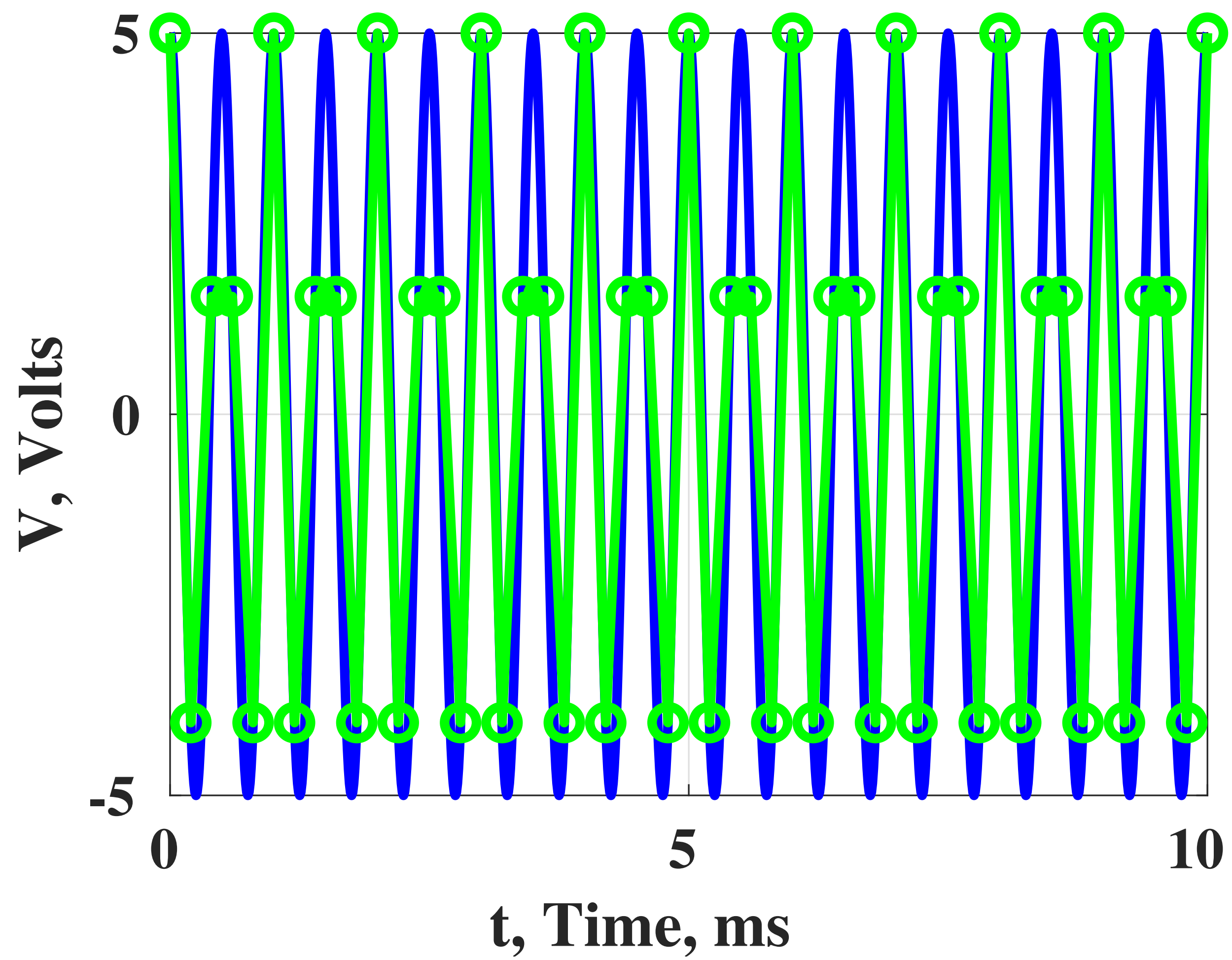
Way Oversampled Sine Wave



$f_0 = 2$Hz (Sampled at f_s10.085kHz)

