ADAPTIVE LINEAR OBSERVER FOR NONLINEAR SYSTEMS

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Abstract: In this paper, we describe the use of Luenberger state estimators for general nonlinear, time-varying systems. Since, in general, it is difficult to determine globally stable pre-specified observer gains for nonlinear systems, we propose using an adaptive observer gain vector that will allow learning of stable values throughout the state estimation process. To this end, we will derive a stochastic gradient adaptation algorithm for the observer gains based on the mean-square error of the estimated outputs. The performance of the adaptive observer scheme will be tested on linear and non-linear systems, including the chaotic Lorenz attractor.

Keywords: Adaptive observer, nonlinear systems, state estimation.

1. INTRODUCTION

Following the celebrated theory of Kalman and the associated Kalman filter, and inspired by the concepts laid by Luenberger on the design of linear observers (state estimators), there have been some work on the design of stable observer schemes for general nonlinear and time-varying systems (Elmas and Zelaya de la Parra,1996;Du *et al.,* 1995). These extensions, under the influence of the classical observer design theory, were focused on analytical design techniques of the nonlinear observer. The main approach followed in this line of practice is to choose the observer gains such that the overall linearized error dynamics matrix consisting of the gain vector and the Jacobians of the state dynamics and the output mapping has stable eigenvalues over a closed subset of the state-space. Under these conditions and for state trajectories that remain in this subset at all times, convergence could be proven (Elmas and Zelaya de la Parra,1996;Du *et al.*, 1995). This procedure of analytical extended Luenberger observer (ELO) design has been applied successfully to realistic nonlinear system models (Elmas and Zelaya de la Parra,1996;Du *et al.,* 1995;Orlowska-Kowalska, 1989;Song *et al.*, 2000;Du and Brdys, 1993). The same approach has also been utilized in designing nonlinear state feedback stabilizers for nonlinear systems (Rodrigues-Millan *et al.*, 1997;Delepaut *et al.*, 1989). In contrast to this classical approach of tackling the observer design problem analytically, we introduce the adaptive system approach, where the observer gains are continuously learned throughout the process of state estimation.

The organization of this paper is as follows. In Sec. 2, we present the structure of the extended Luenberger observer for nonlinear and time-varying systems. In Sec. 3, we describe the stochastic gradient algorithm for the adaptation of the observer gains on a sample-by-sample basis. Finally in Sec. 4, we study the performance of the proposed adaptive Luenberger observer scheme on a variety of dynamical systems.

2. EXTENDED LUENBERGER OBSERVER

For linear time-invariant (LTI) systems described by the dynamic equations

$$\begin{aligned} x_{k+1} &= Ax_k + Bu_k \\ y_k &= Cx_k + Du_k \end{aligned} \tag{1}$$

where x_k is the state vector, u_k is the input vector and y_k is the output vector, the Luenberger observer, provided that the pair (A,C) is observable, is given by

$$\begin{aligned}\tilde{x}_{k+1} &= A\tilde{x}_k + Bu_k + L(y_k - \tilde{y}_k) \\ \tilde{y}_k &= C\tilde{x}_k + Du_k\end{aligned} \quad (2)$$

The global asymptotic stability of the observer can be guaranteed by setting the gain vector L to a value such that the state estimation error dynamics defined by

$$(x_{k+1} - \tilde{x}_{k+1}) = (A - LC)(x_k - \tilde{x}_k) \quad (3)$$

has stable eigenvalues (Kailath, 1980).

The extension of the Luenberger observer to nonlinear systems is straightforward. Given a nonlinear dynamical system, possibly time-varying, defined by the equations

$$\begin{aligned}x_{k+1} &= f(x_k, u_k, k) \\ y_k &= h(x_k, u_k, k)\end{aligned} \quad (4)$$

the extended Luenberger observer is characterized by the dynamic equations

$$\begin{aligned}\tilde{x}_{k+1} &= f(\tilde{x}_k, u_k, k) + L(y_k - \tilde{y}_k) \\ \tilde{y}_k &= h(\tilde{x}_k, u_k, k)\end{aligned} \quad (5)$$

Although there is a solid theory behind the linear Luenberger observer in (2) and there are simple and rigorous analytical methods for selecting the observer gain vector L, such results are not yet available for the extended version in (5), yet. However, in the next section, we will demonstrate an approach to overcome this difficulty by letting L adapt on-line during the course of estimation.

