ECEU692 Intro to SSI Course Notes

Some selected topics in linear algebra for SSI(1)

Profs. DiMarzio and Brooks
Northeastern University

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Outline of those Selected Topics

- Some subsurface problems that lead to linear systems of equations
- Some different ways of looking at n equations in n unknowns: in particular, some geometric interpretations
- The geometry of when linear systems have
  - exact solutions,
  - no solutions, and/or
  - non-unique solutions
- Problems where you have more equations than unknowns
- Problems where you have more unknowns than equations
- A quick discussion of solving linear systems in Matlab
- The Singular Value Decomposition to analyze linear systems

Why are We Interested in Linear Systems for SSI?

Single-layer problems (like you did in lab, questionably sub-surface)

- Reflectance, calculated from measurement, is approximately equal to Albedo

$$R = \frac{\mu_s'}{\mu_s' + \mu_a}$$

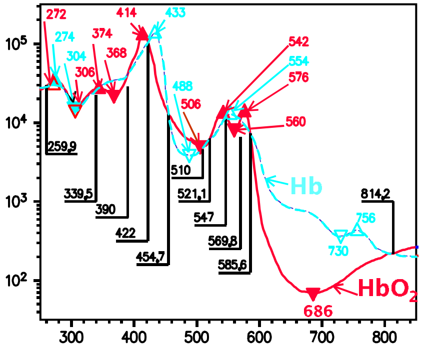
- We want to recover concentration of various chromophores
- Scattering assumed approximately constant with wavelength
- Absorption varies with wavelength over range of interest
- Absorption is weighted linear sum of chromophore concentrations

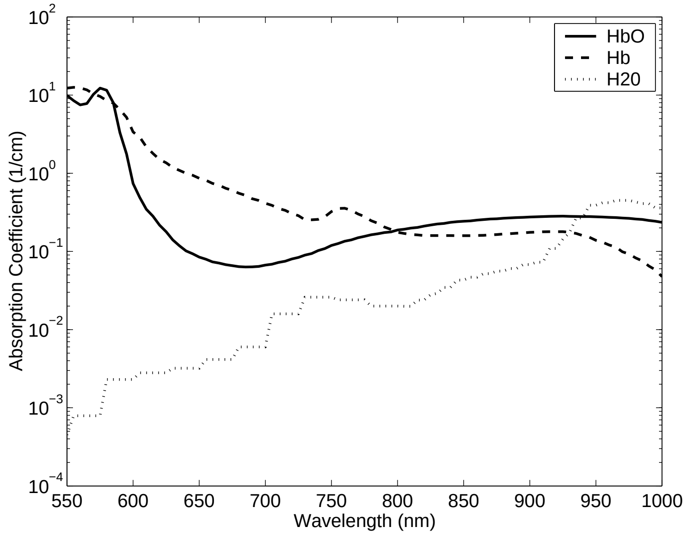
$$\mu_a = \kappa_1 c_1 + \kappa_2 c_2 + \ldots \kappa_n c_m,$$

  κ's give absorption of each chromophore and c's the concentrations.

- Note concentrations not absolute

Where do the κ's come from?





- Could be 4, 5, or more relevant chromophores
- Might need an additional term for background

Examples

- Paint (like you looked at in the lab)
- Looking at the earth from a satellite.
- Pulse oximetry (after some preliminary tricks).
- Key assumptions:
  - Everything is known (calibrated) except the single layer
  - Single layer is "infinitely" thick
  - No specular component

Getting this to look like a linear system of equations

- Inverting the albedo expression (and showing the frequency dependence explicitly at the ith wavelength):

$$\frac{1}{R(\lambda_i)} = \frac{\mu_s' + \mu_a(\lambda_i)}{\mu_s'}$$

$$b(\lambda_i) = \frac{1}{R(\lambda_i)} - 1 = \frac{\mu_a(\lambda_i)}{\mu_s'}$$

- Assume either μ_s' is known or we recover fractional absorption. Plug in for μ_a in terms of previous expression:

$$b(\lambda_i) = \sum_{k=1}^{m} \kappa_k(\lambda_i) c_k$$

- So: at each frequency, a linear equation relating measurements to unknown desired concentrations through known "coefficients".
- c_k depend on chromophore but not wavelength
- Take measurements at n wavelengths, form system of equations