Keeps Nyquist Happy and Just Plotting the Points Looks Pretty Good

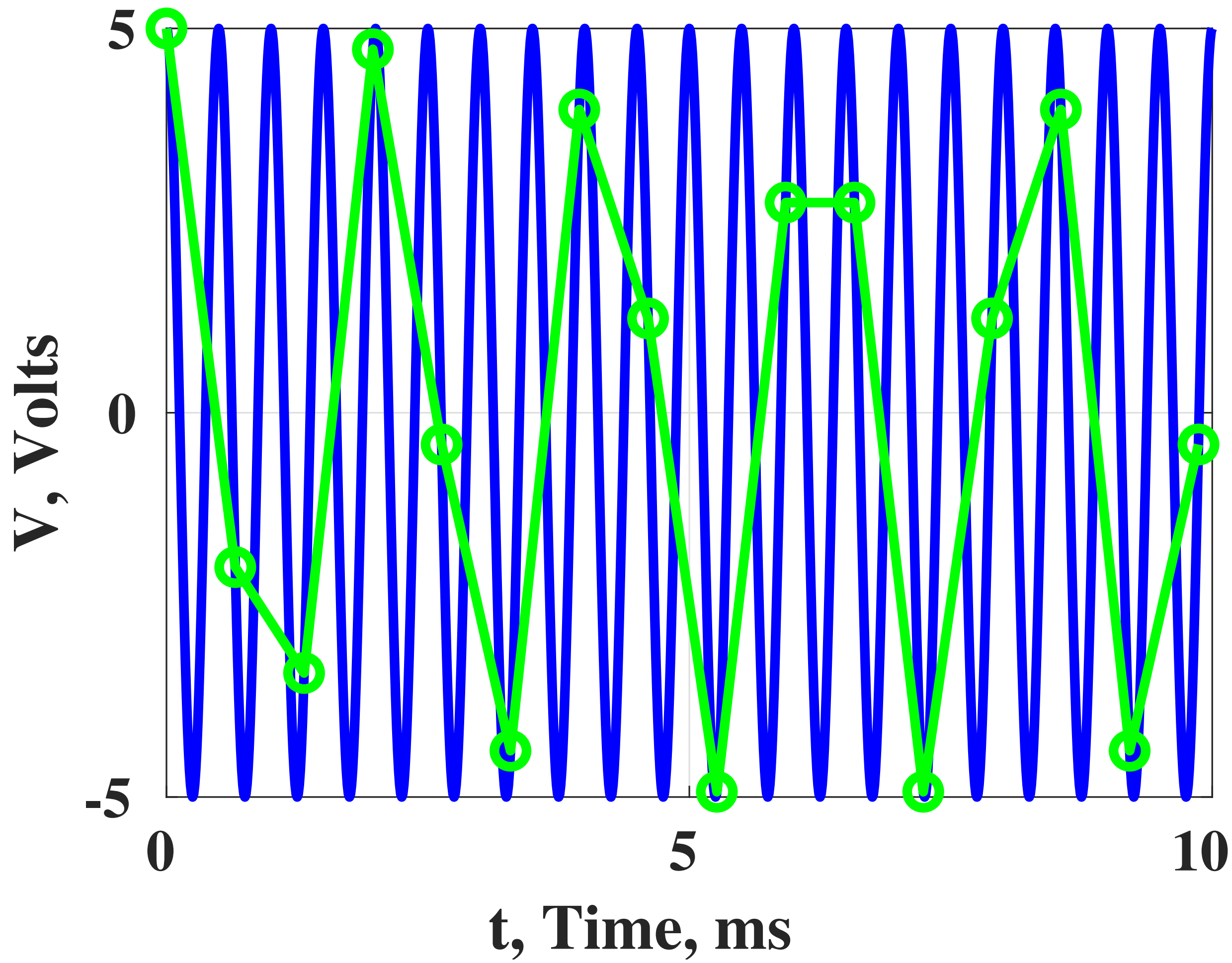
Sampled Sine Wave



$f_0 = 2$Hz, $f_s = 5$kHz: Keeps Nyquist Happy but Looks Bad

Nyquist Says we can recover the signal: Not that it's easy.

Undersampled Sine Wave



$f_s = 1520$Hz: Very Unhappy Nyquist: Wrong Frequency.
There is no way to recover the signal.

