G205 Fundamentals of Computer Engineering

CLASSES 25, Wed. Dec. 3 2003

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M-W, 9:50am-11:30am, 410 Ell

Maximum Flow, 1 (a Light Introduction)

- Another problem that can be modeled through graphs
- Answers questions about “material flows” along a network
  - Is given a graph G=(V,E), a source and a sink
  - Source produces material (steady rate)
  - Sink receives it (same rate)
- The “flow” is the rate at which the material moves

Maximum Flow, 2

- Each directed edge = conduit for the material to go through
- Given the rate, each conduit has a stated capacity
- Maximum flow problem = compute the greatest rate material can be sent from source to sink within the stated capacities

Flow Networks

- A flow network $G=(V,E)$ is a directed graph such that
  - Each $(u,v) \in E$ has a capacity $c(u,v) \geq 0$
  - If $(u,v) \notin E$ then $c(u,v) = 0$
  - There is a source s and a sink t
  - Every other vertex v is such that $s \leadsto v \leadsto t$
    → the network is connected and $|E| \geq |V|-1$

Flows, 1

- A flow in a flow network G is a function f: V x V $\rightarrow$ **R** that satisfies:
  - Capacity constraints: For all u,v $\in$ V $f(u,v) \leq c(u,v)$
  - Skew symmetry: For all u,v $\in$ V $f(u,v) = -f(v,u)$
  - Flow conservation: For all u $\in$ $V \setminus \{s,t\}$ $\mathrm{SUM}(v \in V)\ f(u,v) = 0$

Flows, 2

- The quantity f(u,v) can be =,> or < 0 and is called the flow from u to v
- The value of a flow f is defined as

  $|f| = \text{SUM}(v \in V)\ f(s,v)$ (total flow out of the source)
- Maximum flow = Given a flow network G with source s and sink t we want to find a flow f of maximum value

The Ford-Fulkerson Method, 1

- Method ≠ algorithm
- Three basic ideas:
  - Residual networks
  - Augmenting paths
  - Cuts
- Iterative method: Start with all $f(u,v)=0$ and at each iteration the flow is augmented

The Ford-Fulkerson Method, 2

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Ford-Fulkerson-Method(G,s,t)
  initialize flow to 0
  while exists and augmenting path p do
    augment flow f along p
  return f
```

Residual Networks, 1

- Given a flow network G and a flow f, a residual network consists of edges that can admit more flow
- Residual capacity of (u,v):

$$c_f(u,v) = c(u,v) - f(u,v)$$

(It is the additional flow we can push from u to v before exceeding the capacity c(u,v))

Residual Networks, 2

- Given a flow network G and a flow f the residual network of G induced by f is $G_f=(V,E_f)$ such that:
  - $E_f=\{(u,v) \in VxV: c_f(u,v)>0\}$
  - Careful about E_f: it can be tricky ;-)
- Edges in E_f are either in E or their reversals

Augmenting Paths

- An augmenting path in a flow network G with flow f is a simple path from s to t in G_f
- We can always increase a flow from s to t along an augmenting path
- The residual capacity of an augmenting path p is the maximum amount by which we can increase the flow on each of its edges

$$c_f(p)=\min\{c_f(u,v): (u,v) \text{ is on } p\}$$

Cuts of Flow Networks

- A cut (S,T) of a flow network G=(V,E) is a partition of V into S and T=V\S such that $s \in S$ and $t \in T$
- If f is a flow, the net flow across the cut (S,T) is: $f(S,T)=SUM(x \in S)SUM(y \in T)f(x,y)$
- The capacity of a cut is $c(S,T)=SUM(x \in S)SUM(y \in T)c(x,y)$
- A minimum cut of a network is a cut whose capacity is minimum over *all* cuts of the network

Max-Flow Min-Cut Theorem

- If f is a flow in a flow network $G=(V,E)$ with source s and sink t, then the following conditions are equivalent

1. f is a maximum flow in G
2. G_f contains no augmenting paths
3. $|f| = c(S,T)$ for some cut (S,T) of G

The Basic Ford-Fulkerson Algorithm

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Ford-Fulkerson(G,s,t)
  for each (u,v) ∈ E do
    f[u,v]=f[v,u]=0
  while there is a p=s⇝t in G_f do
    c_f(p)=min{c_f(u,v): (u,v) in p}
    for each (u,v) in p do
      f[u,v] =f[u,v]+c_f(p)
      f[v,u]=-f[u,v]
```

Analysis

- Depends on how an augmenting path is determined
- If arbitrarily chosen: $O(E f^*)$, where f^* is the output maximum flow
- An augmenting path can be found with a BFS (Edmund-Karp algorithm): $O(V E^2)$

Assignments

- Textbook, Chapter 26, pages 643—664
- Updated information on the class web page:

  www.ece.neu.edu/courses/eceg205/2003fa