Matrix-vector Formulation

$$\begin{bmatrix} b_{\lambda_1} \\ b_{\lambda_2} \\ \vdots \\ b_{\lambda_n} \end{bmatrix} = \begin{bmatrix} \kappa_1(\lambda_1) & \kappa_2(\lambda_1) & \dots & \kappa_m(\lambda_1) \\ \kappa_1(\lambda_2) & \kappa_2(\lambda_2) & \dots & \kappa_m(\lambda_2) \\ \vdots & \vdots & \dots & \vdots \\ \kappa_1(\lambda_n) & \kappa_2(\lambda_n) & \dots & \kappa_m(\lambda_n) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_m \end{bmatrix}$$

$$\mathbf{b} = \mathbf{K} \cdot \mathbf{c}$$

- Each column has samples of spectrum of one chromophore at n wavelengths.
- Each row has samples of m chromophore spectra at the same wavelength.
- Some Questions:
  - How many wavelengths to use? $n = m$? $n > m$?
  - Which wavelengths to choose?
  - Should we impose other constraints, and if so how?
    - Sum of concentrations should be 1
    - All concentrations should be ≥ 0.
  - What if we don't know the spectra? or only know some of them?

Two-Layer Problem

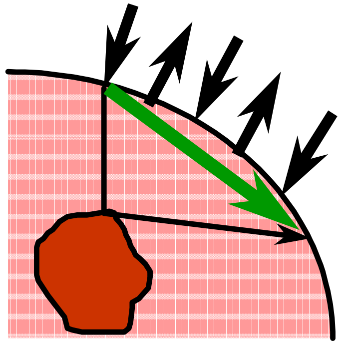
- Start with equation from Chuck's slide 10471-8-15:

$$R = R_0 + \frac{T^2 \mu_s'}{\mu_s' + \mu_{a0} + \mu_a}.$$

- Again assume only μ_a varies with frequency.
- As previously pointed out, non-linear problem.
- However if we can
  - measure R_0 (reflectance of top layer), and
  - either measure other parameters or accept only recovering scaled versions of μ_a,

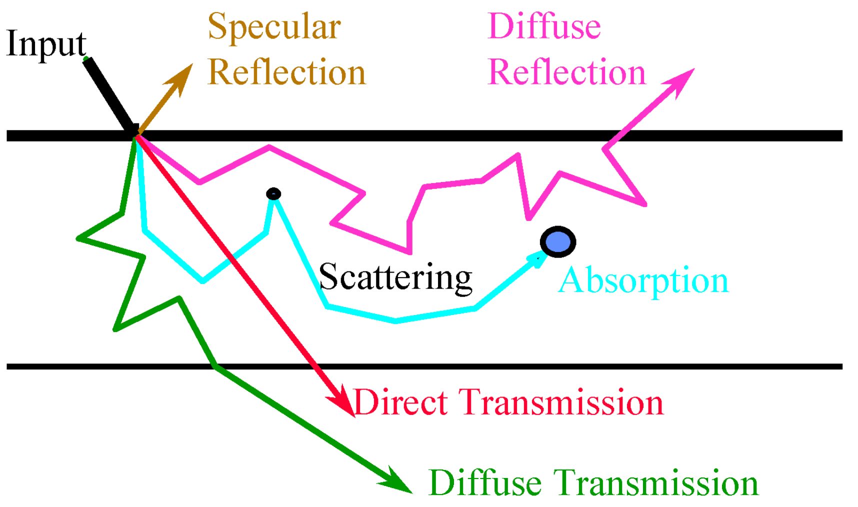
  then reduces to a similar problem to the one above, with complicated function of data on right-hand side
- Same type of questions as before
- Examples
  - Looking at earth through atmosphere
  - Use dark target area to get R_0