Complex Numbers

- The Complex Unit (New Concept)

$$i = j = \sqrt{-1}$$

- Real and Imaginary Parts

$$Re(z) = x \qquad Im(z) = y \qquad z = x + jy$$

- Addition

$$z_1 + z_2 = x_1 + x_2 + j\,(y_1 + y_2)$$

- Powers of the Complex Unit

$$j = \sqrt{-1} \qquad j^2 = -1 \qquad j^3 = -j \qquad j^4 = 1 \qquad \textit{etc.}$$

Deriving Euler's Formula

- Taylor Series for Exponential

$$e^{j\phi} = 1 + \frac{(j\phi)}{1!} + \frac{(j\phi)^2}{2!} + \frac{(j\phi)^3}{3!} + \dots$$

$$e^{j\phi} = 1 + \frac{j\phi}{1!} + \frac{-\phi^2}{2!} + \frac{-j\phi^3}{3!} + \dots$$

- Cosine and Sine

$$\cos\phi = 1 - \frac{\phi^2}{2!} + \dots \qquad j\sin\phi = j\frac{\phi}{1!} - j\frac{\phi^3}{3!} + \dots$$

- Euler's Formula

$$e^{j\phi} = \cos\phi + j\sin\phi$$

i or j: In EE, it's j

- Imaginary Unit

$$\sqrt{-1} = \pm i$$

- But in EE, i is Current
- We use j

$$\sqrt{-1} = \pm j$$

WPJ

- Euler's Formula

$$e^{j\theta} = \cos\theta + j\sin\theta$$

$$e^{j\omega t} = \cos\omega t + j\sin\omega t$$

- Cosine and Sine

$$\cos\omega t = \frac{e^{j\omega t} + e^{-j\omega t}}{2}$$

$$\sin\omega t = \frac{e^{j\omega t} - e^{-j\omega t}}{2j}$$

- Why Exponentials?
  - Compact Notation
  - Easy Math

Some Things are Easier

- Euler's Formula

$$e^{j\theta} = \cos\theta + j\sin\theta \qquad e^{j\omega t} = \cos\omega t + j\sin\omega t$$

- Including a Phase Shift is Simpler: Who remembers the cosine of a sum?

$$\cos(\omega t + \phi) = \frac{e^{j(\omega t+\phi)}}{2} + \frac{e^{-j(\omega t+\phi)}}{2}$$

- Derivatives Are Simple

$$\frac{de^{j\omega t}}{dt} = j\omega e^{j\omega t} \qquad \frac{de^{-j\omega t}}{dt} = -j\omega e^{-j\omega t}$$

- Derivatives Are Important for Capacitors and Inductors

Math Operations

- Addition/Subtraction Again (Think of z as v or i)

$$z_1 + z_2 = x_1 + x_2 + j\left(y_1 + y_2\right)$$

- Multiplication

$$z_1 z_2 = \left(x_1 + jy_1\right) + \left(x_2 + jy_2\right)$$

$$z_1 z_2 = x_1 x_2 - y_1 y_2 + j\left(x_1 y_2 + x_2 y_1\right)$$

- Complex Conjugate (New Concept)

