

G205

# Fundamentals of Computer Engineering

CLASS 19, Mon. Nov. 15 2004

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M-W, 1:30pm-3:10pm

# Maximum Flow, 1 (a Light Introduction)

- ◆ Another problem that can be modeled through graphs
- ◆ Answers questions about “material flows” along a network
  - Is given a graph  $G=(V,E)$ , a source and a sink
  - Source produces material (steady rate)
  - Sink receives it (same rate)
- ◆ The “flow” is the rate at which the material moves

# Maximum Flow, 2

- ◆ Each directed edge = **conduit** for the material to go through
- ◆ Given the rate, each conduit has a stated **capacity**
- ◆ Maximum flow problem = compute the greatest rate material can be sent from source to sink within the stated capacities

# Flow Networks

- ◆ A flow network  $G=(V,E)$  is a directed graph such that
  - Each  $(u,v) \in E$  has a capacity  $c(u,v) \geq 0$
  - If  $(u,v) \notin E$  then  $c(u,v) = 0$
  - There is a source  $s$  and a sink  $t$
  - Every other vertex  $v$  is such that  $s \rightsquigarrow v \rightsquigarrow t$   
→ the network is connected and  $|E| \geq |V| - 1$

# Flows, 1

- ◆ A flow in a flow network  $G$  is a function  $f: V \times V \rightarrow \mathbf{R}$  that satisfies:
- **Capacity constraints:** For all  $u, v \in V$   
 $f(u, v) \leq c(u, v)$
  - **Skew symmetry:** For all  $u, v \in V$   
 $f(u, v) = -f(v, u)$
  - **Flow conservation:** For all  $u \in V \setminus \{s, t\}$   
 $\text{SUM}(v \in V) f(u, v) = 0$

# Flows, 2

- ◆ The quantity  $f(u,v)$  can be  $=, >$  or  $< 0$  and is called the **flow from  $u$  to  $v$**
- ◆ The **value of a flow  $f$**  is defined as  $|f| = \text{SUM}(v \in V) f(s,v)$  (total flow out of the source)
- ◆ **Maximum flow** = Given a flow network  $G$  with source  $s$  and sink  $t$  we want to find a flow  $f$  of maximum value

# The Ford-Fulkerson Method, 1

- ◆ Method  $\neq$  algorithm
- ◆ Three basic ideas:
  - Residual networks
  - Augmenting paths
  - Cuts
- ◆ Iterative method: Start with all  $f(u,v)=0$  and at each iteration the flow is augmented

# The Ford-Fulkerson Method, 2

Ford-Fulkerson-Method( $G,s,t$ )

initialize flow to 0

while exists an augmenting path  $p$  do

    augment flow  $f$  along  $p$

return  $f$

# Residual Networks, 1

◆ Given a flow network  $G$  and a flow  $f$ , a **residual network** consists of edges that can admit more flow

◆ Residual capacity of  $(u,v)$ :

$$c_f(u,v) = c(u,v) - f(u,v)$$

(It is the additional flow we can push from  $u$  to  $v$  before exceeding the capacity  $c(u,v)$  )

# Residual Networks, 2

- ◆ Given a flow network  $G$  and a flow  $f$  the residual network of  $G$  induced by  $f$  is  $G_f = (V, E_f)$  such that:
  - $E_f = \{(u, v) \in V \times V : c_f(u, v) > 0\}$
  - Careful about  $E_f$ : it can be tricky ;-)
- ◆ Edges in  $E_f$  are either in  $E$  or their reversals

# Augmenting Paths

- ◆ An augmenting path in a flow network  $G$  with flow  $f$  is a simple path from  $s$  to  $t$  in  $G_f$
- ◆ We can always increase a flow from  $s$  to  $t$  along an augmenting path
- ◆ The **residual capacity** of an augmenting path  $p$  is the maximum amount by which we can increase the flow on each of its edges

$$c_f(p) = \min\{c_f(u,v) : (u,v) \text{ is on } p\}$$

# Cuts of Flow Networks

- ◆ A **cut**  $(S, T)$  of a flow network  $G=(V, E)$  is a partition of  $V$  into  $S$  and  $T=V \setminus S$  such that  $s \in S$  and  $t \in T$
- ◆ If  $f$  is a flow, the net flow across the cut  $(S, T)$  is:  $f(S, T) = \sum_{x \in S} \sum_{y \in T} f(x, y)$
- ◆ The capacity of a cut is  $c(S, T) = \sum_{x \in S} \sum_{y \in T} c(x, y)$
- ◆ A **minimum cut** of a network is a cut whose capacity is minimum over *all* cuts of the network

# Max-Flow Min-Cut Theorem

- ◆ If  $f$  is a flow in a flow network  $G=(V,E)$  with source  $s$  and sink  $t$ , then the following conditions are equivalent
  1.  $f$  is a maximum flow in  $G$
  2.  $G_f$  contains no augmenting paths
  3.  $|f| = c(S,T)$  for some cut  $(S,T)$  of  $G$

# The Basic Ford-Fulkerson Algorithm

Ford-Fulkerson( $G, s, t$ )

for each  $(u, v) \in E$  do

$f[u, v] = f[v, u] = 0$

while there is a  $p = s \rightsquigarrow t$  in  $G_f$  do

$c_f(p) = \min\{c_f(u, v) : (u, v) \text{ in } p\}$

for each  $(u, v)$  in  $p$  do

$f[u, v] = f[u, v] + c_f(p)$

$f[v, u] = -f[u, v]$

# Analysis

- ◆ Depends on how an augmenting path is determined
- ◆ If arbitrarily chosen:  $O(E f^*)$ , where  $f^*$  is the output maximum flow
- ◆ An augmenting path can be found with a BFS (Edmund-Karp algorithm):  $O(V E^2)$

# Assignments

- ◆ Textbook, Chapter 26, pages 643—664
- ◆ Updated information on the class web page:

[www.ece.neu.edu/courses/eceg205/2004fa](http://www.ece.neu.edu/courses/eceg205/2004fa)