Diffuse Optical Tomography



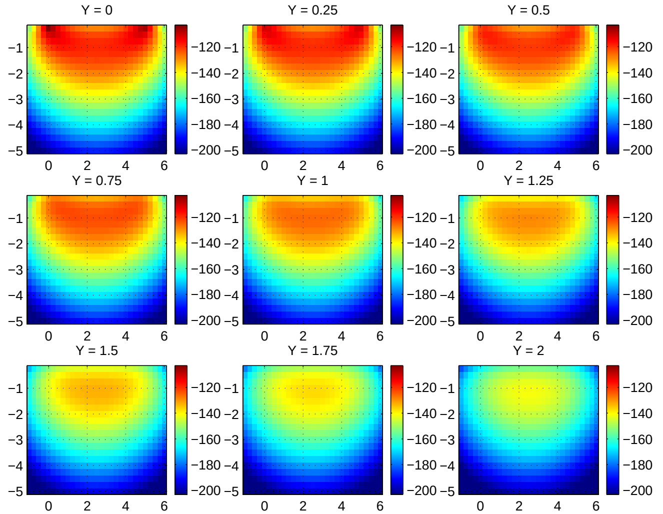
- Measure highly scattered light with an array of sensors
- Illuminate (one at a time or equivalent) with an array of sources
- Try to use all measurements together to infer distribution of scattering and absorption in medium
- Let's concentrate on absorption only

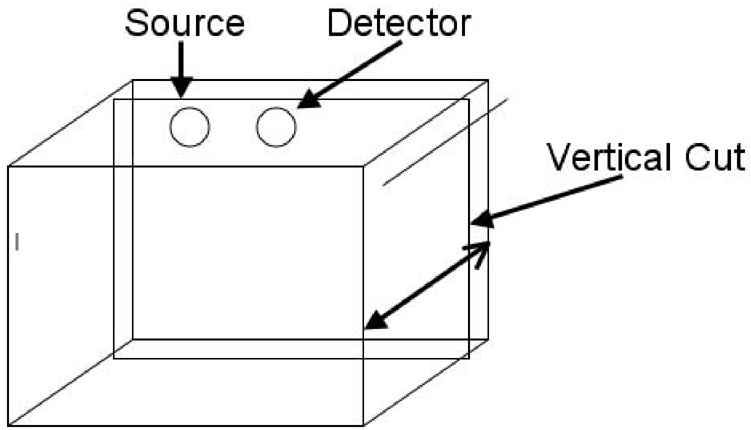
What Does Each Detector See?



- Discretize space into little cubes ("voxels")
- Each voxel sees an amount of light depending on its position relative to the source
- Each detector seems an amount of light from each voxel depending on voxel location and voxel optical properties
- Ignore optical interaction between voxels

