DEPARTMENT OF ELECTRICAL AND COMPUTER ENGINEERING

NORTHEASTERN UNIVERSITY

ECE U692 INTRO TO SUBSURFACE SENSING AND IMAGING Spring 2004

Homework Set 3a

Problem discussion: In class Mon. Feb. 23
Due: By the start of class on Wed. Feb. 25

For each problem please turn in a *complete* answer. In particular you should include any narrative required to understand your method and comment on your results where appropriate, label and place captions on any figures you include, and in general think of your homework as a short report. The intent is *not* to have you do a lot of writing for the sake of writing, and you can certainly assume that we know what the problem statement was, but your homework should be professionally done and we should not have to guess at how to interpret what you hand in.

Problem 1:Linear Algebra

This problem contains some exercises related to the Linear Algebra we've been discussing in class.

(a) Given the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 4 \end{bmatrix}.$$

Answer these 4 questions:

1. What is the dimension of its nullspace?
2. Find a basis for its nullspace.
3. Verify that every vector in the nullspace is orthogonal to the rowspace.
4. Given the vector $\mathbf{x} = [3, 3, 3]^T$, split $\mathbf{x}$ into two vectors $\mathbf{x}_1$ and $\mathbf{x}_2$, such that $\mathbf{x} = \mathbf{x}_1 + \mathbf{x}_2$, $\mathbf{x}_1$ is in the rowspace of $\mathbf{A}$ and $\mathbf{x}_2$ is in its nullspace.

(b) Given the matrix $\mathbf{A}$ and the vectors $\mathbf{x}$ and $\mathbf{b}$ as given below:

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} u \\ v \end{bmatrix}, \text{ and } \mathbf{b} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}.$$

1. Note that there are 3 possible ways to choose two rows of this system to get square (2×2) linear systems. Solve each of these system for the corresponding vector $\mathbf{x}$, and in each case compute the squared error of the original 3×2 system, that is the error between $\mathbf{Ax}$ and $\mathbf{b}$ (which is $||\mathbf{Ax}-\mathbf{b}||_2^2$ as defined in Prof. Brooks' notes) for each solution.

2. In this part you are asked to explicitly set up the least-squares solution in the following manner:

   (i) Write out a general formula for the squared error $\|\mathbf{Ax} - \mathbf{b}\|_2^2$ in terms of the specific numerical values of the elements of $\mathbf{A}$ and $\mathbf{b}$ and the symbolic elements of $\mathbf{x}$, u and v.
   (ii) Take derivatives of this error with respect to u and v, and
   (iii) Setting the error equal to zero, reassemble the resulting two equations into a linear system.

3. How does this linear system compare to what you would have expected from the solution given in class?

4. Solve this system and compare the squared error for this solution to the three values of squared error you got in the first part of the problem. Is the result what you expected ?

(c) In this part of the problem we'll return to the idea of recovering a signal from its projections, which we encountered once already in the guise of the Projection-Slice Theorem. This time we'll look at it in a Linear Algebra context. Let's think of the unknown "slice" of the body that we want to image as being a square array of "pixels", or an "image matrix". For simplicity, we'll assume that this unknown image matrix is only 2×2. Assume you can measure the projections along the rows and along the columns of this matrix, so that you have 4 measurements (2 from the sums along each row and 2 from the sums along each column).

   (i) Can you recover the elements of the matrix uniquely from these 4 measurements? If so, how? If not, how do you know? Note that you must prove your answer using linear algebra, not simply report a result from Matlab or some other program.
   (ii) Now suppose you can measure not only the row sums and the column sums, but also the sums down the diagonal and the anti-diagonal. This gives you 6 measurements. How does this change the situation compared to the previous one? What would you suggest to do here?

Problem 2: Two–Layer Scattering

Let's take a look at a two–layer problem using our simple model to illustrate the difficulties of subsurface sensing. Download the file 10472-3.zip from the website, unzip the files, open Matlab, and run hb.m. This will create arrays for the wavelength, w, and specific absorption, *kappa_ox* and *kappa_dox* for oxy– and deoxy–hemoglobin respectively.

Let's assume that our camera has eight bits, so the results will be integers from 0 (dark) to 255 (bright), and that we have five counts of random noise. We will always adjust the aperture stop of the camera so that the maximum count is 250.

Let the concentration of hemoglobin be 0.12×10^{-3} Molar, which is typical of human dermis under normal conditions. Let's assume a reduced scattering coefficient of 100/cm.

(a) Begin with $R_0 = 0$ and $T_e = 1$, which is a simple single–layer model in which the reflectivity is the albedo. Plot the reflectivity for oxygen saturation values of 1, 0, and 0.5.

(b) Now increase R_0 and decrease T_e, and repeat the plot.

(c) Discuss how the various parameters affect our ability to measure hemoglobin and its oxygen saturation.