DEPARTMENT OF ELECTRICAL AND COMPUTER ENGINEERING

NORTHEASTERN UNIVERSITY

ECE U692 INTRO TO SUBSURFACE SENSING AND IMAGING Spring 2004

Example for Homework Set 3 Problem 1a

Given the matrix

$$\mathbf{A} = \begin{bmatrix} 3 & 0 & 2 \\ 0 & 5 & 1 \end{bmatrix}.$$

Answer these 4 questions:

1. What is the dimension of its nullspace?
   Since

   (a) the first and 2nd columns are independent (since we can't write one as a scaled version of the other, for instance we can't get the 5 in the 2nd row of the 2nd column from the 0 in the 2nd row of the first column),

   (b) the first and 3rd columns are independent, and

   (c) the 2nd and 3rd columns are independent,

   then the dimension must be at least 2, i.e. the 3 columns point in different directions. But since the columns are only length 2 vectors, the dimension of the space they span cannot be more than 2. This means that the nullspace must have dimension $3 - 2 = 1$, that is, there must be one direction in $\mathcal{R}^3$ so that any vector in that direction is perpendicular to both rows, or in other words a vector pointed in that direction makes the columns add to 0.

2. Find a basis for its nullspace.
   By looking at the matrix we can see that $-(2/3)$ times the first column plus $-(1/5$ times the second plus 1 times the third gives 0. So the vector $[-2/3, -1/5, 1]^T$, or equivalently, $[-10, -3, 15]^T$, is a basis for the nullspace.

3. Verify that every vector in the nullspace is orthogonal to the rowspace.
   Any vector in the nullspace can be written as a scalar times a basis vector for the nullspace, e.g. $\alpha[-10, -3, 15]^T$. By simply computing the dot product of such a vector times boht rows you can see that these dot products are zero, or in other words that these vectors are orthogonal to the span of the rows because they're orthogonal to both rows themselves.

4. Given the vector $\mathbf{x} = [-4, 2, 20]^T$, split $\mathbf{x}$ into two vectors $\mathbf{x}_1$ and $\mathbf{x}_2$, such that $\mathbf{x} = \mathbf{x}_1 + \mathbf{x}_2$, $\mathbf{x}_1$ is in the rowspace of $\mathbf{A}$ and $\mathbf{x}_2$ is in its nullspace.
   If we get lucky, we notice that if we simply take the nullspace vector from above, $[-10, -3, 15]^T$, and subtract it from the given vector, we get $[6, 5, 5]^T$, and that this vector is 2 times the first row plus 1 times the second, and is thus in the rowspace.

   A more analytical approach would be to notice that we must have the following constraints: $\mathbf{x}_1$ must equal some number times the first row plus some other number times the second, while $\mathbf{x}_2$ must equal some third number times a nullspace vector. Finally we know the sum $\mathbf{x}_1 + \mathbf{x}_2$ because it's given in the problem statement. So we have three unknown numbers. If we write down all this information it turns out we have 3 equations for these 3 numbers and can solve this 3×3 system (say, using Matlab) to get the answer.