3. STOCHASTIC MSE GRADIENT TO ADAPT THE OBSERVER GAINS

The stochastic gradient approach has probably been popularized by Widrow's derivation of the LMS algorithm and its incredible success in solving difficult problems with ease in spite of its computational simplicity (Widrow and Stearns, 1985). In selecting the Luenberger observer gains we try to minimize the mean-square-error (MSE) between the actual output y_k and the estimated output $\tilde{y}_k$, which is defined by $E[(y_k - \tilde{y}_k)^T (y_k - \tilde{y}_k)]$. Employing the stochastic gradient approach (going only one step back in time), the instantaneous squared-error becomes the stochastic approximation to this cost function.

$$\begin{aligned}&\hat{J} = (y_k - \tilde{y}_k)^T (y_k - \tilde{y}_k) \\ &\frac{\partial \hat{J}}{\partial L_{.j}} = -2(y_k - \tilde{y}_k)^T h_x(\tilde{x}_k, u_k, k) \frac{\partial \tilde{x}_k}{\partial L_{.j}} \\ &\frac{\partial \tilde{x}_k}{\partial L_{.j}} = \begin{bmatrix} f_x(\tilde{x}_{k-1}, u_{k-1}, k-1) \\ -L_{.j} \cdot h_x(\tilde{x}_{k-1}, u_{k-1}, k-1) \end{bmatrix} \frac{\partial \tilde{x}_{k-1}}{\partial L_{.j}} + (y_{j,k-1} - \tilde{y}_{j,k-1}) \cdot I_{nxn} \\ &L_{.j} \leftarrow L_{.j} - \eta \cdot \frac{\partial \hat{J}}{\partial L_{.j}}\end{aligned} \quad (6)$$

For multiple output systems, L will have multiple columns. Using (5), we can derive the (approximate) gradient of the instantaneous cost function with respect to each of the columns of L, denoted by $L_{.j}$. This gradient is given in (6) and is approximate because the dependency of the current weight vector on the previous value of the weight vector is omitted.

It is well known and is trivial to show that this stochastic gradient forces the weight vector L to converge to the MSE-optimal values in the mean. Using a sufficiently small step size, which is also essential for the accuracy of the approximation, the misadjustment may be kept down.

4. CASE STUDIES

In this section, we study the performance of the proposed adaptive observer scheme on linear and nonlinear systems, including the Van der Pol oscillator and the chaotic Lorenz attractor. In all simulations, the step size is 0.01.

4.1 Linear Time-Invariant System

Our first case study uses a stable single-input single-output LTI system excited by zero-mean white Gaussian noise (WGN). The system matrices are selected to be

$$A = \begin{bmatrix} 0.9 & 0.1 \\ 0 & 0.8 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ -0.9 \end{bmatrix}, \quad c = \begin{bmatrix} 1 & 0 \end{bmatrix} \quad (7)$$

Two simulation results are presented in Fig. 1; one with noiseless measurements and one with measurement noise at 25dB signal-to-noise ratio (SNR). Notice in Fig. 1a, where there is no measurement noise, the state stimation errors decay exponentially, whereas under measurement noise the estimate accuracy is bounded by the noise power, as expected. For the LTI system case, it is possible to determine an upper bound for the step size of the stochastic steepest descent to guarantee the asymptotic stability of the algorithm and the observer. However, this derivation will be omitted.

4.2 Van der Pol Oscillator

Our second case study involves the discretized (first-order difference) Van der Pol oscillator dynamics. This is characterized by the following equations.

$$\begin{aligned}x_{1,k+1} &= x_{1,k} + T \cdot x_{2,k} \\ x_{2,k+1} &= x_{2,k} - 9T \cdot x_{1,k} + \mu \cdot T(1 - x_{1,k}^2) x_{2,k}\end{aligned} \quad (8)$$

In (8), T is the sampling time used in the discretization and the smaller T is the better the approximation. Note that although the continuous Van der Pol dynamics are globally stable, the first-order discretization causes some portion of the state-space to become unstable. If the state-trajectory passes through this unstable region, the observer might fail to follow the diverging state trajectory. However, as long as the state trajectory remains in the stable region, the observer converges smoothly.