$$z^* = x - jy \qquad (= \texttt{conj(z)} \text{ in Matlab})$$

The Complex Conjugate

- Some Rules that Make Life Easier

$$(z_1 + z_2)^* = z_1^* + z_2^*$$

$$(z_1 z_2)^* = z_1^* z_2^*$$

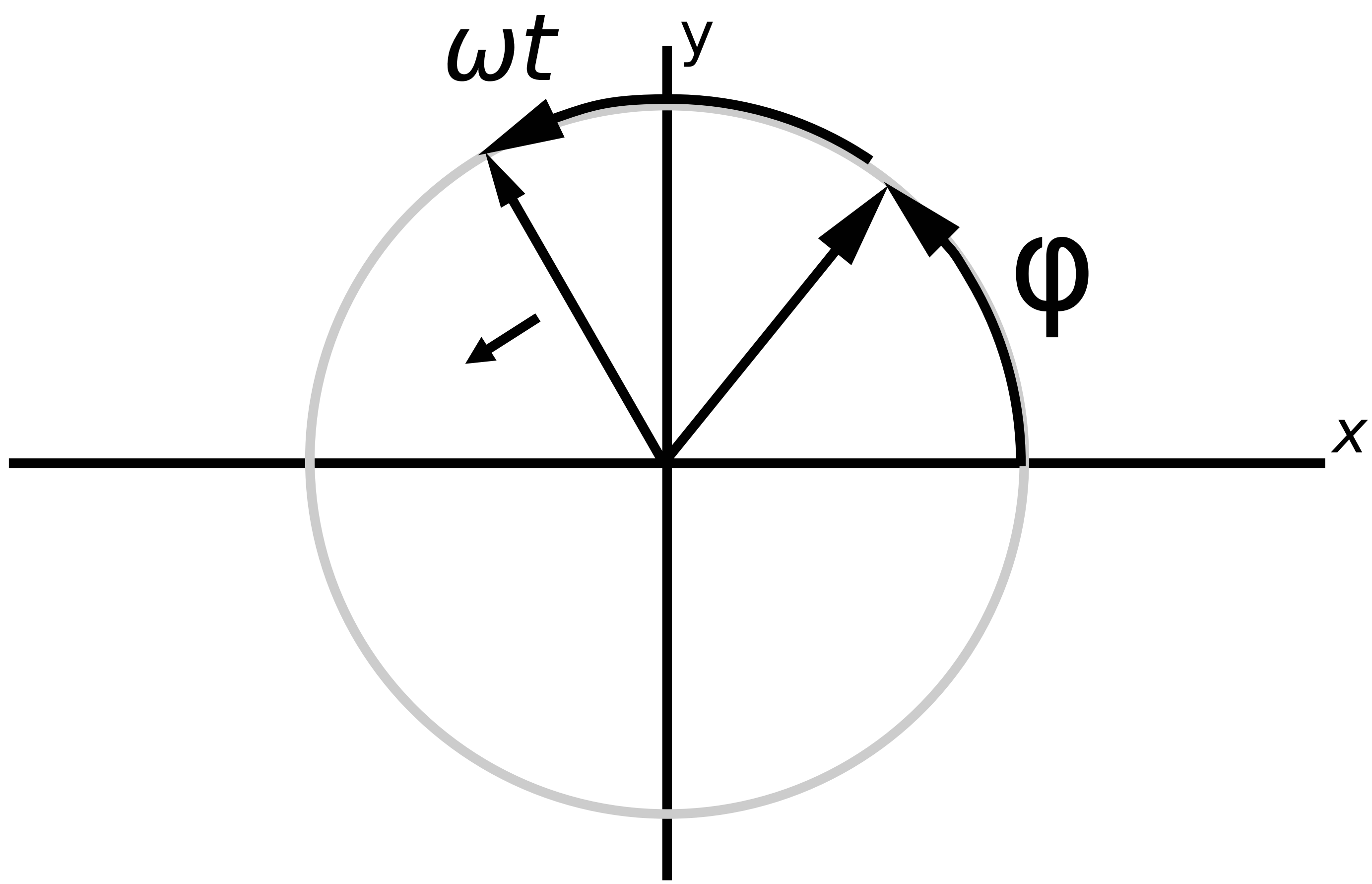
- Some Real Useful Results
  - Useful for getting real results

$$z + z^* = (x + jy) + (x - jy) = 2x = 2Re(z)$$

  - Useful for talking about power

$$zz^* = |z|^2$$

Phasors



Polar Form

- Amplitude

$$|z|^2 = x^2 + y^2 = zz^* \qquad |z| = \sqrt{x^2 + y^2}$$

- Phase (naturally in radians)