A Simulation of How This Works

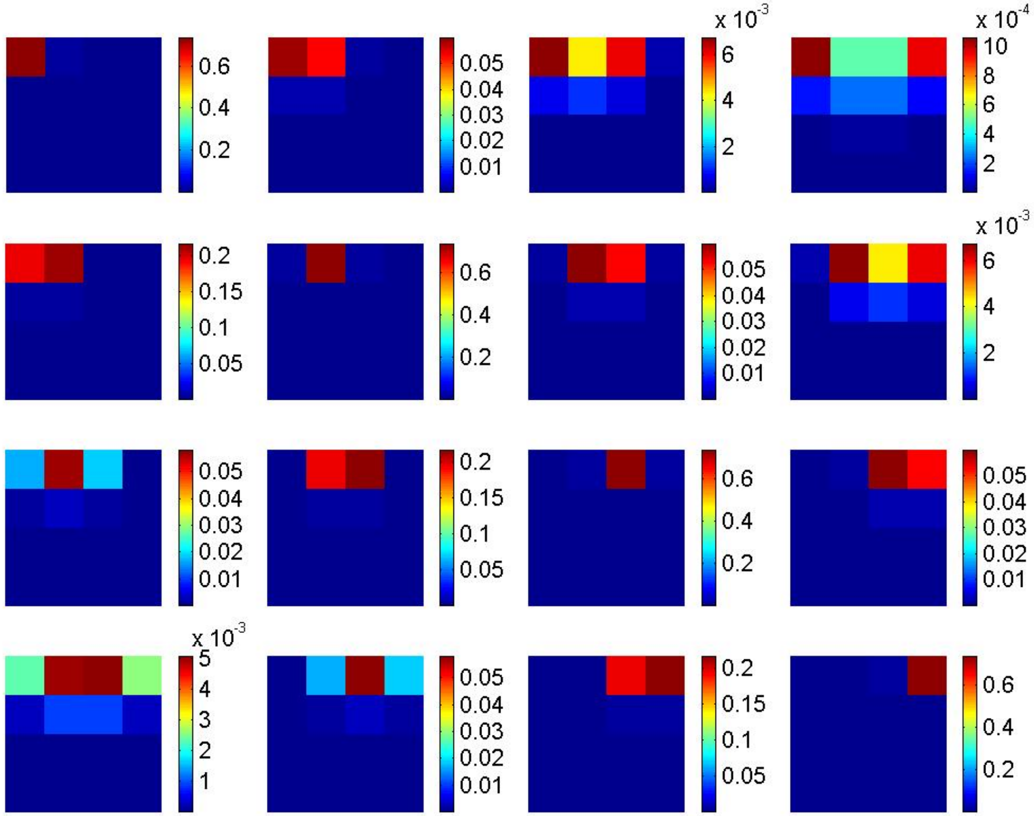




Sensitivity profiles of vertical slices through a cube as you move away from the place where the source-detector pair are $(\boldsymbol{Y} = \mathbf{0})$

A 4 $\times$ 4 Geometry with 4 Sensor/Detector Pairs

For each source-detector pair, what is its sensitivity to each voxel ?



Linear DOT

- Under certain assumptions (We will discuss in detail later)
  - Known, constant background of scattering and absorption
  - Absorption perturbations
  - Scattering much greater than absorption
  - Discretize known geometry
  - Some other details
- Can approximate a function of measured fluence for each source/detector pair as linear combination of
  - absorption of each discrete "voxel" times
  - a geometric quantity depending where voxel is with respect to that particular source/dectector pair
- Now stack these equations for each source/detector pair
- Note that the absorption of the voxel is the same for each such pair

DOT System of Equations

$$\begin{bmatrix} b_{p_1} \\ b_{p_2} \\ \vdots \\ b_{p_n} \end{bmatrix} = \begin{bmatrix} g_1(p_1) & g_2(p_1) & \dots & g_m(p_1) \\ g_1(p_2) & g_2(p_2) & \dots & g_m(p_2) \\ \vdots & \vdots & \dots & \vdots \\ g_1(p_n) & g_2(p_n) & \dots & g_m(p_n) \end{bmatrix} \begin{bmatrix} \mu_{a_1} \\ \mu_{a_2} \\ \vdots \\ \mu_{a_m} \end{bmatrix} \quad (1)$$

$$\mathbf{b} = \mathbf{G} \cdot \mathbf{m}$$

- Each column is for one voxel and gives its “path-weight” across all source-detector pairs
- Each row is for one source-detector pair and gives the “path-weights” of each voxel from that pair
- Thus each row corresponds to a “vectorized” version of each small image in slide 12.
- Small matrix entry means that voxel is not seen well by a particular source-detector pair

Some Comments on This Result

- This is a difficult linear system to solve accurately:
  - Due to scattering and absorption, differences in absorption are hard to "see" in measurements
  - Therefore need to "amplify" relatively small differences in measurements to recover differences in solution
  - Put another way, relatively large differences in absorption get blurred and attenuated into relatively small differences in measurements
  - If we have
    1. noise in data (always!)
    2. error in model (always!)

    then we amplify noise too !
- Many other linearized SSI modalities have similar set-up and problems:
  - acoustics
  - impedance tomography
  - bio-electric source imaging
  - hyperspectral source imaging

Other SSI Applications that Involve Linear Equations

It's a long list, including:

- Linear convolution problems (including one view of the apodization problem we've seen)
- Discrete Fourier Transform (including another view of the apodization problem we've seen)
- Subsampling
- De-convolution
- Source recovery (example, inverse electrocardiography)
- Linearizations of non-linear inverse problems (most tomography, including optical, impedance, acoustic, . . .)