Once again, we present two simulation results corresponding to noiseless and noisy measurements.

In the noisy output measurement case, the SNR is again 25 dB. These results are shown in Fig. 2. In these simulations we have assumed a sampling time of $T = 0.1$ and the oscillator parameter is selected to be $\mu = 0.5$. The system output is taken as the first state variable. The state estimation errors exhibit similar behavior to that observed in the LTI system case. In the noiseless case, the errors decay exponentially and in the noisy case they are bounded by a value controlled by the measurement noise.

4.3 Lorenz Attractor

The Van der Pol oscillator states converge to a limit cycle and one suspects if this periodicity helps the observer exhibit good performance. In order to clear these doubts, we test the observer scheme on a chaotic system that has very high Lyapunov exponents, thus without any correctional terms, the slightest difference in initial conditions will lead to a very large divergence in the state trajectories. The discretized Lorenz attractor dynamics, are given by

$$\begin{aligned} x_{1,k+1} &= (1 - T \cdot \sigma) x_{1,k} + T \cdot \sigma \cdot x_{2,k} \\ x_{2,k+1} &= (1 - T) \cdot x_{2,k} + T \cdot x_{1,k} \cdot (r - x_{3,k}) \\ x_{3,k+1} &= (1 - T \cdot b) \cdot x_{3,k} + T \cdot x_{1,k} \cdot x_{2,k} \end{aligned} \quad (9)$$

where the sampling time is taken as $T = 0.01$ and the attractor parameters are chosen to be $\sigma = 10,\ r = 28,\ b = 8/3$. The system output is assumed to be the first state variable. Two simulation results corresponding to noiseless measurement and 25dB-SNR measurement noise are shown in Fig. 3.

5. NOISE ANALYSIS FOR LTI SYSTEMS

In the case studies presented in the previous section, we have observed that the additive measurement noise causes the state estimate accuracy to be bounded. In fact this is expected and the Kalman filter addresses this problem and offers the optimal gain matrix to yield the best state estimates in terms of minimum estimation MSE. In this section, we will present an analytical analysis of the effect of noise on the asymptotic signal-to-error-ratio of the proposed adaptive observer scheme. For simplicity, we will assume a single-input single-output (SISO) LTI system. However, the approach can be easily generalized to multi-input multi-output (MIMO) linear systems and can also be employed to approximately determine the effect of noise for nonlinear systems using linearization.

Now, without loss of generality, suppose that the input to our SISO system (A,b,c) is zero mean white Gaussian noise with power σ_u^2. The output measurement noise is also zero-mean white Gaussian with σ_v^2. We observe that

$$\begin{aligned} E[x_{k+1}] &= AE[x_k] + bE[u_k] = AE[x_k] \\ E[y_k] &= cE[x_k] \end{aligned} \quad (10)$$

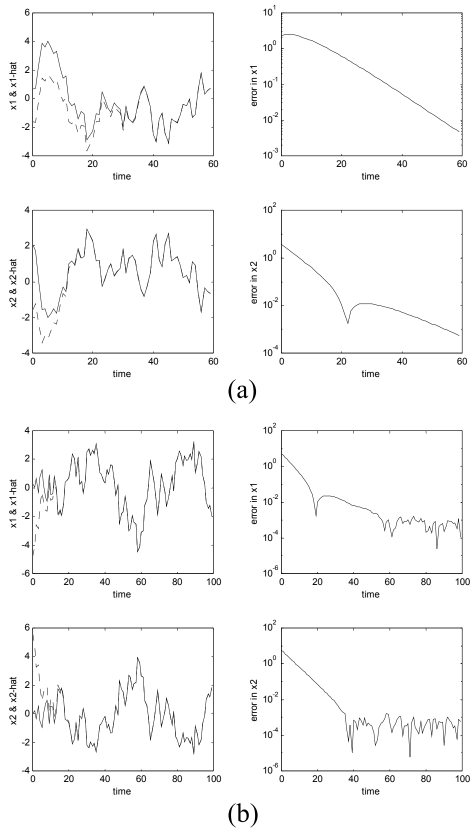

Figure 1. The LTI system states, their estimates, and the estimation errors for a) the noiseless measurement case b) the noisy measurement case.