$\phi = \arctan\left(y/x\right)$ but be careful which quadrant

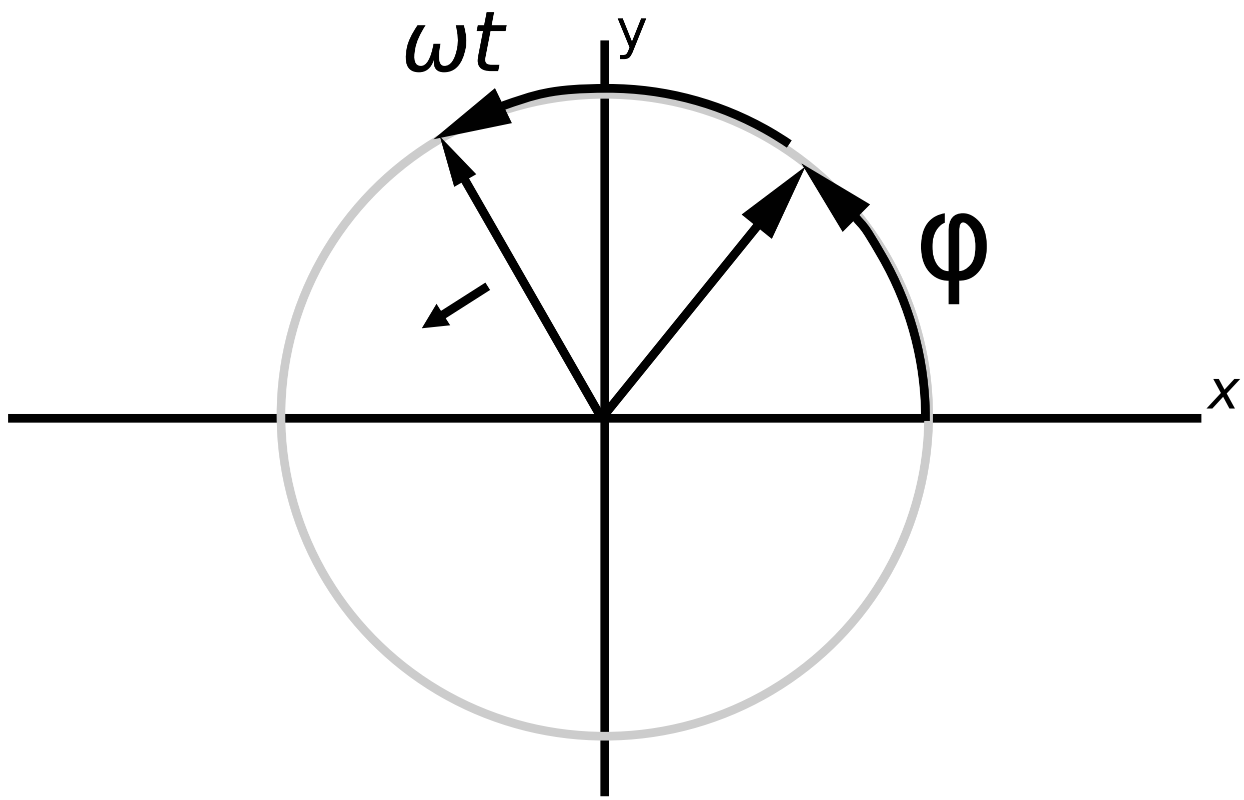
`angle(z)` in Matlab does the right thing.)

- Don't use `phase(z)` in Matlab.

- Notation for Polar Form (Be careful of radians and degrees)