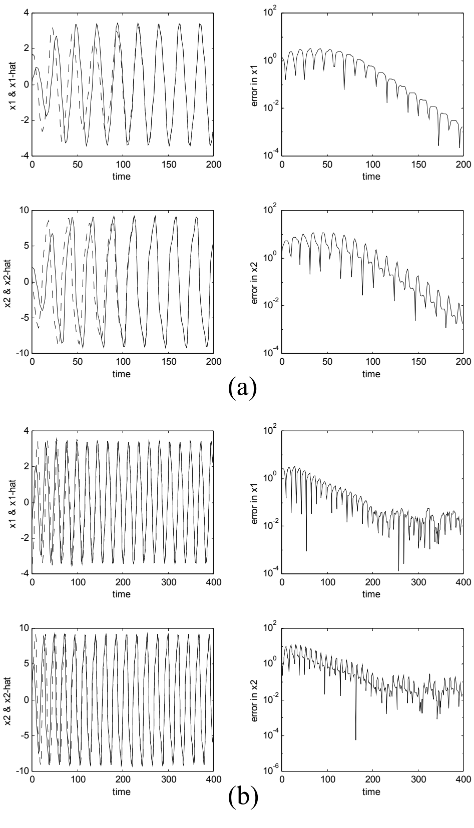

Figure 2. The Van der Pol oscillator states, their estimates, and the estimation errors for a) the noiseless measurement case b) the noisy measurement case.

from which we notice that if the eigenvalues of A are inside the unit circle, $\lim_{k\to\infty} E[x_k]=0$, and likewise $\lim_{k\to\infty} E[y_k]=0$. Now let's consider

$$E[x_{k+1}x_{k+1}^T]=AE[x_k x_k^T]A^T+\sigma_u^2 bb^T \qquad (11)$$

which, along with the fact that the means converging to zero, defines the asymptotic behavior of the state covariance matrix. The state covariance matrix is defined as $\Sigma_{xk}=E[(x_k-m_{xk})(x_k-m_{xk})^T]$. We notice from (11) that this covariance matrix, provided that A has stable eigenvalues, asymptotically converges to a value $\Sigma_{x\infty}$ that satisfies the following equation.

$$\Sigma_{x\infty}=A\Sigma_{x\infty}A^T+\sigma_u^2 bb^T \qquad (12)$$

By recursively substituting the right hand side of (12) in for $\Sigma_{x\infty}$ in the right hand side, we can obtain an explicit solution for $\Sigma_{x\infty}$.

$$\Sigma_{x\infty}=\sigma_u^2\sum_{i=0}^{\infty}(A^i b)(A^i b)^T=\sigma_u^2\beta \qquad (13)$$

From this, we can determine the asymptotic output variance in terms of the Markov parameters cA^ib as

$$\sigma_{y\infty}^2=c\Sigma_{x\infty}c^T=\sigma_u^2\sum_{i=0}^{\infty}(cA^i b)(cA^i b)^T=\sigma_u^2\gamma \qquad (14)$$

Assuming that the observer in (3) has access to the noisy measurement $\bar{y}_k=y_k+v_k$, where v_k is zero-mean WGN with variance σ_v^2, we see that the observer error dynamics are given by

$$e_{k+1}=(A-L_k c)e_k-L_k v_k \qquad (15)$$

where $e_k=x_k-\tilde{x}_k$. Asymptotically, the error covariance matrix converges to

$$\Sigma_{e\infty}=\sigma_v^2\sum_{i=0}^{\infty}((A-L_\infty c)^i L_\infty)((A-L_\infty c)^i L_\infty)^T=\sigma_v^2\bar{\beta} \qquad (16)$$

Under these circumstances, the asymptotic signal-to-error ratio (SER) for the estimation of the j^{th} state variable can be calculated in decibels as

$$SER_j=10\log_{10}\frac{\sigma_u^2\beta_{jj}}{\sigma_v^2\bar{\beta}_{jj}} \qquad (17)$$

The measurement signal-to-noise ratio is similarly

$$SNR=10\log_{10}\frac{\sigma_u^2\gamma}{\sigma_v^2} \qquad (18)$$

From (17) and (18), we notice that $SER_j=SNR+C$, where C is some constant that depends on A, b, and c. In order to demonstrate this result experimentally, we have performed a series of Monte Carlo simulations. Given the system described in (7), we have run 10 Monte Carlo simulations for each of the various SNR levels ranging from –10dB to 25dB. The SER and SNR values corresponding to each simulation are estimated using the last 500 samples of each 1000-sample run and these values were averaged over these 500 samples as well as the 10 Monte Carlo simulations, which used randomly selected initial state vectors.

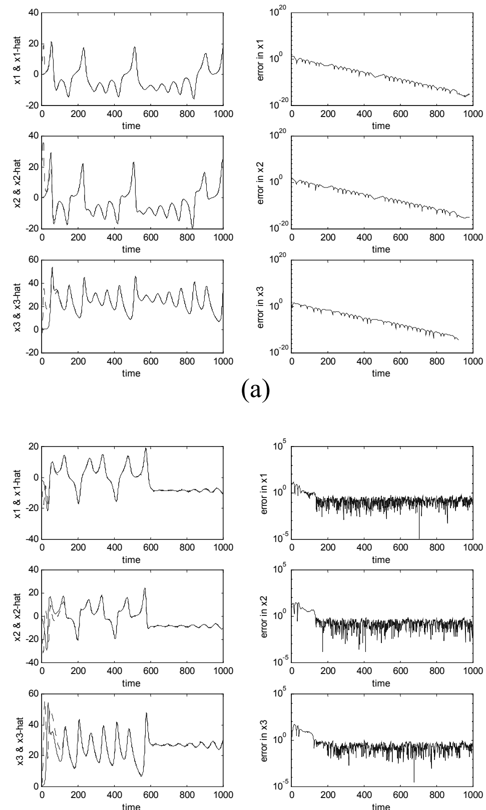

Figure 3. The three states, their estimates, and the estimation errors for a) the noiseless measurement case b) the noisy measurement case for the Lorenz attractor.

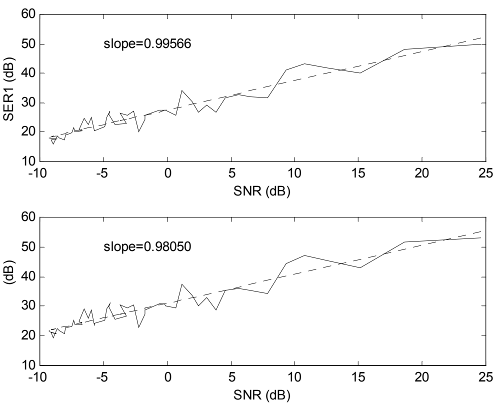

Figure 4. The SER (dB) versus SNR (dB) plots using Monte Carlo simulation results for the two state estimates of an LTI system.

The results of these Monte Carlo simulations are summarized in Fig. 4. In the two subplots, we show the SER for the estimation of each state variable versus the SNR of the output measurements. The slopes of the best lines fit using least squares are consistent with our expectations. They are 0.99566 and 0.98050, which are very close to the predicted value of 1.

6. CONCLUSIONS

In this paper, we have introduced an adaptive extended Luenberger observer scheme for state estimation in nonlinear, time-varying systems. We have demonstrated the effectiveness and the accuracy of the state estimator on an LTI system, the Van der Pol oscillator, and the Lorenz attractor. The proposed adaptive observer scheme is therefore a promising approach for state estimation in nonlinear dynamical plants.

The adaptive observer tries to obtain the state estimates that minimize the output estimation error. In the case of linear systems, we believe that the observer gains converge to those of the Kalman filter, although we do not have a rigorous proof. For externally excited nonlinear systems, we should not expect the observer gains to converge to any specific value; in order to maintain the optimality of the output estimates, the observer gains will keep adapting based on the current local Jacobian matrices of the state and output equations.

We have studied analytically the noise rejection capability of the observer in the LTI system case and showed that the power of the state estimation errors depend on the output measurement signal to noise ratio, input power, and the Markov parameters of the system. Using Monte Carlo simulations, we have demonstrated that the signal-to-estimation-error ratios accurately and rapidly converge to their theoretical asymptotic values. An extension of this noise analysis to predict the estimation-error power in the general nonlinear system case lies as a future line of research.

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