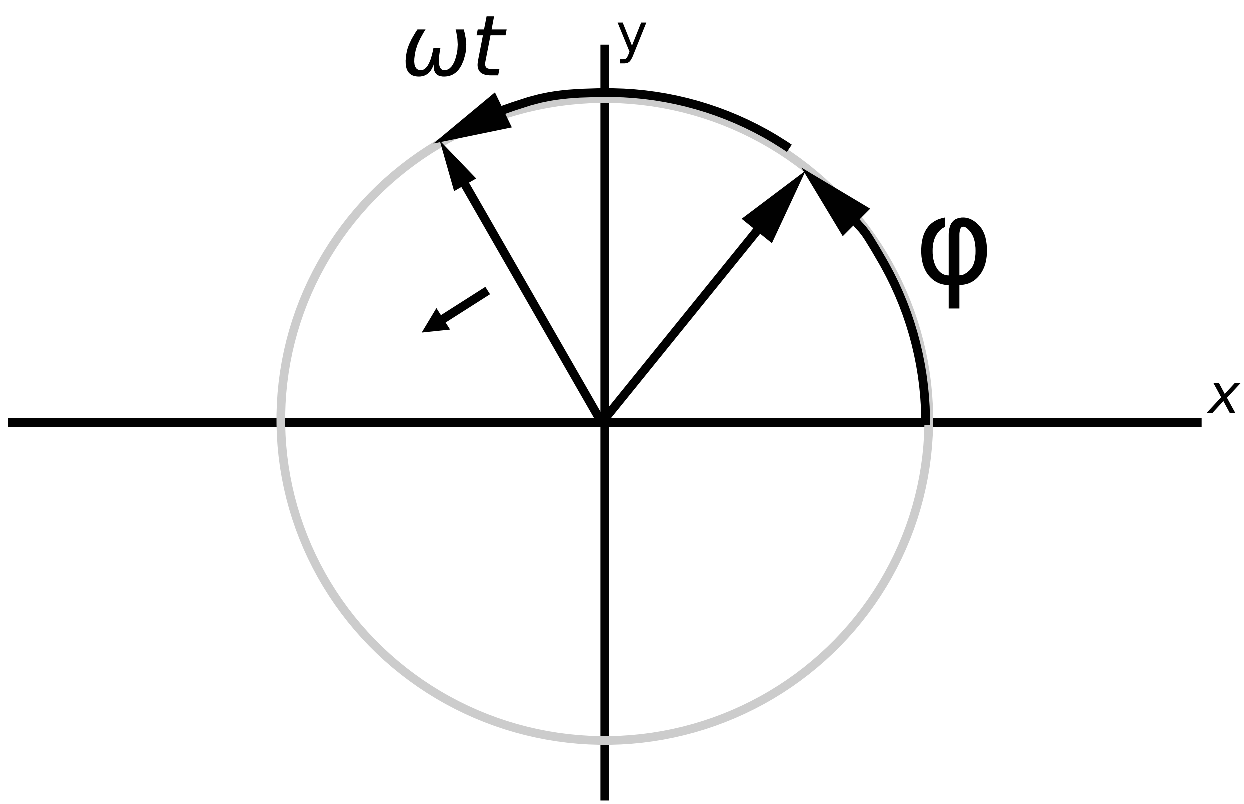
$z = |z|\angle\phi$ *e.g.* $17\angle\frac{\pi}{3}$ or $A\angle\phi$ or $(4\text{mA})\angle 23^\circ$

Be Careful with Phase



$x + jy$ and $-x - jy$ have different phases
although arctan $(y/x) =$ arctan $(-y/-x)$
Remember that `angle(z)` in Matlab does the right thing.

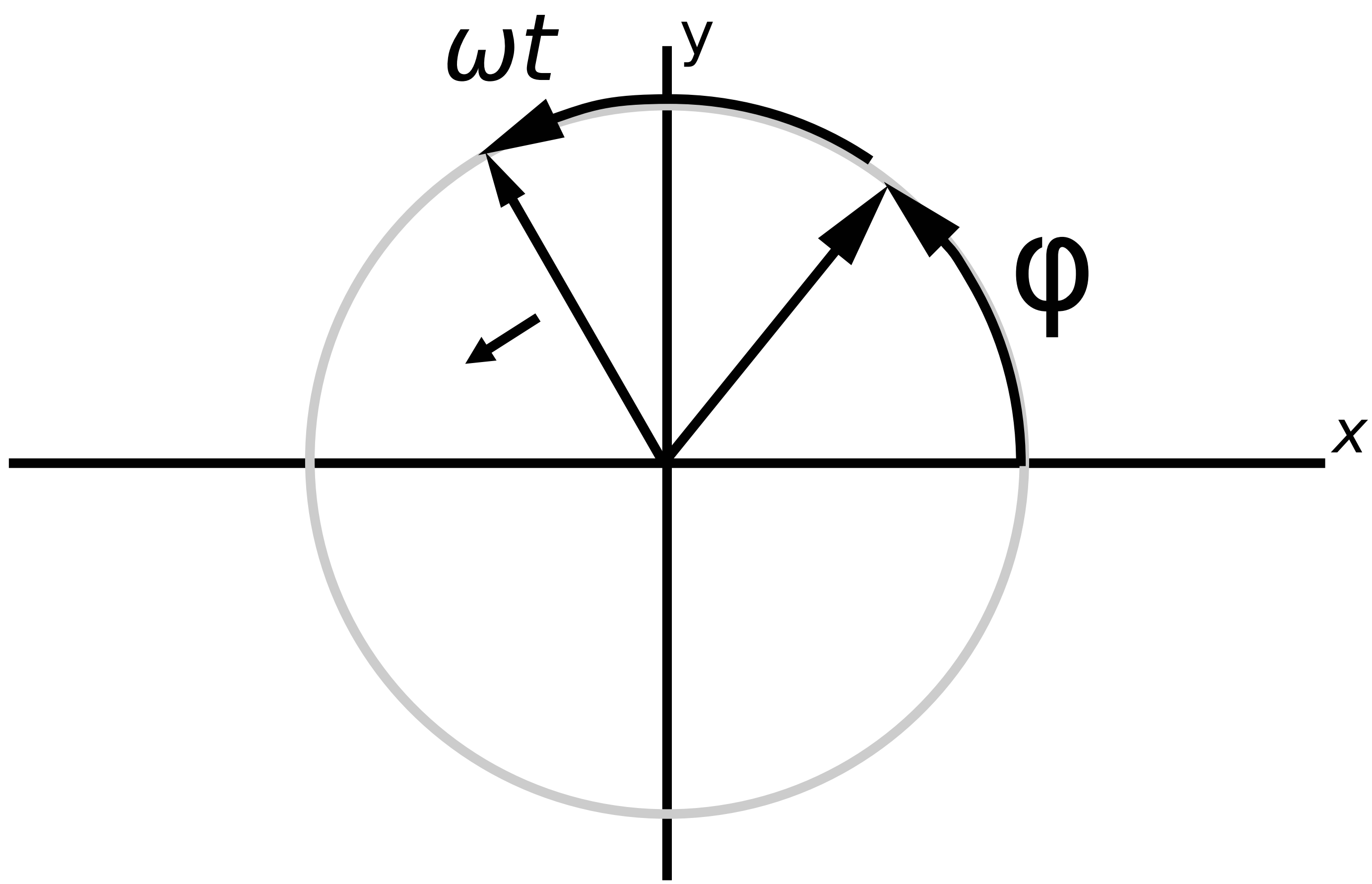
Rectangular Form



$$x = Re(z) = |z| \cos \phi$$

$$y = Im(z) = |z| \sin \phi$$

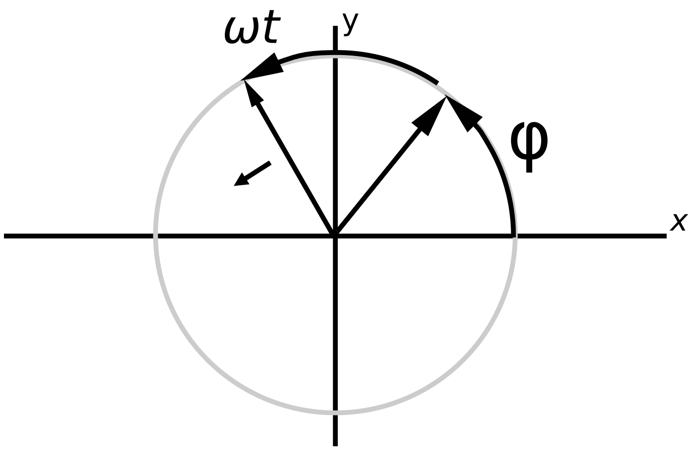
Sinusoids and Phasors

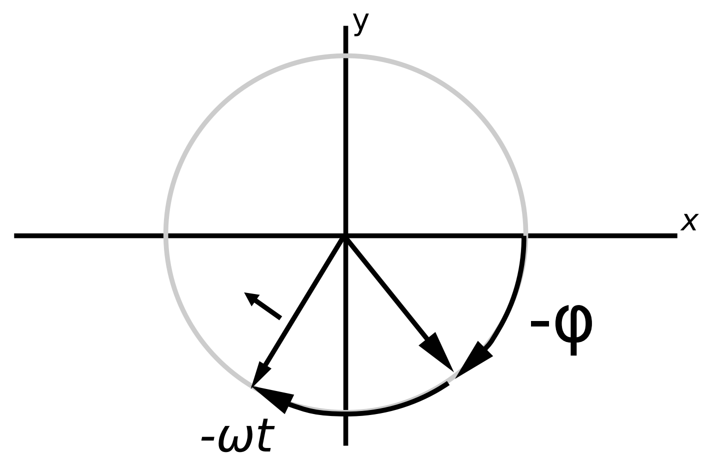


If $\phi = 0$ real part is $|z| \cos \omega t$

In general, real part is $|z| \cos (\omega t + \phi)$.

Using Negative Frequencies





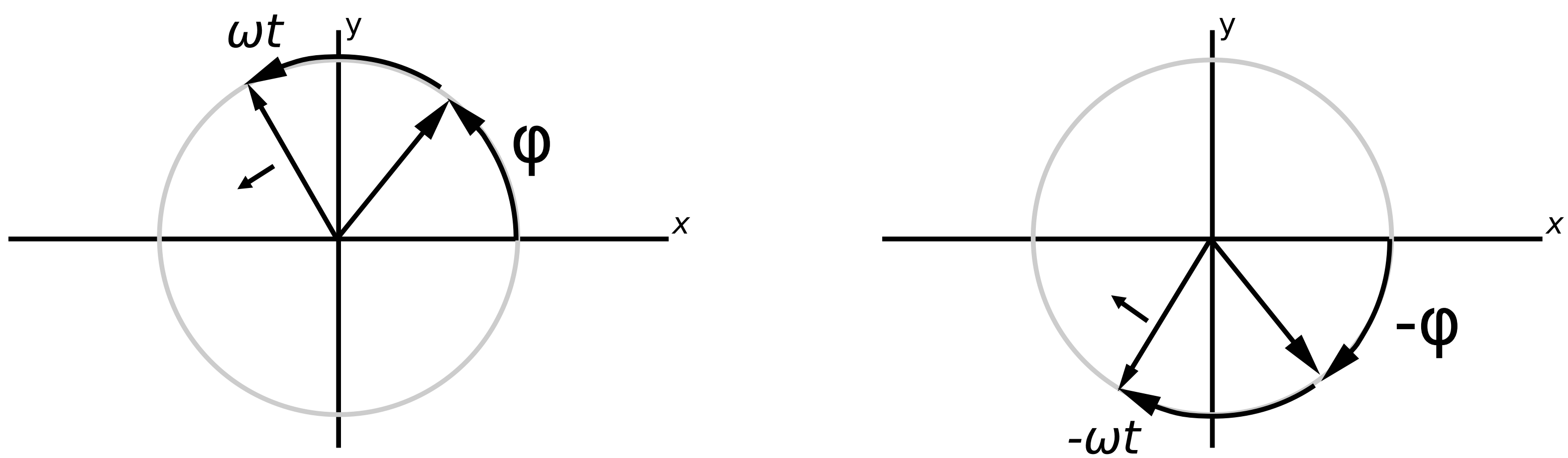
$$|z|e^{j(\omega t+\phi)} \quad + \quad |z|e^{-j(\omega t+\phi)}$$

$$z + z^* = 2Re(z)$$

If $\phi = 0$ real part is $2|z|\cos\omega t$.

In general, real part is $2|z|\cos(\omega t + \phi)$.

A Little Confusion



$$|z|e^{j(\omega t+\phi)} \quad + \quad |z|e^{-j(\omega t+\phi)}$$

$$z + z^* = 2Re(z)$$

For linear problems we can just solve the positive frequency part.

Then the negative–frequency part is just the conjugate.

Some people like to add the complex conjugate.

Some like to take the real part.

It's easy to make errors of factors of 2.

Be Able to. . .

- Perform Math on Complex Numbers . . .
- Convert Between Polar and Rectangular . . .
- . . . on Paper
- . . . on Your Computer
- . . . on Your Calculator
- Find the Phasor for a Sinusoid
- Find the Sinusoid from a Phasor